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MODELY STOCHASTICKÉ A FRAKCIONÁLNÍ STOCHASTICKÉ VOLATILITY

Diplomová práce

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STOCHASTIC AND FRACTIONAL STOCHASTIC VOLATILITY MODELS

Master Thesis

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Declaration

I do hereby declare that the entire dissertation is solely my original work and that I have used only the cited sources.

Plzeň, 22th May 2014   Signature:

"Human beings, who are almost unique in having the ability to learn from the experience of others, are also remarkable for their apparent disinclination to do so."

Douglas N. Adams

Abstract

Stochastic and Fractional Stochastic Volatility Models

by Tomáš Sobotka

The main subject of the thesis is to study and implement selected stochastic volatility models alongside the newly proposed approximative fractional stochastic volatility model (FSV) that was firstly introduced by Intarasit and Sattayatham in 2011 [35]. After the semi-closed form solution of a generic pricing PDE is derived, we compare these modern approaches on the task of market calibration. This is done using both synthetic and the real market data. We also inspect a long-range dependence in market realized volatilities and we comment on suitability of the FSV approach with respect to the option market calibration.

Plzeň, May 2014.

Keywords: Hurst exponent, fractional Brownian motion, financial modelling, European options, stochastic volatility, market calibration.

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Acknowledgements

Glossary of Notation

$AAE(\Theta)$	Average absolute error for parameters Θ and corresponding model
$AARE(\Theta)$	Average absolute relative error for parameters Θ and corresponding model
ACV	Auto-covariance function
Aggvar	Aggregate Variance method
BSM	Black-Scholes-Merton model
GA	Genetic algorithm
GPH	Geweke-Porter-Hudak estimator
H	Hurst exponent
i	Imaginary unit $\sqrt{-1}$
i.i.d.	Independent and identically distributed
K	Strike price of an option
$L^2(\Omega)$ conv.	Mean square convergence, Definition 1.7
LRD	Long-range dependence, Definition 4.1
LSQ	Least-square optimisation routine *lsqnonlin()*
$MAE(\Theta)$	Maximum absolute error for parameters Θ and corresponding model
$\mathcal{N}(a, b^2)$	Gaussian distribution with mean a and variance b^2
ODE	Ordinary differential equation
PDE	Partial differential equation
Per	Periodogram analysis
r	Continuously compounded risk-free rate
R/S	Rescaled range analysis
S_t	Stock price process
$(S_T - K)^+$	$\max\{S_T - K, 0\}$
SA	Simulated annealing
SDE	Stochastic (ordinary) differential equation
τ	Time to maturity, $\tau := T - t$
Θ	Parameter set of a model
V_t	Value of an option at time t

Contents

Chapter 1

Introduction

1.1 Motivation for standard and fractional stochastic volatility models

The first attempt to model financial returns is dated back to 1900 when a French mathematician, Louis Bachelier, published his PhD thesis called *The Theory of Speculation* (English translation in [15]). He suggested a new stochastic process for stock-price modelling and option pricing. The process is the well known Wiener process [1] and it will be described in the next section. Bachelier's approach to stock modelling wasn't fully appreciated until late 1970s.

The idea was first revisited by Paul Samuelson who incorporated a deterministic drift term of stock returns. However, the main breakthrough in financial modelling and option pricing started due to F. Black, M. Scholes and R. C. Merton in 1973. They came up with the new framework for option pricing, a still widely popular Black-Scholes model (BSM) [5]. Their framework attains an explicit price for European-style options under certain assumptions. Therefore, it is very straightforward to value any European-style contract, while having the market data and the value of a single model parameter. This led to an immense increase in option trading, but also it raised a wave of criticism. The criticism involved so called *stylised facts* that are common to a variety of financial markets.

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[1] According to Ben-el-Mechaiekh and Dimand (foreword in [17]), W. Feller suggested naming it the Wiener-Bachelier process.

Empirical stylised facts

After market crash in 1987, empirical studies showing severe discrepancies between the markets and the BSM started to appear. These discrepancies were lately known as the stylized facts and we illustrate some of the facts on a data set that will accompany us throughout the thesis.

Even before Black, Scholes and Merton presented their main result, Mandelbrot in [41] argued that stock returns are not Normally distributed. In Figures 1.1a - 1.1c we compare the empirical distribution of logarithmic returns of the FTSE 100 index with assumed Gaussian distribution. Both the Q-Q plot 1.1b and Figure 1.1c shows that our sample exhibits fat-tails. Implications of fat-tails are well discussed, for instance, in a popular book written by N. Taleb [58].

(A) Sample distribution of logarithmic returns.

(B) Log-returns vs Normal distribution.

(C) "*Fat tails*" of the sample.

(D) Returns of the FTSE 100 index.
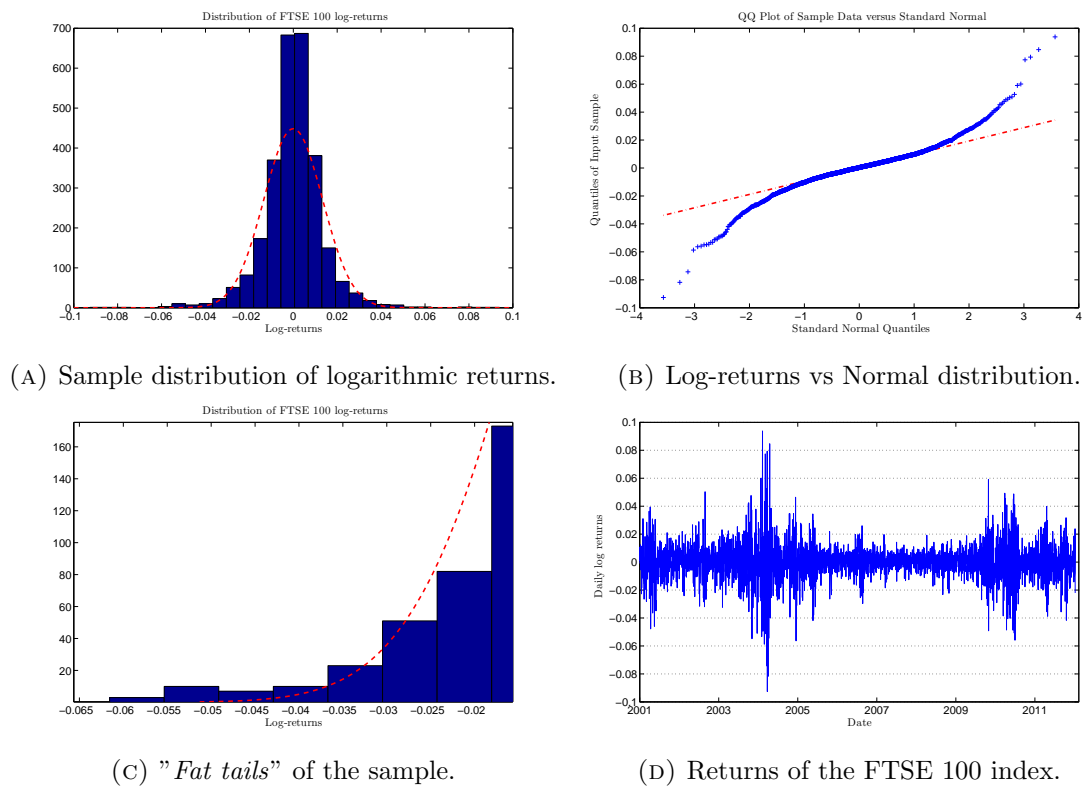
FIGURE 1.1: A typical distribution of log-returns for financial markets (FTSE 100: 2001-2011).

Another observed effect is called the volatility leverage. For lower than average values of the spot, the realized volatility tends to reach higher levels than for above average prices of the spot (see Figure 4.3).

The last empirical fact, we will mention, is known as the volatility clustering. Mandelbrot was one of the first who made the following observation [41]:

"...large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes."

Mandelbrot also suggests existence of long-range dependence in volatility time-series. This will be inspected in more detail in Chapter 4.

Beyond the Black-Scholes paradigm

Lately, both practitioners and academics pointed out that tasks of modern financial modelling require procedures that go well beyond the Black-Scholes model assumptions and that can mimic most of the introduced stylised facts [37]. In this thesis we focus on the tasks of option pricing and market calibration for continuous time models only.

Among the most popular option pricing tools are stochastic volatility models. These approaches neglect constant volatility assumption of the BSM. In fact, as the name suggests, the realized volatility is modelled as a stochastic process. This enables fitting the whole option price surface with reasonable errors (Chapter 6) and the models might mimic the volatility leverage (using coefficient ρ), fat-tails and a higher peaked distribution of returns (compared to the Gaussian distribution).

To explain the volatility clustering phenomena, one might use a stochastic volatility model that attains a long-range dependence in the volatility process. These approaches will be referred to as the fractional stochastic volatility models.

Structure of the thesis

In Chapter 2, we introduce the most popular stochastic volatility approaches and several fractional volatility models. Thereafter, we focus on the notion of approximative fractional volatility which is described in more detail.

We derive a generic pricing PDE that attains an explicit semi-closed form solution for the Bates, Heston and the approximative FSV model. The solution is expressed in terms of characteristic functions (see Appendix A) and only the Fourier transform integral (3.20) has to be computed numerically to price a given European call option.

The forth chapter includes estimation of long-range dependence in volatility data. To this purpose we employ both synthetic and the real market data consisting of 30, 60, 90-day realized volatilities of FTSE 100 returns.

Derivatives of our main interest, European options, are introduced in Chapter 5. We also describe the process of calibration to an option market and we briefly mention optimisation routines that will be applied to this aim.

In Chapter 6, the numerical results of a market calibration are presented for all three models and for several different calibration methods. We use both synthetically generated data and the data from FTSE 100 option market.

Last but not least, we conclude all key results of this thesis. We comment on applicability of the newly proposed FSV model in practise and we suggest several aspects for the future research.

1.2 Preliminaries

In this section we briefly introduce preliminaries that will be used later in the text. We will follow S. E. Shreve [57], B. Maslowski [44], L. C. Evans [21] and Biagini et al. [4]. See aforementioned references for a more detailed and involving overview. We start with a generic probability space, that will accompany us in the most parts of the thesis.

1.2.1 Probability space and objects defined on it

Definition 1.1 ($\sigma-$algebra)**.** Let Ω be a non-empty set and $\mathcal{F}$ be a collection of sets in Ω. If

- (i) $\emptyset \in \mathcal{F}$;

- (ii) $\mathcal{F}$ is closed under the complements, i.e. $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}$;

- (iii) $\mathcal{F}$ is closed under all countable unions, i.e. $A_1,\ A_2, ... \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$;

than we say $\mathcal{F}$ forms a $\sigma-algebra$. Particularly, the smallest $\sigma-$algebra containing all open subsets of $\mathbb{R}$ is called the Borel $\sigma-$algebra and is denoted as $\mathcal{B}$.

Definition 1.2 (Probability space)**.** Let Ω be a non-empty set, $\mathcal{F}$ be a collection of sets in Ω that forms a $\sigma-$algebra. A probability measure $\mathbb{P}$ is a function assigning to each $A \in \mathcal{F}$ a number in $[0, 1]$. $\mathbb{P}$ has to satisfy the following conditions:

- (i) $\mathbb{P}(\Omega) = 1$,

- (ii) For disjoint sets $A_1,\ A_2, ... \in \mathcal{F}$; we have $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$.

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.

First of all, we shall discuss the terminology connected with probability spaces. The elements of $\mathcal{F}$ are called *events*, $\omega \in \Omega$ is a *sample point* (or an elementary event) and the measure $\mathbb{P}(A)$ is interpreted as the *probability* of the event $A \in \mathcal{F}$. A property that is true, except for events with zero probability, is said to hold almost surely (or *a.s.*). For all the following definitions, unless specified otherwise, we assume a probability space denoted by $(\Omega, \mathcal{F}, \mathbb{P})$.

