

Computational Problem of the Determinant Matrix Calculation

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Abstract A new more accurate formula to calculate condition number of the determinant of matrix is proposed. The effectiveness of the new formula was confirmed by the examples of the Hilbert matrix. In order to validate the new approach comparison with statistical Monte Carlo calculation was used.

Keywords condition number of matrix, determinant of Hilbert matrix, Frobenius's norm, standard deviation.

I. INTRODUCTION

The determinant of matrix is an important element for numerical analysis, optimization and design of various electrical circuits. Cramer's rule is an explicit formula for the solution of a system of linear equations (SLE). For each variable, the denominator is the determinant of the matrix of coefficients, while the numerator is the determinant of a matrix in which one column has been replaced by the vector of constant terms [1].

In the numeric methods the condition number of matrix is playing important role. Extended notation of the value of condition number is given in the form of the product of norm of original ($\|A\|$) and inverse ($\|A^{-1}\|$) matrices or relationship of eigenvalues of matrix A [1], [2].

$$|\lambda_{\max}| / |\lambda_{\min}| \geq \nu(A) \leq \|A\| \cdot \|A^{-1}\| \quad (1)$$

The bigger value of $\nu(A)$, the bigger instabilities occur while solving SLE or during the determinant calculation.

II. COMPUTATIONAL PROBLEM

The accuracy of the numerical methods is a fundamental problem [3]. The Hilbert matrices are canonical examples of ill-conditioned matrices, making them notoriously difficult to use in numerical computation and of the determinant calculation.

As an example (table 1) of the strange "surprises" we show the calculation of the determinant of the matrix Hilbert (DetH) of ranks $n = 9..15$ with programs MAPLE V5 R4, MathCad 6, MathCad 15, MathLab, and author's PASCAL-program. Correct results are the MAPLE program's, whose precision is based on the fractional arithmetic.

The bold font indicates the exact value of digits of mantissa. The table shows that some programs incorrectly calculated the determinants of Hilbert matrix for $n > 12$. Even negative values (for $n > 13$) were calculated. This means that those programs use only the *double*-format.

TABLE I
COMPARISON OF HILBERT MATRIX DETERMINANT CALCULATION

n	det H _n ×	Maple 5 R4 (precisely)	MathCad 6 (double)	MathCad 15 (double)	MathLab (double)	PASCAL (extended)
9	×10 ⁻⁴³	9.720234312	9.72027340	9.7202716438	9.7203	9.72023431
10	×10 ⁻⁵³	2.164179226	2.16438968	2.1644385324	2.1644	2.16417933
11	×10 ⁻⁶⁵	3.019095334	3.02799443	3.0265052167	3.0273	3.01910241
12	×10 ⁻⁷⁸	2.637780651	2.70240144	2.8503634897	2.8581	2.63796503
13	×10 ⁻⁹²	1.442896518	1.87070764	3.0135327695	4.4480	1.445036161
14	×10 ⁻¹⁰⁸	4.940314914	-532.618393	-351.16701385	-39.220	5.039172101
15	×10 ⁻¹²⁴	1.058542743	-40701.4442	-39397.218029	-21903	9.554398512

III. NEW FORMULA FOR CONDITION NUMBER OF DETERMINANT MATRIX

To effectively solve the problem of estimating of the accuracy the determinant calculation we use a statistical approach (Monte-Carlo) [2]. Random change of matrix elements leads to a dispersion value of its determinant. Greater dispersion of determinant means that the matrix is more difficult for the calculations. The relative error is calculated as the following formula

$$\det A = \det A_0 \pm \Delta_{3\sigma}, \delta(\det A) = |\Delta_{3\sigma} / \det A_0|,$$

where $\det A_0$ - mathematical expectation of $\det A$, $\Delta_{3\sigma}$ - standard deviation of the width 3σ .

Monte-Carlo calculations to determine the required precision matrix determinant is relatively complicated and lengthy process. Therefore author proposed a new formula for the calculation of the condition number of the matrix determinant (CNMD) without the Monte-Carlo calculus.

$$K_T(A) = \|A^{-1} \cdot A^T\|_F \quad (2)$$

where $\|A^{-1} \cdot A^T\|_F$ is the Frobenius norm of Hadamard's product of inverse and transposed matrix A . We will assess the new formula on the basis of the calculation of the Hilbert matrices (table II). The condF column by means the CNMD (1) using the Frobenius norm. The $K_T(A)$ column by means CMND according to (2). The $K_S(A)$ column by means CMND using Monte-Carlo calculation.

The little difference between $K_T(A)$ and $K_S(A)$ confirms the suitability of the new formula (2).

TABLE II
COMPARISON OF CONDITION NUMBER OF THE MATRIX DETERMINANT

n	detH _n ×	condF	$\frac{ \lambda_{\max} }{ \lambda_{\min} }$	$K_T(H_n)$	$K_S(H_n)$
5	×10 ⁴	48.085..	47.661..	4.6781	4.6809
8	×10 ⁸	154.94..	152.58..	8.3703	8.3726
12	×10 ¹⁶	175.18..	171.32..	5.6787	5.8304
50	×10 ⁷¹	1500.9..	1422.9..	9.7697	-
100	×10 ¹⁴⁸	405.37..	-	1.2283	-

IV. REFERENCES

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