Although a probability space is one of the key constructs for random experiments, we cannot directly observe it. Therefore, we are interested in the mapping from Ω to $\mathbb{R}$, the values of which we can observe.

Definition 1.3 (Random variable)**.** A real-valued *random variable* is a function X defined on Ω; $X : \omega \mapsto \mathbb{R}$ satisfying for all $B \in \mathcal{B}$

$$\{\omega \in \Omega : X(\omega) \in B\} = X^{-1}(B) \in \mathcal{F}.$$

This means that the set of all ω for which $X(\omega)$ maps into subset of Borel $\sigma-$algebra $\mathcal{B}$ is stored in $\mathcal{F}$. Equivalently, we say $X(\omega)$ is $\mathcal{F}-$measurable.

Next definition introduces a notion of stochastic independence for various objects.

Definition 1.4.

(i) Events $A, B \in \mathcal{F}$ are said to be stochastically independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

(ii) $\sigma-$algebras $\mathcal{G}_1 \subset \mathcal{F}$ and $\mathcal{G}_2 \subset \mathcal{F}$ are stochastically independent if each two events $A \in \mathcal{G}_1$, $B \in \mathcal{G}_2$ are stochastically independent.

(iii) Random variables $X(\omega), Y(\omega)$ defined on the same probability space are said to be stochastically independent if $\sigma-$algebras $\mathcal{G}_1 := \left\{X^{-1}(B); B \in \mathcal{B}\right\}$ and $\mathcal{G}_2 := \left\{Y^{-1}(B); B \in \mathcal{B}\right\}$ are stochastically independent.

We briefly recall basic properties of random variables.

Definition 1.5. Mathematical expectation (expected value, mean value) of random variable $X(\omega)$ is defined by the following Lebesgue integral:

$$\mathbb{E}[X(\omega)] := \int_{\Omega} X(\omega)d\mathbb{P},$$

whenever the integral exists. *Conditional expectation* (conditional expected value, conditional mean value) given $\sigma-$algebra $\mathcal{H} \subset \mathcal{F}$ is defined as

$$\mathbb{E}[X(\omega)|\mathcal{H}] := \int\limits_{H \in \mathcal{H}} X(\omega)d\mathbb{P},$$

provided the integral exists. We can understand a conditional expectation as the best estimate of X given "all information" in $\mathcal{H}$ (and nothing else). Variance of random variable $X(\omega)$ takes the form:

$$\mathrm{Var}[X(\omega)] := \int\limits_{\Omega} \left(X(\omega) - \mathbb{E}[X(\omega)]\right)^2 d\mathbb{P} = \mathbb{E}\left[(X(\omega) - \mathbb{E}[X(\omega)])^2\right].$$

We will also use characteristic functions of random variables, especially for option pricing.

Definition 1.6 (Characteristic function). Let $X(\omega)$ be a real-valued random variable. Characteristic function of $X(\omega)$ is defined as

$$f_X(\phi) := \mathbb{E}\left[e^{i\phi X(\omega)}\right], \quad \text{for } \phi \in \mathbb{R},$$

where i is the imaginary unit.

Proposition 1.1. *Joint characteristic function of two stochastically independent random variables $X(\omega)$, $Y(\omega)$ takes the form*

$$f_{X+Y}(\phi) = f_X(\phi)f_Y(\phi). \tag{1.1}$$

Proof.

$$\begin{aligned}
f_{X+Y}(\phi) &= \mathbb{E}\left[e^{i\phi(X(\omega)+Y(\omega))}\right] \\
&= \mathbb{E}\left[e^{i\phi(X(\omega)}e^{i\phi(Y(\omega)}\right] \\
&= \mathbb{E}\left[e^{i\phi X(\omega)}\right]\mathbb{E}\left[e^{i\phi Y(\omega)}\right] \quad \text{(by independece)} \\
&= f_X(\phi)f_Y(\phi).
\end{aligned}$$

$\square$

Having a sequence of random variables defined on the same probability space, we might wonder how to define the notions of convergence/divergence. One of the possible ways is due to the next definition.

Definition 1.7 (Mean square convergence)**.** Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables and X be a random variable defined on the same probability space as all X_n. We say X_n converge to X in the mean squares (or in $L^2(\Omega)$ norm) if $\mathbb{E}[X_n^2] < \infty$, for all $n \in \mathbb{N}$, $\mathbb{E}[X^2] < \infty$ and if

$$\lim_{n \to \infty} \mathbb{E}\left[(X_n - X)^2\right] = 0. \tag{1.2}$$

This will be the most useful mean of convergence in this thesis. For another types, as for instance, a.s. convergence, convergence in probability or in distribution see [2], Section I.§5. For practical use, one might want to inspect the "evolution in time" with respect to random experiments.

Definition 1.8 (Stochastic process)**.** A real-valued stochastic process X_t, $t \in \Lambda$, is a collection of real random variables $(X_t(\omega))_{t \in \Lambda}$, defined on the sample space Ω. Λ is an index set which is in our case interpreted as time and it can be either a discrete time $(t \in \mathbb{N}, t \in \mathbb{Z})$ or continuous time set, $t \in [0, T]$ for some $T > 0$. in this thesis, we assume the latter case.

If we fix ω we obtain a function of t

$$X(t) : [0, T] \longmapsto \mathbb{R} \ \text{ for any fixed } \omega,$$

that is called a *sample path* (or trajectory) of the process X. Conversely, fixing t gives us a random variable

$$X(\omega) : \Omega \longmapsto \mathbb{R} \ \text{ for } t \in [0, T].$$

From this point on, we will lose ω in our notation of stochastic processes (and random variables), as for instance, a process $X_t(\omega)$ will be denoted as X_t. Not displaying a dependence on sample points follows a habit of probabilistic literature and it also eases (a little bit) the complexity of notation used in this text.

Definition 1.9 (Filtration)**.** Let T be (as previously) a fixed positive number and let us assume that for each $t \in [0, T]$ there is a $\sigma-$algebra $\mathcal{F}_t \subset \mathcal{F}$. Assume further that for every $0 \le s \le t$, $\mathcal{F}_s$ is included in $\mathcal{F}_t$. Then we call the collection of $\sigma-$algebras $(\mathcal{F}_t)_{t \in [0,T]}$ a *filtration*.

We can think of the filtration $(\mathcal{F}_t)_{0 \le t \le T}$ as of a non-decreasing collection of all available information at time t. To utilise this object we introduce a notion of stochastic process being adapted to a given filtration.

Definition 1.10 (Adapted stochastic process)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with filtration $\mathcal{F}_t$, let X_t be a stochastic process and let $t : \ 0 \le t \le T$ for

some positive T. X_t is said to be an adapted stochastic process if for each t a random variable X is $\mathcal{F}_t-$measurable.

We recall that if a random variable is $\mathcal{F}_t$-measurable it means that we have all information needed to retrieve the value of X at time t. For more information on measurability with respect to random variables and processes we link the reader to [57] or [2]. In mathematical finance, processes that are called martingales play an eminent role.

Definition 1.11 (Martingale)**.** Let M_t be a continuous-time stochastic process adapted to filtration $\mathcal{F}_t$ [2]. We say M_t is a *martingale* if $\mathbb{E}[|M_t|] < \infty$ and

$$\mathbb{E}\left[M_t|\mathcal{F}_s\right] = M_s \quad a.s.;$$

for $0 \le s < t \le T$. $\mathbb{E}\left[M_t|\mathcal{F}_s\right]$ denotes a conditional expectation with respect to the filtration $\mathcal{F}_s$ and intuitively we can understand M_s as the best estimate of the future outcome at t given the history upto time t.

One can also define super-martingales or sub-martingales and discrete-time martingales, but for our purposes a localised martingale property would be more useful. Before defining local martingales we look at stopping times (or Markov times).

Definition 1.12 (Stopping time)**.** A non-negative random variable τ defined on a probability space with filtration $\mathcal{F}_t$, $\tau : \Omega \mapsto [0, +\infty]$, is a *stopping time* (or a Markov time) provided it is $\mathcal{F}_t-$measurable for all $t \ge 0$, i. e.

$$\{\tau \le t\} \in \mathcal{F}_t \quad \forall t \ge 0.$$

Definition 1.13 (Local martingale)**.** A stochastic process L_t adapted to $\mathcal{F}_t$ is said to be a *local martingale* if there exists a sequence of stopping times $(\tau_n)_{n \ge 0}$ such that

(i) the sequence (τ_n) is a.s. increasing and a.s. diverges, i.e. $\mathbb{P}\left(\tau_{n+1} > \tau_n\right) = 1$ and $\mathbb{P}\left(\lim_{n \to +\infty} \tau_n = +\infty\right) = 1$;

(ii) the stopped process, $L_{\min\{\tau_n, t\}}$ is a martingale for every n.

Remark 1.1. Clearly, every martingale is also a local martingale. Converse statement, however, does not need to be true (see [47]).

Definition 1.14 (Semi-martingale)**.** A stochastic process Z_t defined on a probability space with filtration $\mathcal{F}_t$ is said to be a *semi-martingale* if it can be represented in the form

$$Z_t = L_t + V_t,$$

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[2]For simplicity one might consider a natural filtration of M_t which is a $\sigma-$algebra generated by all previous outcomes, denoted as $\sigma(M_u, 0 \le u \le t)$.

where L_t is a local martingale on filtration $\mathcal{F}_t$ and V_t is an $\mathcal{F}_t-$adapted stochastic process with *càdlàg* [3] sample paths and with a locally bounded variation,

$$\int_0^t |dV_s| < \infty \ \ \text{for all } \omega \in \Omega \text{ and for } 0 \le t \le T,$$

where the integral is defined as in [14].

1.2.2 Examples of stochastic processes

We will start with the main building block of mathematical finance - a standard *Wiener process* [4].

Definition 1.15. A real-valued continuous time stochastic process W_t, $0 \le t \le T$, is called a Wiener process if

(i) $W_0 = 0$ a.s.,

(ii) $W_t - W_s$ is normally distributed with zero mean and variance $t - s$ for all $0 \le s \le t$,

(iii) for all times $0 < t_1 < t_2 < ... < t_n$, random variables $W_{t_1}, W_{t_2} - W_{t_1}, ..., W_{t_n} - W_{t_{n-1}}$ are stochastically independent.

Lemma 1.2. *Wiener process W_t is a martingale.*

Proof. Let times $0 \le s < t$ are given.

$$\begin{aligned}
\mathbb{E}\left[W_t|\mathcal{F}_s\right] &= \mathbb{E}\left[W_t - W_s + W_s|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[W_t - W_s|\mathcal{F}_s\right] + \mathbb{E}\left[W_s|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[W_t - W_s|\mathcal{F}_s\right] + W_s \\
&= 0 + W_s.
\end{aligned}$$

In the third line we used the fact that W_s is known at time s and since the conditional expectation of increments $W_t - W_s$ with respect to the filtration $\mathcal{F}_s$ is zero, we obtain:

$$\mathbb{E}\left[W_t|\mathcal{F}_s\right] = W_s.$$

Because $\mathbb{E}[|W_t|] < \infty$ for all $t \ge 0$, we conclude the proof. $\qquad\square$

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[3] From French *"continue à droite, limite à gauche"*, meaning right continuous with left limit.
[4] Also called a Brownian motion.

We can also observe that Wiener process has stationary increments, i.e. $W_{t+s} - W_t$ has the same distribution as W_s. The process attains many interesting properties and there exist several ways how to define and construct W_t. A curious reader should look at [57], on how to construct the process via the time-step limit of scaled random walks, at [44], for interesting historical references and for existence of W_t and also at [2], for more advance properties.

Another continuous-time process we would like to briefly introduce is a fractional Brownian motion.

Definition 1.16 (Fractional Brownian motion)**.** A real-valued continuous time process B_t^H, $0 \leq t \leq T$, called the fractional Brownian motion, is a Gaussian process [5] such that

(i) $B_0^H = 0$ a.s.,

(ii) $\mathbb{E}\left[B_t^H\right] = 0$, for all $t \in [0, T]$,

(iii) $\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left\{t^{2H} + s^{2H} - (t-s)^{2H}\right\}$, for $0 \leq s < t$,

where H is known as the Hurst parameter (or equivalently the Hurst exponent) and H ranges in $[0, 1]$.

In the next paragraph we list several properties of the fractional Brownian motion that are of our main interest [4]:

- For $H = \frac{1}{2}$, B_t^H is the standard Wiener process.

- For $H \neq \frac{1}{2}$, the fractional Brownian motion is not a semi-martingale.

- For $H > \frac{1}{2}$, increments $B_t^H - B_s^H$ for any $0 \leq s < t$ are positively correlated and conversely for $H < \frac{1}{2}$ the increments are negatively correlated.

- For $H > \frac{1}{2}$, the fractional Brownian motion attains a long-range dependence which is defined later in Chapter 4 (Definition 4.1).

The use of fractional Brownian motion has been proposed in so-called fractional Black-Scholes models, for instance in [31] or [19], and also in fractional stochastic volatility models.

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[5]Which means that all its finite-dimensional distributions, defined e.g. in [44], are Gaussian.
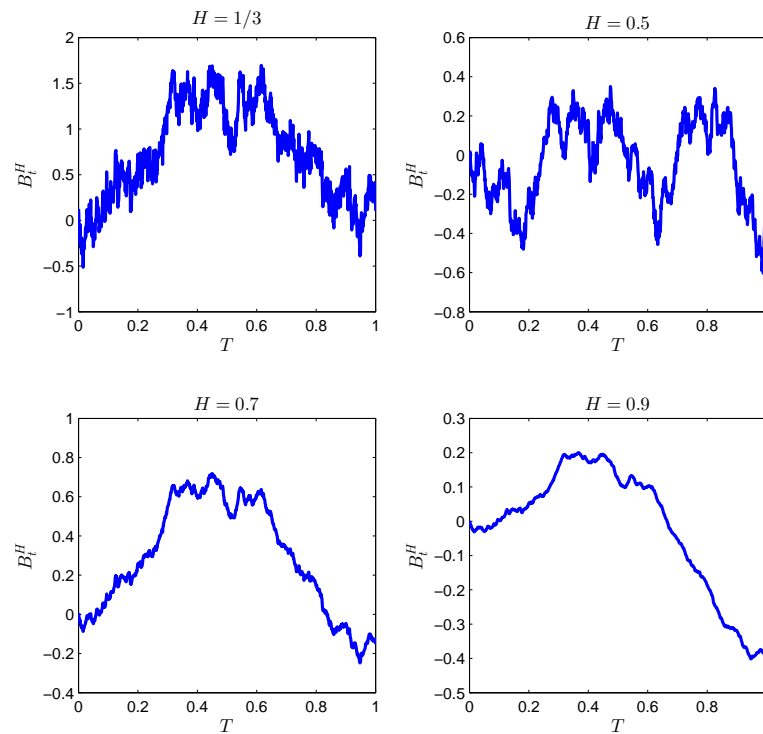
FIGURE 1.2: Sample paths of a fractional Brownian motion for different H.

A stochastic counting process G_t, $0 \leq t \leq T$, keeps count of the number of events that have happened up to time t [6]. G_t will be integer-valued, non-negative and also non-decreasing process.

Definition 1.17 (Poisson process). A *Poisson process* N_t, $0 \leq t \leq T$, is a counting process with the following properties:

(i) $N_0 = 0$ a.s.,

(ii) the process has stationary and stochastically independent increments,

(iii) $\mathbb{P}(N_t = n) = \frac{(\lambda_J t)^n}{n!} e^{-\lambda_J t}$ for $n \in \mathbb{N}_0$ and $\lambda_J \in \mathbb{R}^+$.

A constant λ_J is known as the intensity of jump occurrences (or the hazard rate). The expected number of jumps in a time interval of length τ equals to $\lambda_J \tau$ and the jump times are exponentially distributed. Clearly, for $\lambda_J \neq 0$, N_t is not a martingale.

Proposition 1.3. *Let N_t be a Poisson process with intensity λ_J, then the compensated Poisson process C_t, $0 \leq t \leq T$, defined as*

$$C_t = N_t - \lambda_J t,$$

is a martingale.

Proof. We utilise the independence of $N_t - N_s$ on the filtration $\mathcal{F}_s$ for every $0 \leq s < t \leq T$ and a fixed $T > 0$. Also the expected value of the increment is $\lambda_J(t - s)$.

$$
\begin{aligned}
\mathbb{E}[C_t|\mathcal{F}_s] &= \mathbb{E}\left[C_t - C_s + C_s|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[C_t - C_s|\mathcal{F}_s\right] + \mathbb{E}\left[C_s|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[N_t - N_s - \lambda_J(t - s)|\mathcal{F}_s\right] + C_s \\
&= \mathbb{E}\left[N_t - N_s\right] - \lambda_J(t - s) + C_s \\
&= C_s.
\end{aligned}
$$

$\mathbb{E}[|C_t|] < \infty$ for all $0 \leq t \leq T$, hence C_t is a martingale. $\qquad\square$

The Poisson process can take only values in $\mathbb{N}_0$ (all jumps are of a unit size) which motivates us to define a compound Poisson process.

Definition 1.18 (Compound Poisson process)**.** A stochastic process U_t, $0 \leq t \leq T$, is said to be a compound Poisson process if it can be decomposed as

$$
U_t = \sum_{i=0}^{N_t} Y_i,
$$

where N_t is the Poisson counting process and Y_i are identically, independently distributed (i.i.d.) random variables. These variables are also independent of N_t.

The previous definition does not specify the distribution of Y_i. In this thesis, we will assume (unless mentioned otherwise) that the random variables Y_i are log-normally distributed. Similarly as for the standard Poisson process, we can also construct a compensated compound Poisson process. This proves to be especially useful for the task of stock evolution modelling.

1.2.3 Stochastic differential equations and Itô integrals

We will define an integral with respect to the Wiener process and its filtration. For an approximative fractional model, we will employ a modification of this approach that was developed by Thao and Nguyen [60]. This stochastic integral is introduced in the second part of Chapter 2. Firstly, we present a well known stochastic integral which is named after Japanese mathematician Kiyoshi Itô.

Definition 1.19 (Itô integral)**.** Let W_t, $0 \leq t \leq T$, be the standard Wiener process adapted to the filtration $\mathcal{F}_t$. Let Δ_t is also an $\mathcal{F}_t-$adapted stochastic process such that

$$\mathbb{E}\left[\int_0^T \Delta_t dt\right] < \infty. \tag{1.3}$$

Let $\Pi_n = \{0 = t_0, t_1, ..., t_n = T\}$ be the partitioning set and the norm $\| \Pi_n \|$ is defined as

$$\| \Pi_n \| = \max_{k=1,2,...,n-1} (t_{k+1} - t_k).$$

The Itô integral I_T is then obtained by the following expression

$$I_T = \int_0^T \Delta_t dW_t := \lim_{\|\Pi_n\| \to 0} \sum_{i=0}^{n-1} \Delta_{t_i}(W_{t_{i+1}} - W_{t_i}) \tag{1.4}$$

I_T defined above is a random variable and the following theorem lists some of its properties.

Theorem 1.4. *Let $T > 0$, let Δ_t and Γ_t, $0 \leq t \leq T$, be $\mathcal{F}_t-$adapted stochastic processes that satisfy (1.3). Itô integral I_t defined by (1.4) has the following properties:*

(i) *(Adaptivity) For each t, I_t is $\mathcal{F}_t-$measurable.*

(ii) *(Continuity) The sample paths of I_t are continuous.*

(iii) *(Itô isometry) $\mathbb{E}\left[I_t^2\right] = \mathbb{E}\int_0^t \Delta_s^2 ds$.*

(iv) *(Linearity) $\int_0^t (a\Delta_s + b\Gamma_s)dW_s = a\int_0^T \Delta_s dW_s + b\int_0^T \Gamma_s dW_s$ for $a, b \in \mathbb{R}$.*

(v) *(Martingale) $\mathbb{E}[I_t] = 0$ and I_t is a martingale.*

Proof. For the proof, see Section 4.2 of [57] for simple process integrands and for more general integrands, see Section 4.3 in the same book. $\qquad \square$

Itô integrals are of key importance for the stochastic differential equations (SDE's). In fact, they appear in what is defined as the strong solution of an SDE. Let us start with the definition of this class of differential equations.

Definition 1.20 (SDE)**.** Let $\mathcal{F}_t$ be a filtration generated by a Wiener process W_t, let T be a positive constant. Also let $\mu : \mathbb{R} \times [0, T] \mapsto \mathbb{R}$, $\sigma : \mathbb{R} \times [0, T] \mapsto \mathbb{R}$ be measurable functions and X_0 be $\mathcal{F}_0-$measurable random variable. A stochastic differential equation

can be expressed in the form of stochastic differential

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \quad t \in [0, T] \tag{1.5}$$

$$X_0 = x_0 \in \mathbb{R}. \tag{1.6}$$

Moreover, if $\int_0^T \mu(X_t, t)dt < \infty$ and $\int_0^T \sigma^2(X_t, t)dt < \infty$ then a continuous $\mathcal{F}_t-$adapted process X_t satisfying

$$X_t = x_0 + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s)dW_s, \quad t \in [0, T], \tag{1.7}$$

is said to be a (strong) solution of the SDE (1.5)-(1.6).

Remark 1.2. The first integral on the right hand side of (1.7) is a Lebesgue integral and the second one is an Itô integral. The functions $\mu(X_t, t)$, $\sigma(X_t, t)$ are called the drift and diffusion respectively. X_t defined as (1.7) is also known as an Itô process.

Definition 1.21 (Pathwise uniqueness). Let X_t, $0 \leq t \leq T$, and Y_t, $0 \leq t \leq T$, are adapted stochastic processes defined on the same probability space with filtration $\mathcal{F}_t$. If both processes satisfy (1.5)-(1.6) and also if

$$X_t = Y_t \quad a.s. \text{ for all } t \in [0, T],$$

we say that the processes X_t, Y_t are indistinguishable. The solution of (1.5) is (pathwise) unique, if any two processes satisfying the SDE with respect to the same initial condition are indistinguishable.

For many practical applications (e.g. pricing by Monte-Carlo simulation), it might be important to inspect when an SDE, given by a model, attains a unique strong solution. In particular, we will have the following SDE representation of volatility process for both Heston and Bates models (see Chapter 2).

$$v_t = -\kappa(v_t - \bar{v})dt + \xi\sqrt{v_t}dW_t, \tag{1.8}$$

$$v_0 = v, \tag{1.9}$$

where, for now, assume $\kappa, \bar{v} \in \mathbb{R}^+$ and $\xi \in \mathbb{R} - \{0\}$. Also the initial condition v is a (deterministic) constant in $[0, 1]$. The classical existence and uniqueness theorem introduced by K. Itô requires the Lipschitz continuity for both drift and diffusion (see [44]). The term $\xi\sqrt{v_t}$, however, is not Lipschitz continuous and hence this result cannot be applied. This brings us to the next theorem that is due to Yamada and Watanabe (1971) [64].

Theorem 1.5 (Pathwise existence and uniqueness)**.** *[26] Consider SDE (1.5)-(1.6), where $\mu(X_t, t) = \mu(X_t)$ and $\sigma(X_t, t) = \sigma(X_t)$. If*

$$|\mu(x) - \mu(y)| \leq C|x - y|, \tag{1.10}$$

$$|\sigma(x) - \sigma(y)|^2 \leq g(|x - y|), \quad for\ all\ x, y \in \mathbb{R}, \tag{1.11}$$

where $C > 0$ is a constant, g represents a strictly increasing function on $[0, \infty]$ such that $g(0) = 0$ and $\lim\limits_{n \to 0^+} \int_n^\infty g^{-2}(x)dx = \infty$, then the SDE has a unique strong solution.

Proof. The proof can be found in the original paper [64]. □

Important corollary is made in [34], page 168. For the one-dimensional case (as in (1.8)), it is enough to show that the drift is Lipschitz continuous function (i.e. as in (1.10)) and diffusion is Hölder continuous with exponent $1/2$. This implies the existence and uniqueness of a strong solution. For the SDE (1.8), Hölder continuity assumption means that we are able to find a constant $C > 0$ such that

$$|\xi\sqrt{v_t(\omega_1)} - \xi\sqrt{v_t(\omega_2)}| \leq C\sqrt{|v_t(\omega_1) - v_t(\omega_2)|},$$

for all $t \in [0, \infty]$ and for $v_t(\omega_1), v_t(\omega_2) \in \mathbb{R}^+$. Since ξ is a constant, we can clearly obtain C (as assumed $v_t(\omega_1), v_t(\omega_2) \geq 0$, so we would set $C := |\xi|$). Moreover, the drift in (1.8) is Lipschitz continuous, hence we conclude that the initial value problem (1.8) has a unique strong solution (with respect to parameter bounds and non-negativity of v_0).

A useful tool for solving certain SDE's is the famous Itô-Doeblin formula [6].

Lemma 1.6 (Itô-Doeblin)**.** *Let X_t, $0 \leq t \leq T$, be an Itô process defined by (1.7). Let $V = V(x, t), V : \mathbb{R} \times [0, \infty] \mapsto \mathbb{R}$, be a function with continuous partial derivatives $\frac{\partial V}{\partial t}$, $\frac{\partial V}{\partial x}$, $\frac{\partial^2 V}{\partial x^2}$. Then the process $Y_t = V(X_t, t)$ has the stochastic differential*

$$dY_t = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\sigma^2(X_t, t)dt. \tag{1.12}$$

Proof. The proof can be found in [21], Section 4.D. □

When deriving partial differential equations that mimic value of a self-financing portfolio (Chapter 3), we will also need a two-dimensional version of Itô-Doeblin lemma which is conveniently described in Section 4.6. of [57].

---

[6]The Lemma is more commonly known as the Itô formula. Due to the recent discovery of the original work of Wolfgang Döblin, several authors suggested re-naming this lemma [57].

To model dynamics of volatile markets, several authors proposed adding a jump term governed by a Poisson process [7]

$$X_{t-}Y_t dN_t,$$

where $X_{t-} = \lim_{s \to t^-} X_s$. The symbolical notation $Y_t dN_t$ is, in fact, the Poisson sum

$$Y_t dN_t = \sum_{i=1}^{dN_t} Y_i,$$

and to recall definition of a compound process, Y_i are *i.i.d.* random variables. Combining a drift and a diffusion from (1.7) with the previously defined sum, one would obtain the jump-diffusion SDE. This class of stochastic differential equations is not discussed in the thesis, mainly because we set up the pricing PDE's for diffusion processes only. After the characteristic function with respect to option pricing are derived we may (using Proposition 1.1 and independence) retrieve the joint characteristic function as a product of diffusion and pure-jump characteristic functions. This way we obtain a characteristic functions of option pricing models with jump-diffusion dynamics without additional complexity brought by the jump term. The procedure of derivation of characteristic function with respect to option pricing is assessed more thoroughly in Chapter 3.

1.2.4 Theorems used for a derivation of pricing PDE's

In this section we present two important theorems (without proofs) for the option pricing task.

Theorem 1.7 (Girsanov). *Let W_t, $0 \le t \le T$, be a Wiener process adapted to the filtration $\mathcal{F}_t$. Also let Θ_t, $0 \le t \le T$, be an adapted process. Define*

$$Z_t = \exp\left\{-\int_0^t \Theta_s dW_s - \frac{1}{2}\int_0^t \Theta_s^2 ds\right\}, \tilde{W}_t = W_t + \int_0^t \Theta_s ds, \qquad (1.13)$$

and assume that

$$\mathbb{E}\left[\int_0^T \Theta_s^2 Z_s^2 ds\right] < \infty.$$

Then $\mathbb{E}[Z_T] = 1$ (T is fixed) and under the probability measure $\tilde{\mathbb{P}}$, defined as

$$\tilde{\mathbb{P}}(A) = \int_A Z_T d\mathbb{P} \text{ for all } A \in \mathcal{F}, \qquad (1.14)$$

the process $\tilde{W}_t, 0 \le t \le T$, is a Wiener process.

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[7]Respectively governed by a compensated Poisson process (under the risk-neutral measure).

Definition 1.22 (Risk-neutral measure). A probability measure $\tilde{\mathbb{P}}$, defined by (1.14) where $\mathbb{E}[Z_T] = 1$, is said to be risk-neutral if

(i) measure $\tilde{\mathbb{P}}$ is equivalent to the probability measure $\mathbb{P}$, i.e. if
$$\tilde{\mathbb{P}}(A) = 0 \Leftrightarrow \mathbb{P}(A) = 0, \ A \in \mathcal{F};$$
$$\tilde{\mathbb{P}}(B) = 1 \Leftrightarrow \mathbb{P}(B) = 1, \ B \in \mathcal{F},$$

(ii) under $\tilde{\mathbb{P}}$, the discounted stock price process $S_t \exp\{-\int_0^t R_s ds\}, 0 \le t \le T$, where both the interest rate process R_t and the stock price process S_t are adapted, is a martingale.

First requirement for the risk-neutral measure is the following. We want both measures $\tilde{\mathbb{P}}$ and $\mathbb{P}$ to agree on sets with measure zero and on the sets that will happen almost surely.

In this thesis we focus on pricing of option contracts and on market calibration. Thus, we utilise a single probability measure - the risk-neutral one. However, since the market volatility is not tradable, models introduced in the next chapter will not have a unique risk-neutral measure [8]. This would lead to the notion known as the market price of volatility risk, which we discuss in Section 3.2. We are not specifying $\tilde{\mathbb{P}}$ nor showing the relation between the two measures, one would certainly need to be more specific when using the calibrated models for risk-management purposes. In that case, one obtains model parameters that are with respect to $\tilde{\mathbb{P}}$ (i.e. from the market calibration procedure), but one also needs to evaluate risk-measures (e.g. Vaule-at-Risk, see [63]) under the original probability measure.

Theorem 1.8 (Feynman-Kac). *Consider SDE (1.5) and let $h(y)$ be a Borel-measurable function. For a fixed $T > 0$ let $t \in [0, T]$. Define the function*

$$g(x, t) = \mathbb{E}\left[h(X(T))|X_t = x\right]. \tag{1.15}$$

Assume $\mathbb{E}\left[h(X(T))|X_t = x\right] < \infty$ for all x, t. Then $g(x, t)$ satisfies the partial differential equation

$$\frac{\partial g(x, t)}{\partial t} + \mu(x, t)\frac{\partial g(x, t)}{\partial x} + \frac{1}{2}\sigma^2(x, t)\frac{\partial^2 g(x, t)}{\partial x^2} = 0, \tag{1.16}$$

with the terminal condition

$$g(x, T) = h(x) \quad \text{for all } x.$$

---

[8]This follows directly from the incompleteness of markets under the stochastic volatility models, see Theorem 5.4.9 in [57]

Proof. The proof depends on a lemma that states the process $g(X_t, t)$ defined by (1.15) is a martingale. For the whole outline of the proof see [57], Section 6.4. $\qquad\square$

For financial applications, we might rather use a modification of Feynman-Kac theorem where $g(x, t) = \mathbb{E}\left[e^{-r(T-t)}h(X(T))|X_t = x\right]$, $r \in \mathbb{R}$ and instead of a homogeneous right hand side of (1.16) we would obtain a term $rg(x, t)$. Function $g(x, t)$ represents a discounted mean value of random variable $h(X(T))$ given $X_t = x$. If we set $h(y)$ as a pay-off of a financial derivative conditional on the spot reaching y at time T, in fact, we obtain a fair market price as the solution of (1.16) for $t = 0$ and for the spot price process X_t. The solution can be retrieved either numerically or, as in our case, analytically. This depends on the choice of underlying process X_t. The model overview, that might help us choose the spot price process, follows in the next chapter. Last but not least, for an $n-$dimensional version of the Fenman-Kac theorem we link the reader to [57], Section 6.6.

Chapter 2

Overview of dynamic volatility modelling

Both stochastic and fractional stochastic volatility models will be introduced in this chapter. We will briefly describe the most popular approaches in terms of option pricing alongside recently proposed fractional volatility modelling. Following Intarasit & Sattayatham [35], we will also show how dynamics of an approximative fractional model can be transformed into the standard stochastic volatility settings.

All considered models share the following assumptions ([63] Part I, Chapter 5 and Part IV, Chapter 51):

- No arbitrage opportunities occur, thus the risk-free rate r is unique. Moreover, r is constant during the life of the given option;

- Any fraction of a stock can be bought and trading of securities is continuous in time;

- Short selling of any security is allowed at the considered market.

By short selling we mean that an investor can sell any available asset even the one she does not own at the moment. Later she re-purchase the asset to cover the transaction. For simplicity, we also omit dividend payments.

2.1 Stochastic volatility models

Market dynamics is modelled by a system of two SDE's. The first differential equation describes evolution of a stock price process (hence denoted by S_t). The second equation corresponds most commonly to variance [9] of the underlying price. For option pricing tasks, the state space SDE's are accompanied by deterministic initial value constraints, $S_0, \; v_0 \in \mathbb{R}^+$.

One of the first models came up in 1987, developed by John Hull and Alan White [32]. The model assumes the following market dynamics,

$$dS_t \;\; = \;\; rS_t dt + \sqrt{v_t} S_t dW_t, \tag{2.1}$$

$$dv_t \;\; = \;\; C_1 v_t dt + C_2 v_t dW_t^\star, \tag{2.2}$$

where r, C_1 and C_2 are parameters of the model. The increments of Wiener processes W_t, $W_t^\star$ are assumed to be stochastically independent under the original model. Wiggins [62] suggested the use of correlation coefficient ρ, i.e. $\mathbb{E}[dW_t dW_t^\star] = \rho dt$, which would be the case of all further stochastic volatility models. The variance process v_t follows geometrical Brownian motion which implies that some of the interesting statistical properties are explicitly known to us [36]:

$$\mathbb{E}[\sqrt{v_t}] = \mathbb{E}[\sigma_t] \;\; = \;\; \sigma_0 \exp\left\{\frac{1}{2}C_1 t - \frac{1}{8}C_2^2 t\right\},$$

$$Var[\sigma_t] \;\; = \;\; \sigma_0^2 \left[1 - \exp\left\{-\frac{1}{4}C_2^2 t\right\}\right] \exp\left\{C_1 t\right\}.$$

Option pricing under the *Hull-White* market dynamics can be implemented using Monte-Carlo simulations or (for European options) by analytical expression firstly derived in [32].

Chin [9] argues, using empirical analysis by Cont [13], that a model with variance process v_t expressed by the geometric Brownian motion (2.2) cannot reflect observed decay rate of implied volatilities. To deal with this shortcoming of the *Hull-White* model, mean-reverting approaches for volatility have been developed by Scott [55]. We will introduce a modified version of the model from an article by Chesney and Scott [8]:

$$dS_t \;\; = \;\; rS_t dt + e^{y_t} S_t dW_t, \tag{2.3}$$

$$dy_t \;\; = \;\; -\kappa(y_t - \bar{y})dt + \xi dW_t^\star. \tag{2.4}$$

---

[9]Hence the name stochastic volatility models.

Unlike in most of the models, volatility under the Chesney & Scott dynamics is expressed as $\sigma_t = e^{y_t}$. There are two parameters within the drift term of dy_t; κ describes a reversion rate and $\bar{y}$ denotes an average level around which process y_t fluctuates. The diffusion term is represented by a constant and instantaneous volatility cannot attain negative values. According to [36], the model needs strong negative correlation to reflect observed properties of market volatility.

The most popular mean-reverting model is the one by Steven Heston [29].

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t, \tag{2.5}$$

$$dv_t = -\kappa(v_t - \bar{v})dt + \xi\sqrt{v_t}dW_t^\star, \tag{2.6}$$

where $\bar{v}$ represents a long term variance and the last parameter ξ denotes volatility of v_t. A popularity of the model comes from its tractability and from the existence of a semi-closed solution for European option prices. Unless the *Feller's* condition is satisfied, $2\kappa\bar{v} \geq \xi^2$ [22], the discretized variance process can reach negative values, which is an issue that has to be dealt with. This model will be implemented for calibration, option pricing tasks and will be compared with the jump-diffusion and approximative fractional models introduced later in this text.

Another frequently used approach has been developed by Hagan et al [28].

$$dS_t = \sigma_t S_t^\beta dW_t^{(1)}, \tag{2.7}$$

$$d\sigma_t = \alpha\sigma_t dW_t^{(2)}. \tag{2.8}$$

Stochastic Alpha, Beta, R model (SABR) is increasingly popular for portfolio hedging and risk management purposes. Many other modelling approaches have been introduced, for instance, models by Stein and Stein / Schöbl and Zhu in [54].

Jump-diffusion and stochastic volatility

To improve flexibility of models and to enhance market calibration, many academics and professionals suggested a jump-diffusion modification of the stock price process S_t.

The first model to utilise jump-diffusion processes in finance was introduced by Robert C. Merton in 1976. A jump process alongside stochastic volatility has been proposed by

Bates [1].

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t + Y_t S_{t-} dN_t, \qquad (2.9)$$

$$dv_t = -\kappa(v_t - \bar{v})dt + \xi\sqrt{v_t}dW_t^\star, \qquad (2.10)$$

where Wiener processes are, as in previous cases, correlated with coefficient ρ. N_t is a compensated Poisson process with annual frequency λ_J and $Y_t S_{t-}$ represents an amplitude of the jump at time t. Drift and diffusion terms of dv_t are the same as in the Heston's approach - the model retains mean-reverting property and despite having more parameters, it is still tractable for pricing tasks. This will be inspected in Chapter 6.

One can add a jump process to the second equation as well.

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t + Y_t^{(1)} S_{t-} dN_t, \qquad (2.11)$$

$$dv_t = -\kappa(v_t - \bar{v})dt + \xi\sqrt{v_t}dW_t^\star + Y^{(2)}dN_t^\star. \qquad (2.12)$$

The model was introduced by Duffie et al. in [18] and there are several implementations either with correlated or independent Poisson processes [10]. As empirical studies have shown (e.g. [23], [25]), this approach might suffer from over fitting. While having four more parameters, it might not provide as good market fit as the Bates model.

2.2 Fractional stochastic volatility models

Comte and Renault pioneered the use of a fractional Brownian motion in stochastic volatility [11]. Their asset pricing model is a modification of the original Hull-White approach [32].

$$dY_t = \sigma_t dW_t, \qquad (2.13)$$

$$d(\ln \sigma_t) = \kappa \ln \sigma_t + \gamma dB_t^H; \qquad (2.14)$$

$$B_t^H = \int_0^t \frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)}dW_s, \qquad (2.15)$$

where $\int_0^t dB_s^H$ is understood in a path-wise sense, Y_t describes a logarithmic price $(\ln S_t)$ at time t and both κ, γ are parameters of the model. Option prices can be retrieved by a Monte-Carlo simulation framework. For the option pricing and portfolio hedging tasks, Comte and Renault proposed a new affine fractional model driven by the following

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[10]The later case means that increments $N_t - N_s$ are stochastically independent on $N_t^\star - N_s^\star$ for any $0 \le s < t \le T$.

SDE's:

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^{(1)}, \tag{2.16}$$

$$dX_t = -\kappa(X_t - \bar{v})dt + \gamma X_t dW_t^{(2)}, \tag{2.17}$$

$$v_t = \theta + X_t^{\alpha}. \tag{2.18}$$

where X_t^{α} can be formally expressed using the following notation,

$$X_t^{\alpha} = \int_{-\infty}^{t} \frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)} X_s ds. \tag{2.19}$$

For the definition of fractional integral (2.19) see [12]. Authors have discussed how to discretize the integral and, as previously, the prices of derivatives are obtained by the means of simulation.

A model with jump-diffusion stock evolution alongside approximative fractional volatility was introduced by Intarasit and Sattayatham [35] in 2011.

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t + Y_t S_{t-} dN_t, \tag{2.20}$$

$$dv_t^{\varepsilon} = -\kappa(v_t^{\varepsilon} - \bar{v})dt + \xi v_t^{\varepsilon} dB_t^{\varepsilon}. \tag{2.21}$$

Parameters and processes Y_t, N_t and, of course, W_t are defined similarly as in case of the Bates model. We will discuss the notation of dB_t^{ε} in more detail later in this text. The main advantage of this approach, according to the authors, should lie in its tractability with respect to option pricing. The model can be transformed into the standard volatility settings (Section 2.3) and also the standard hedging arguments and assumptions can be used to derive a semi-closed form solution for European option prices (which we will asses in Chapter 3).

2.3 Approximate fractional stochastic volatility

To describe approximative FSV modelling, we will closely follow [35] and [59]. Instead of the fractional Brownian motion, we will consider a process,

$$B_t = \int_0^t (t-s)^{H-1/2} dW_s, \tag{2.22}$$

where H is a constant ranging from $[0,1]$ and as in case of the fractional Brownian motion, also known as the Hurst exponent. For $H > 1/2$ process B_t has a long memory

[53]. Moreover, we can approximate B_t by

$$B_t^\varepsilon = \int_0^t (t - s + \varepsilon)^{H-1/2} dW_s, \tag{2.23}$$

such that B_t^ε converges to B_t in $L^2(\Omega)$ as ε tends to 0 (for proof see [60]). Also B_t^ε is a semimartingale with respect to the filtration $\mathcal{F}_t$ generated by the standard Wiener process W_t (following Proposition 2.1 and Remark 2.1). The use of approximation B_t^ε instead of the fractional Brownian motion provides several advantages. Firstly, there is no arbitrage opportunity under the approximative model dynamics for a wide class of simple and self-financing portfolios [11]. Secondly, if we exploit (2.29) as the driving process of dv_t, we can use a standard Itô stochastic calculus instead of more advanced mathematical techniques for derivation of pricing PDE's.

To describe approximative fractional approaches, we start with dynamics of v_t as in [35].

$$dv_t = -\kappa(v_t - \bar{v})dt + \xi v_t dB_t. \tag{2.24}$$

Before transforming the approximative version of (2.24), we need to define the fractional integral that comes up in the solution,

$$v_t = -\kappa \int_0^t (v_s - \bar{v})ds + \xi \int_0^t v_s dB_s. \tag{2.25}$$

Definition of fractional integral $\int f(t, \omega)dB_t$

To define integration with respect to B_t, we start with a deterministic integrand $f(t)$ [53].

Definition 2.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with filtration $\mathcal{F}_t$ generated by a standard Wiener process. Let a deterministic function $f(t)$ with bounded variation on $t \in [0, T]$ be given for a fixed $T > 0$. Then the fractional integral $\int_0^t f(s)dB_s$ is defined as

$$I_t := \int_0^t f(s)dB_s = f(t)B_t - \int_0^t B_s df(s). \tag{2.26}$$

---

[11]If S_t is not a semimartingale, there exist arbitrages in this class. For more details on arbitrages, see [16].

If we recall definition of B_t as in (2.22), we note that the integral on the right-hand side of (2.26) exists in the Riemann-Stieltjes sense for almost all ω [53]. Now we will consider an integrand that is represented by a stochastic process.

Definition 2.2. Let a stochastic process $X_t, 0 \leq t \leq T$, have sample paths of bounded variation for almost all ω. Then we define the fractional integral of X_t,

$$I_t := \int_0^t X_t dB_s = X_t B_t - \int_0^t B_s dX_t - [X, B]_t \,, \tag{2.27}$$

provided the right-hand side integral exists in a path-wise (Riemann-Stieltjes) sense and $[X, B]_t$ represents the quadratic variation [12] of X_t and B_t.

In a similar fashion, we are able to define an integral with respect to the semimartingale (2.23). Furthermore, the following theorem was introduced by Ngueyen and Thao in [60].

Theorem 2.1. *Let X_t be a stochastic process with continuous sample paths of bounded variation on $[0; T]$ and $\mathbb{E} \int_0^t (X_s)^2 ds < \infty$. Let B_t^ε be a process defined by (2.23). The stochastic integral,*

$$I_t^\varepsilon = \int_0^t X_t dB_s^\varepsilon, \tag{2.28}$$

converges uniformly in $L^2(\Omega)$ to I_t (2.27) as $\varepsilon \to 0$ for $t \in [0, T]$.

Proof. Proof of the theorem is shown in [53]. $\qquad \square$

Transformation into the standard stochastic volatility settings

Proposition 2.1. *B_t^ε (2.23) can be decomposed as*

$$B_t^\varepsilon = (H - 1/2) \int_0^t \varphi_s^\varepsilon ds + \varepsilon^{H-1/2} W_t, \tag{2.29}$$

where φ_t^ε represents Itô integral,

$$\varphi_t^\varepsilon = \int_0^t (t - s + \varepsilon)^{H-3/2} dW_s. \tag{2.30}$$

Proof. [59], [53]: We start with a process φ_t^ε where $1/2 < H < 1$ (or $0 < H < 1/2$) and we set $a := H - 1/2$. We integrate the process with respect to dt. Then a stochastic

---
[12]Quadratic variation of stochastic processes is discussed e.g. in [14]

version of the Fubini's theorem is applied to interchange a Lebesgue integral and a stochastic integral.

$$
\begin{aligned}
\int_0^t \varphi_s^\varepsilon ds &= \int_0^t \int_0^u (t-u+\varepsilon)^{a-1} dW_u ds \\
&= \int_0^t \left(\int_u^t (t-s+\varepsilon)^{a-1} ds \right) dW_u \\
&= \int_0^t \left(\frac{(t-u+\varepsilon)^a}{a} - \frac{\varepsilon^a}{a} \right) dW_u \\
&= \frac{1}{a} \int_0^t (t-u+\varepsilon)^a dW_u - \frac{1}{a} \int_0^t \varepsilon^a dW_u \\
&= \frac{1}{a} \left(B_t^\varepsilon - \varepsilon^a W_t \right).
\end{aligned}
$$

Hence $B_t^\varepsilon = \varepsilon^a W_t + (H - 1/2) \int_0^t \varphi_s^\varepsilon ds$. □

Remark 2.1. Since the standard Wiener process is a martingale, B_t^ε forms up a semi-martingale.

Using the result which was first shown by Thao [59], we can transform volatility process v_t^ε,

$$
\begin{aligned}
dv_t^\varepsilon &= -\kappa(v_t^\varepsilon - \bar{v})dt + \xi v_t^\varepsilon dB_t^\varepsilon, \\
&= \left[(a\xi\varphi_t^\varepsilon - \kappa)v_t^\varepsilon - \kappa\bar{v} \right] dt + \xi v_t^\varepsilon \varepsilon^a dW_t, \\
&= \left[(a\xi\varphi_t^\varepsilon - \kappa)v_t^\varepsilon - \theta \right] dt + \xi v_t^\varepsilon \varepsilon^a dW_t,
\end{aligned} \tag{2.31}
$$

where $a = H - 1/2$ and $\theta = \kappa\bar{v}$ is a constant. Because of our interest in the long-memory property, we will consider only H ranging from $(1/2, 1)$ [13]. Then the transformed model takes the following form

$$
\begin{aligned}
dS_t &= rS_t dt + \sqrt{v_t} S_t dW_t + Y_t S_{t-} dN_t, &\tag{2.32} \\
dv_t &= \left[(a\xi\varphi_t^\varepsilon - \kappa)v_t - \theta \right] dt + \xi v_t \varepsilon^a dW_t^\star, &\tag{2.33}
\end{aligned}
$$

where a correlation coefficient ρ can be artificially added, such that $\mathbb{E}[dW_t dW_t^\star] = \rho dt$.

---

[13] The transformed model exhibits a long memory in v_t for $H > 0.5$, for arguments see [53].

Chapter 3

Derivation of the pricing solution for FSV model

In this chapter we will derive a semi-closed form solution to price European-style options. At the beginning, we will follow [23] and [63], to derive a general valuation PDE for stochastic volatility models without jumps. For description of system dynamics, we will use notation of Gatheral [23], i.e.

$$
\begin{align}
dS_t &= rS_tdt + \sqrt{v_t}S_t dW_t^{(1)}, \tag{3.1}\\
dv_t &= \alpha dt + \beta\sqrt{v_t}dW_t^{(2)}, \tag{3.2}\\
\mathbb{E}\left[dW_t^{(1)}dW_t^{(2)}\right] &= \rho dt, \tag{3.3}
\end{align}
$$

where $\alpha = \alpha(S_t, v_t, t)$, $\beta = \beta(S_t, v_t, t)$ and ρ is an instantaneous correlation between standard Wiener processes $W_t^{(1)}, W_t^{(2)}$ for $t \geq 0$.

3.1 Self-financing portfolio

Let $\Pi = \Pi_t$ be the value of self-financing portfolio [14] π at time t. Let π be *delta* (i.e. $\frac{\partial \Pi}{\partial S_t} = 0$) and *vega* hedged ($\frac{\partial \Pi}{\partial v_t} = 0$) and let it consist of one option priced $V = V(S_t, v_t, t)$, $(-\Delta)$ units of the underlying stock with a price $S = S_t$ and $(-\Delta_1)$ units of another option with $V_1 = V_1(S_t, v_t, t)$. Then the portfolio value is determined by the following expression

$$
\Pi = V - \Delta S - \Delta_1 V_1. \tag{3.4}
$$

---

[14]We cannot withdraw nor add funds to the portfolio in $t > 0$. The change in portfolio value Π is thus given only by changes in prices of the underlying assets for constant positions.

The portfolio is self-financing and thus a change in its value is given by

$$d\Pi = dV - \Delta dS - \Delta_1 dV_1. \tag{3.5}$$

Using Itô lemma, we can derive expressions for differentials dV and dV_1.

$$\begin{aligned} dV = &\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial v_t}dv_t + \frac{1}{2}v_t S^2 \frac{\partial^2 V}{\partial S^2}dt + \frac{1}{2}v_t \beta^2 \frac{\partial^2 V}{\partial v_t^2}dt \\ &+ \rho v_t \beta S \frac{\partial^2 V}{\partial v_t \partial S}dt, \end{aligned} \tag{3.6}$$

$$\begin{aligned} dV_1 = &\frac{\partial V_1}{\partial t}dt + \frac{\partial V_1}{\partial S}dS + \frac{\partial V_1}{\partial v_t}dv_t + \frac{1}{2}v_t S^2 \frac{\partial^2 V_1}{\partial S^2}dt + \frac{1}{2}v_t \beta^2 \frac{\partial^2 V_1}{\partial v_t^2}dt \\ &+ \rho v_t \beta S \frac{\partial^2 V_1}{\partial v_t \partial S}dt. \end{aligned} \tag{3.7}$$

Having explicitly expressed dV and dV_1, we substitute the differentials into equation (3.5).

$$\begin{aligned} d\Pi = &\left[\frac{\partial V}{\partial t} + \frac{1}{2}v_t S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}v_t \beta^2 \frac{\partial^2 V}{\partial v_t^2} + \rho v_t \beta S \frac{\partial^2 V}{\partial v_t \partial S}\right]dt \\ &- \left[\frac{\partial V_1}{\partial t} + \frac{1}{2}v_t S^2 \frac{\partial^2 V_1}{\partial S^2} + \frac{1}{2}v_t \beta^2 \frac{\partial^2 V_1}{\partial v_t^2} + \rho v_t \beta S \frac{\partial^2 V_1}{\partial v_t \partial S}\right]\Delta_1 dt \\ &+ \left[\frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} - \Delta\right]dS + \left[\frac{\partial V}{\partial v_t} - \Delta_1 \frac{\partial V_1}{\partial v_t}\right]dv_t. \end{aligned} \tag{3.8}$$

Assumption that π is delta and vega hedged directly implies

$$\left[\frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} - \Delta\right] = 0, \quad \text{and} \quad \left[\frac{\partial V}{\partial v_t} - \Delta_1 \frac{\partial V_1}{\partial v_t}\right] = 0.$$

Hence, we are able to express Δ and Δ_1:

$$\Delta_1 = \frac{\partial V/\partial v_t}{\partial V_1/\partial v_t}; \quad \Delta = \frac{\partial V}{\partial S} - \frac{\partial V/\partial v_t}{\partial V_1/\partial v_t}\frac{\partial V_1}{\partial S}, \tag{3.9}$$

provided $\partial V_1/\partial v_t \neq 0$. Due to hedging assumptions, the change in portfolio value (3.8) is simplified to:

$$d\Pi = \underbrace{\left[\frac{\partial V}{\partial t} + \frac{1}{2}v_t S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}v_t \beta^2 \frac{\partial^2 V}{\partial v_t^2} + \rho v_t \beta S \frac{\partial^2 V}{\partial v_t \partial S}\right]}_{=A} dt$$
$$- \underbrace{\left[\frac{\partial V_1}{\partial t} + \frac{1}{2}v_t S^2 \frac{\partial^2 V_1}{\partial S^2} + \frac{1}{2}v_t \beta^2 \frac{\partial^2 V_1}{\partial v_t^2} + \rho v_t \beta S \frac{\partial^2 V_1}{\partial v_t \partial S}\right]}_{=B} \Delta_1 dt$$
$$= A dt - B\Delta_1 dt. \tag{3.10}$$

In the last line we used a short-hand notation for derived terms. The value of portfolio is immune to any changes in stock price S_t and thus admits no risk. Furthermore, we assume that there is a unique risk-free rate which we denote by r. We also utilise values of hedging parameters Δ, Δ_1.

$$A dt - B\Delta_1 dt = r(V - \Delta S - \Delta_1 V_1)dt$$
$$A - B\Delta_1 = r(V - \Delta S - \Delta_1 V_1)$$
$$\frac{A - rV + \frac{\partial V}{\partial S}rS}{\partial V/\partial v_t} = \frac{B - rV_1 + \frac{\partial V_1}{\partial S}rS}{\partial V_1/\partial v_t} \tag{3.11}$$

provided $\partial V/\partial v_t \neq 0$.

3.2   Market price of volatility risk

Each side of equation (3.11) depends either on $V(S_t, v_t, t)$ or $V_1(S_t, v_t, t)$. Both sides have to be equal to some function $g = g(S_t, v_t, t)$. In our case, we will closely follow [23] and without loss of generality we set $g = -(\alpha - \phi\beta\sqrt{v_t})$, where according to the CAPM [15], ϕ represents the market price of volatility risk. As we are interested in the price of option V, we use just the left-hand side of (3.11).

$$A - rV + \frac{\partial V}{\partial S}rS = -(\alpha - \phi\beta\sqrt{v_t})\frac{\partial V}{\partial v_t};$$
$$\frac{\partial V}{\partial t} + \frac{1}{2}v_t S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}v_t \beta^2 \frac{\partial^2 V}{\partial v_t^2} + \rho v_t \beta S \frac{\partial^2 V}{\partial v_t \partial S} - rV + rS\frac{\partial V}{\partial S} = -(\alpha - \phi\beta\sqrt{v_t})\frac{\partial V}{\partial v_t}. \tag{3.12}$$

To simplify the last equation, we substitute $\tau = T - t$, where T is the time to maturity of option V. We also express the equation in terms of logarithm of the stock price,

---

[15]CAPM stands for Capital Asset Pricing Model, for more information see [63].

$x_t = \ln(S)$, rather than S.

$$-\frac{\partial V}{\partial \tau} + \frac{1}{2}v_t\frac{\partial^2 V}{\partial x_t^2} + \left(r - \frac{1}{2}v_t\right)\frac{\partial V}{\partial x_t} + \rho\beta v_t\frac{\partial^2 V}{\partial v_t \partial x_t} - rV + \frac{1}{2}v_t\beta^2\frac{\partial^2 V}{\partial v_t^2} = -(\alpha - \phi\beta\sqrt{v_t})\frac{\partial V}{\partial v_t}.$$
(3.13)

As discussed in Chapter 1, the markets under stochastic volatility models are not complete and the price of an option depends on the choice of g (i.e. depends on investor's risk preferences with respect to the portfolio π). To obtain unique option prices, we chose the risk-neutral drift of dv_t, defined as $\hat{\alpha} = \alpha + \phi\beta\sqrt{v_t}$, which rules out ϕ from our equations. This is justified, because we are interested only in fitting models to option prices. We also assume that SDE's for S_t and v_t are in the risk-neutral form. Hence, all calibrated parameters will be under a risk-neutral measure.

$$-\frac{\partial V}{\partial \tau} + \frac{1}{2}v_t\frac{\partial^2 V}{\partial x_t^2} + \left(r - \frac{1}{2}v_t\right)\frac{\partial V}{\partial x_t} + \rho\beta v_t\frac{\partial^2 V}{\partial v_t \partial x_t} - rV + \frac{1}{2}v_t\beta^2\frac{\partial^2 V}{\partial v_t^2} + \alpha\frac{\partial V}{\partial v_t} = 0.$$
(3.14)

3.3 Price of a call option in terms of in-the-money probabilities

Price of a call option has to satisfy (3.14) with initial condition that is given by the pay-off function of the call option.

$$-\frac{\partial V_c}{\partial \tau} + \frac{1}{2}v_t\frac{\partial^2 V_c}{\partial x_t^2} + \left(r - \frac{1}{2}v_t\right)\frac{\partial V_c}{\partial x_t} + \rho\beta v_t\frac{\partial^2 V_c}{\partial v_t \partial x_t} - rV_c + \frac{1}{2}v_t\beta^2\frac{\partial^2 V_c}{\partial v_t^2} + \alpha\frac{\partial V_c}{\partial v_t} = 0;$$
(3.15)

$$V_c(\tau = 0, K) = V_c(t = T, K) = (S_T - K)^+. \quad (3.16)$$

The price can be also expressed as an expectation of the discounted pay-off.

$$\begin{aligned}
V_c(\tau, K) &= e^{-r\tau}E\left[(S_T - K)^+\right] \\
&= S_t P_1(x_t, v_t, \tau) - e^{-r\tau}K P_2(x_t, v_t, \tau) \\
&= e^{x_t}P_1(x_t, v_t, \tau) - e^{-r\tau}K P_2(x_t, v_t, \tau),
\end{aligned}$$
(3.17)

where P_1, P_2 can be interpreted as risk-neutral probabilities that option expires in the money conditional on the value of $x_t = \ln S_t$. We can substitute (3.17) for V_c in (3.15). The equation has to be satisfied for any combination of real parameters r, τ and for any

price S_t. For $K = 0$, $S_t = 1$, we obtain the PDE with respect to P_1 only.

$$-\frac{\partial P_1}{\partial \tau} + \frac{1}{2}v_t\frac{\partial^2 P_1}{\partial x_t^2} + \left(r + \frac{1}{2}v_t\right)\frac{\partial P_1}{\partial x_t} + \rho\beta v_t\frac{\partial^2 P_1}{\partial v_t\partial x_t} + \frac{1}{2}v_t\beta^2\frac{\partial^2 P_1}{\partial v_t^2} + (\alpha + \rho\beta v_t)\frac{\partial P_1}{\partial v_t} = 0.$$

$$(3.18)$$

Following similar arguments, we are able to retrieve the PDE for P_2 by setting $S_t = r = 0$, $K = -1$.

$$-\frac{\partial P_2}{\partial \tau} + \frac{1}{2}v_t\frac{\partial^2 P_2}{\partial x_t^2} + \left(r - \frac{1}{2}v_t\right)\frac{\partial P_2}{\partial x_t} + \rho\beta v_t\frac{\partial^2 P_2}{\partial v_t\partial x_t} + \frac{1}{2}v_t\beta^2\frac{\partial^2 P_2}{\partial v_t^2} + \alpha\frac{\partial P_2}{\partial v_t} = 0. \quad (3.19)$$

3.4 Characteristic functions

Instead of solving the system of two PDEs (3.18)-(3.19) directly, we express characteristic functions of the log-price at maturity T. After characteristic functions $f_j = f_j(\phi, \tau)$ for $j = 1, 2$, are known, we can easily obtain P_j using the inverse Fourier transform.

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathsf{Re}\left(\frac{e^{i\phi\ln(K)}f_j}{i\phi}\right)d\phi. \tag{3.20}$$

As in the original paper by Heston [29], we assume that characteristic functions f_j are of the following form:

$$f_j = \exp\left\{C_j(\tau, \phi) + D_j(\tau, \phi)v_0 + i\phi x\right\}. \tag{3.21}$$

Moreover, as a direct consequence of the discounted version of Feynman-Kac theorem, f_j follows PDE (3.18) and (3.19). Firstly, we substitute assumed expression (3.21) for f_1.

$$-\left(\frac{\partial C_1}{\partial \tau} + v_t\frac{\partial D_1}{\partial \tau}\right)f_1 + \rho\beta v_t i\phi D_1 f_1 - \frac{1}{2}v_t\phi^2 + \frac{1}{2}v_t\beta^2 D_1^2 f_1$$
$$+ \left(r + \frac{1}{2}v_t\right)i\phi f_1 + (\alpha - \rho\beta v_t)f_1 D_1 = 0, \tag{3.22}$$

f_1 is a characteristic function of the log price process and thus cannot be identically equal to zero.

$$-\frac{\partial C_1}{\partial \tau} + v_t\frac{-\partial D_1}{\partial \tau} + \rho\beta v_t i\phi D_1 - \frac{1}{2}v_t\phi^2 + \frac{1}{2}v_t\beta^2 D_1^2$$
$$+ \left(r + \frac{1}{2}v_t\right)i\phi + (\alpha - \rho\beta v_t)D_1 = 0. \tag{3.23}$$

Instead of a general drift of variance process dv_t, we assume a linear drift term with respect to v_t, i.e. $\alpha(S_t, v_t, t) = \theta + \bar{\alpha}(S_t, t)v_t$. After rearranging terms with C_1, D_1 and factoring out v_t we obtain:

$$v_t \left[-\frac{\partial D_1}{\partial \tau} + \rho \beta i \phi D_1 - \frac{1}{2}\phi^2 + \frac{1}{2}\beta^2 D_1^2 + \frac{1}{2}i\phi + (\bar{\alpha} + \rho\beta) D_1 \right]$$
$$-\frac{\partial C_1}{\partial \tau} + ri\phi + \theta D_1 = 0. \qquad (3.24)$$

We assume that instantaneous variance does not hit zero, i.e. $v_t > 0$, for $t : 0 \leq t \leq T$. Since all terms outside the brackets are independent on v_t, we deduce that

$$\frac{\partial D_1}{\partial \tau} = \rho \beta i \phi D_1 - \frac{1}{2}\phi^2 + \frac{1}{2}\beta^2 D_1^2 + \frac{1}{2}i\phi + (\bar{\alpha} + \rho\beta) D_1; \qquad (3.25)$$
$$\frac{\partial C_1}{\partial \tau} = ri\phi + \theta D_1. \qquad (3.26)$$

Following the same steps, one can obtain a system of equations for f_2 as well. Therefore characteristic functions f_j defined by (3.21) have to satisfy the following system of four differential equations

$$\frac{\partial D_1}{\partial \tau} = \rho \beta i \phi D_1 - \frac{1}{2}\phi^2 + \frac{1}{2}\beta^2 D_1^2 + \frac{1}{2}i\phi + (\bar{\alpha} + \rho\beta) D_1; \qquad (3.27)$$
$$\frac{\partial D_2}{\partial \tau} = \rho \beta i \phi D_2 - \frac{1}{2}\phi^2 + \frac{1}{2}\beta^2 D_2^2 - \frac{1}{2}i\phi + \bar{\alpha} D_2; \qquad (3.28)$$
$$\frac{\partial C_j}{\partial \tau} = ri\phi + \theta D_j; \qquad (3.29)$$

with respect to the initial condition

$$C_j(0, \phi) = D_j(0, \phi) = 0, \qquad (3.30)$$

where $j = 1, 2$. The first two equations for D_j are known as the Riccati equations with constant coefficients. Once D_j are obtained, one can solve the last two ODE's by a direct integration.

3.5 Solution of the Ricatti equation with constant coefficients

For notational convenience we will rewrite equations (3.27) and (3.28) using abbreviated form.
$$\frac{\partial D_j(\tau, \phi)}{\partial \tau} = A_j D_j^2 + B_j D_j + K_j, \qquad (3.31)$$

where A_j, B_j and K_j are in general complex constants. Their values will be studied later, now we show how to solve (3.31). Let us also denote:

$$\Delta_j = \sqrt{B_j^2 - 4A_j K_j}; \ Y_j = \frac{-B_j + \Delta_j}{2A_j}; \ g_j = \frac{B_j - \Delta_j}{B_j + \Delta_j}.$$

Proposition 3.1. *Assuming $A_j \neq 0$ for $j = 1, 2$, Ricatti equations (3.31) attain an analytical solution with respect to the initial condition $D_j(0, \phi) = 0$,*

$$D_j(\tau, \phi) = \frac{Y_j \left(1 - e^{\Delta_j \tau}\right)}{1 - g_j e^{\Delta_j \tau}}.$$

Proof. Without loss of generality, we will solve the equation for fixed j and for $y = D_j$. We can rewrite (3.31) as an ODE without the fixed index j.

$$y' = Ay^2 + By + K, \tag{3.32}$$
$$Ay' = (Ay)^2 + ABy + AK, \tag{3.33}$$

Since A, B and K are constant in time (or with respect to τ), we are able to substitute $v = Ay; \ v' = Ay' + A'y = Ay'$.

$$v' = v^2 + Bv + AK, \tag{3.34}$$
$$-\frac{u''}{u} = -B\frac{u'}{u} + AK, \tag{3.35}$$

where $v = -u'/u; v' = -\frac{u''u - (u')^2}{u^2} = v^2 - \frac{u''}{u'}$. The equation can be rewritten in the following form

$$0 = u'' - Bu' + AKu. \tag{3.36}$$

Moreover, we can solve (3.36) explicitly.

$$u(\tau) = I_1 \exp\left\{\frac{B - \sqrt{B^2 - 4AK}}{2}\tau\right\} + I_2 \exp\left\{\frac{B + \sqrt{B^2 - 4AK}}{2}\tau\right\} = I_1 e^{\frac{B-\Delta}{2}\tau} + I_2 e^{\frac{B+\Delta}{2}\tau},$$

where $I_1, I_2 \in \mathbb{R}$ and both can be expressed due to initial condition:

$$u'(0) = I_1\left(\frac{B - \Delta}{2}\right) + I_2\left(\frac{B + \Delta}{2}\right) = 0;$$
$$u(0) = I_1 + I_2 = \gamma; \quad \gamma \in \mathbb{R} - \{0\}.$$

Solving the system of two linear equations we obtain I_1, I_2.

$$I_1 = \gamma \frac{B+\Delta}{2\Delta};$$
$$I_2 = -\gamma \frac{B-\Delta}{2\Delta},$$

and the solution $u(\tau)$:

$$u(\tau) = \gamma \left[\left(\frac{B+\Delta}{2\Delta} \right) e^{\frac{B-\Delta}{2}\tau} - \left(\frac{B-\Delta}{2\Delta} \right) e^{\frac{B+\Delta}{2}\tau} \right]. \tag{3.37}$$

To obtain $y(\tau)$ we go through steps (3.32)-(3.36) backwards.

$$u' = \gamma \left[\left(\frac{B^2 - \Delta^2}{4\Delta} \right) e^{\frac{B-\Delta}{2}\tau} - \left(\frac{B^2 - \Delta^2}{4\Delta} \right) e^{\frac{B+\Delta}{2}\tau} \right];$$
$$= \gamma \left[\frac{AK}{\Delta} e^{\frac{B-\Delta}{2}\tau} - \frac{AK}{\Delta} e^{\frac{B+\Delta}{2}\tau} \right].$$

Since $v = \frac{u'}{u}$:

$$v = \frac{2AK \left(e^{\frac{B-\Delta}{2}\tau} - e^{\frac{B+\Delta}{2}\tau} \right)}{(B+\Delta)e^{\frac{B-\Delta}{2}\tau} - (B-\Delta)e^{\frac{B+\Delta}{2}\tau}}.$$

Using $y = v/A$, one can obtain the solution:

$$y = \frac{2K \left(e^{\frac{B-\Delta}{2}\tau} - e^{\frac{B+\Delta}{2}\tau} \right)}{(B+\Delta)e^{\frac{B-\Delta}{2}\tau} - (B-\Delta)e^{\frac{B+\Delta}{2}\tau}};$$
$$= \frac{2K \left(e^{\frac{B-\Delta}{2}\tau} - e^{\frac{B+\Delta}{2}\tau} \right)}{(B+\Delta)e^{\frac{B-\Delta}{2}\tau} \left(1 - \frac{B-\Delta}{B+\Delta} e^{\Delta\tau} \right)};$$
$$= \frac{\frac{2K}{B+\Delta} \left(1 - e^{\Delta\tau} \right)}{1 - \frac{B-\Delta}{B+\Delta} e^{\Delta\tau}}. \tag{3.38}$$

As we fix $j = 1$ or $j = 2$, we substitute $B = B_j$, $K = K_j$ and Δ becomes Δ_j. Then we arrive at the expression in Proposition 3.1. $\qquad \square$

Remark 3.1. By the proof of 3.1 we have managed to show that proposed expression solves our Ricatti equation. Moreover, this solution is unique which is discussed alongside other properties in a book by W. T. Reid [52]. We also neglect the case when $A_j = 0$, which would imply (equally) zero diffusion term in the volatility process ($\beta \equiv 0$). Thus there would be no other source of randomness apart from the Wiener process W_t in the stock evolution.

3.6 Option pricing formula

Comparing (3.27)-(3.28) with (3.31), we set the values of A_j, B_j and K_j (Table 3.1).

TABLE 3.1: Values of constants A_j, B_j and K_j, notation as in (3.1).

j	1	2
A_j	$\frac{1}{2}\beta^2$	$\frac{1}{2}\beta^2$
B_j	$\rho\beta i\phi + \bar{\alpha} + \rho\beta$	$\rho\beta i\phi + \bar{\alpha}$
K_j	$-\frac{1}{2}\phi(\phi - i)$	$-\frac{1}{2}\phi(\phi + i)$

In the next step, we integrate the right-hand side of (3.29) for $t \in [0, \tau]$ to express C_j.

$$
\begin{aligned}
C_j(\tau, \phi) &= ri\phi\tau + \theta \int_0^\tau D_j(t, \phi)dt \\
&= ri\phi\tau + \theta \int_0^\tau \frac{Y_j\left(1 - e^{\Delta_j t}\right)}{1 - g_j e^{\Delta_j t}}dt \\
&= ri\phi\tau + \theta Y_j \left[\tau + \int_0^\tau \frac{(g_j - 1)e^{\Delta_j t}}{1 - g_j e^{\Delta_j t}}dt\right] \\
&= ri\phi\tau + \theta Y_j\tau - \theta Y_j\frac{g_j - 1}{\Delta_j g_j}\ln\left(\frac{1 - g_j e^{\Delta_j \tau}}{1 - g_j}\right) \\
&= ri\phi\tau + \theta Y_j\tau - \frac{\theta}{A}\ln\left(\frac{1 - g_j e^{\Delta_j \tau}}{1 - g_j}\right).
\end{aligned}
\tag{3.39}
$$

For diffusion stochastic volatility models, we obtain characteristic functions in the form of $f_j(\tau, \phi) = \exp\{C_j(\tau, \phi) + D_j(\tau, \phi)v_0 + i\phi x\}$. In case of the models with jumps, we also need to include a characteristic function of a compound, compensated Poisson process, denoted by ψ (see Appendix A). As in [23], we get the following result

$$
f_j^*(\tau, \phi) = \exp\{C_j(\tau, \phi) + D_j(\tau, \phi)v_0 + i\phi x + \psi(\phi)\tau\},
\tag{3.40}
$$

which would be used to price European calls for the Bates and FSV model. Characteristic functions of each model are summarised in the Appendix A.

Remark 3.2. We recall, that the price of a European call option is retrieved using f_j or f_j^* in the inverse Fourier transform integral (3.20) and then by evaluating expression (3.17).

Chapter 4

Estimation of the Hurst exponent

In this chapter we will introduce several techniques to measure long-range dependence (LRD) in a given time-series. These techniques are well developed and implemented in various programming frameworks. We will use MATLAB codes written by Chu Chen [16] and Rafal Weron [61] (GPH estimator) to analyse a long-time persistence in both synthetic and realized volatility data.

In literature the LRD is defined in various ways. First of all, we will start with the auto-covariance (ACV) function of a given stochastic process. Let $(\Omega, \mathcal{F}, P)$ be a generic probability space on which a stochastic process X_t, $0 \leq t \leq T$ for some finite T, is defined. Assuming $\mathbb{E}[X_t] < \infty$, one can express auto-covariance of X by

$$C_{XX}(t, s) = \mathbb{E}\left[(X_t - \mathbb{E}X_t)(X_s - \mathbb{E}X_s)\right].$$

In case of a second-order stationary process [17] X, the ACV function depends only on the difference between t and s, $k := s - t$ for $0 \leq t \leq s$ and thus it will be denoted as a function of k,

$$\gamma_X(k) = \mathbb{E}\left[(X_t - \mathbb{E}X_t)(X_{t+k} - \mathbb{E}X_{t+k})\right].$$

Definition 4.1 (LRD). A second-order stationary stochastic process X_t, $0 \leq t \leq T$, with finite mean and variance is said to have long-range dependence if

$$\lim_{k \to \infty} \frac{\gamma_X(k)}{Ck^\alpha} = 1, \tag{4.1}$$

---

[16]Codes available under the BSD license at http://www.mathworks.com/matlabcentral/fileexchange/19148-hurst-parameter-estimate. For our purposes, we changed plots and added approximative bounds on estimates.

[17]The first moment and covariance of the process do not vary in time.

where $\gamma_{X_t}(k)$ is an auto-covariance function of the process, both C and α are constants and $\alpha \in (0,1)$. Dependence between X_t and X_{t+k} decays slowly as $k \to \infty$ and the sum of auto-covariances for different lags diverges,

$$\sum_{k=1}^{+\infty} \gamma_X(k) = +\infty. \tag{4.2}$$

One can understand the LRD phenomenon quite intuitively. For increasing lag k, the dependence might be small, but its cumulative effect is not negligible due to (4.2). Moreover, the Definition 4.1 gives us a clue on how to measure degrees of dependence. The auto-covariance function for an arbitrary LRD process follows the power law

$$\gamma_X(k) \approx Ck^\alpha.$$

In this chapter, we will focus on estimation of the Hurst exponent which is related to α, $\alpha = 2H - 1$. If we recall, $\alpha \in (0,1)$, one can easily see that for LRD processes H takes values from $1/2$ to 1 [3].

4.1 Available methods

R/S method

In the original paper, hydrologist E.H. Hurst discovered long-range dependence of water levels at the river Nile. To describe this phenomena on a time-series $D = \{D_i\}_{i=1}^N$, he employed a cumulative mean adjusted series,

$$Z_n = \sum_{i=1}^n (D_i - \mu); \quad \mu = \frac{1}{N} \sum_1^N D_i,$$

and then he defined a range function as

$$R(n) = \max\{Z_1, Z_2, ..., Z_n\} - \min\{Z_1, Z_2, ..., Z_n\}.$$

Rescaled range statistics (R/S) is then obtained by $R(n)/S(n)$, where $S(n)$ is a standard deviation of the first n elements of D. Hurst described an asymptotic behaviour of R/S statistics [33] by

$$\mathbb{E}\left[\frac{R(n)}{S(n)}\right] = Cn^H \quad \text{as } n \to \infty. \tag{4.3}$$

This result has been known as the Hurst effect [42]. To estimate H, one tries to fit $\ln[R(n)/S(n)]$ by a straight line using the least squares method. A slope of the regression line forms out our estimate of H provided none of the linear regression assumptions were violated. R/S method, although being the oldest, is still widely used, mainly for its ease of implementation.

Aggregate Variance method

Given a time-series with standard deviation σ which is generated by fractional Gaussian noise [18], one can obtain a relation between standard deviation of sample mean (σ_n) and the sample size [56]:

$$\sigma_n = \frac{\sigma}{n^{1-H}}. \tag{4.4}$$

The relation is, in fact, just a generalization of a well-known result for a standard Gaussian noise,

$$\sigma_n = \frac{\sigma}{\sqrt{n}}.$$

Hence, an estimate of the Hurst exponent can be retrieved by partitioning the series and comparing a size of partitions with its standard deviations of sample means. Again, we construct a regression line in the least square sense with slope β. Then, the estimate of H can be expressed as

$$\hat{H} = 1 - \beta/2. \tag{4.5}$$

Higuchi method

The Higuchi method for the long-memory estimation is based on a relation between the expectation of a block curve length and fractal dimension of the block data. Firstly, we construct a new time series from the one that is given. Assuming initial time series $D = \{D_i\}_{i=1}^N$ of N elements (i.e. discrete observations with regular intervals), we build a k dimensional series for the kth block as follows

$$D_i^m: \ D_m, \ D_{m+k}, \ D_{m+2k}, ..., \ D_{m+Jk},$$

---

[18]A fractional Gaussian noise is formed out of increments of a fractional Brownian motion defined by 1.16, whereas a standard Gaussian noise consists of the increments generated by a standard Wiener process.

TABLE 4.1: List of all employed estimators.

Estimation procedure	Abbreviation used	Introduced in
Aggregate Variance method	Aggvar	[3]
Geweke-Porter-Hudak estimator	GPH	[24]
Higuchi method	Higuchi	[30]
Peng method	Peng	[48]
Periodogram analysis	Per	[3]
Rescaled range analysis	R/S	[33]

for $J = \left[\frac{N-m}{k}\right]$, where $[x]$ denotes an integer point function and $m = 1, 2, ..., k$. The length of a curve represented by D_i^m can be defined as [30]:

$$L(k) = \left(\sum_{i=1}^{J} |D_{m+ik} - D_{m+(i-1)k}|\right) \frac{N-1}{Jk^2}. \tag{4.6}$$

Higuchi showed that the expectation of $L(k)$ follows the power law

$$\mathbb{E}[L(k)] = k^{-d}, \tag{4.7}$$

where $d = 2 - H$. Hence, to estimate H, we apply the same procedure as in case of the Hurst's R/S analysis.

In the next section, we employ aforementioned estimators alongside others that were described, for instance, in a paper by Serinaldi [56] and a book by Beran [3].

Remark 4.1. To estimate the Hurst exponent from volatility data, our initial time series would be formed out of the increments of (simulated) volatility, i.e. the ith element would be

$$D_i = v_{t_i} - v_{t_{i-1}},$$

for $i = 1, 2, ..., N - 1$ while having N observations of the realized or simulated volatility.

Remark 4.2. Estimates of the Hurst exponent will be further referred to as $\hat{H}$, whereas the simulated or the theoretical value is denoted by H.

4.2 Estimation on synthetic data

To test different estimators we have simulated 10000 sample paths of five long-range dependent processes driven by fractional Brownian motions. Each process was simulated with different value of the Hurst exponent and, with our focus on LRD, we employed

TABLE 4.2: Average estimates $\pm$ sample variance of estimates.

Method	$H = 0.60$	$H = 0.65$	$H = 0.70$	$H = 0.75$	$H = 0.80$
Aggvar	$.5876 \pm .0010$	$.6358 \pm .0008$	$.6845 \pm .0008$	$.7336 \pm .0007$	$.7849 \pm .0006$
GPH	$.5849 \pm .0068$	$.6408 \pm .0061$	$.6997 \pm .0059$	$.7597 \pm .0054$	$.8280 \pm .0051$
Higuchi	$.5983 \pm .0007$	$.6480 \pm .0007$	$.6978 \pm .0006$	$.7467 \pm .0006$	$.7952 \pm .0005$
Peng	$.5963 \pm .0011$	$.6462 \pm .0010$	$.6963 \pm .0010$	$.7464 \pm .0010$	$.7972 \pm .0011$
Per	$.6013 \pm .0008$	$.6541 \pm .0007$	$.7066 \pm .0007$	$.7604 \pm .0006$	$.8149 \pm .0006$
R/S	$.6194 \pm .0008$	$.6618 \pm .0009$	$.7050 \pm .0010$	$.7485 \pm .0010$	$.7905 \pm .0011$

the following set of the Hurst parameter values.

$$H^{(sim)} = \{0.60, \ 0.65, \ 0.70, \ 0.75, \ 0.80\} \tag{4.8}$$

The simulated processes follow a pathwise SDE, where κ, $\bar{v}$, ξ are fixed and $H \in H^{(sim)}$,

$$dv_t = -\kappa(v_t - \bar{v})dt + \xi v_t dB_t^H, \tag{4.9}$$

for these parameters:

$$\kappa = 5, \quad \bar{v} = 0.1, \quad \xi = 2.$$

and for the initial condition $v_0 = 0.05$. Equation (4.9) was discretized using a fractional Milstein-type scheme [19] with $\Delta t = 2^{-10}$. To keep simulated paths non-negative, a reflection rule, $v_t = \max\{v_t, -v_t\}$, was used [20].

After synthetic data were generated, we used procedures to obtain an estimate of the Hurst parameter alongside variance of the estimates. The most satisfying results were obtained using the Huguchi method. This procedure provided us with the lowest average error and estimates for different sample paths were consistent, having the lowest variance out of the tested approaches.

TABLE 4.3: Average absolute relative errors (AARE, for definition see Chapter 6).

Aggvar	GPH	Higuchi	Peng	Per	R/S
2.11%	1.75%	0.39%	0.50%	1.01%	1.43%

Due to the heuristic nature of all estimators, we also analysed a decay in estimate quality for decreasing length of discretized trajectories. This criteria should be of great importance, since only a limited amount of market data is available to us. All estimates are within ± 0.2 bounds from the expected H for a dataset with 8196 observations.

---

[19]This scheme is introduced in [27].

[20]In fact, non-negativity of discrete realizations of (4.9) is not necessary, until we interpret v_t as volatility of S_t.
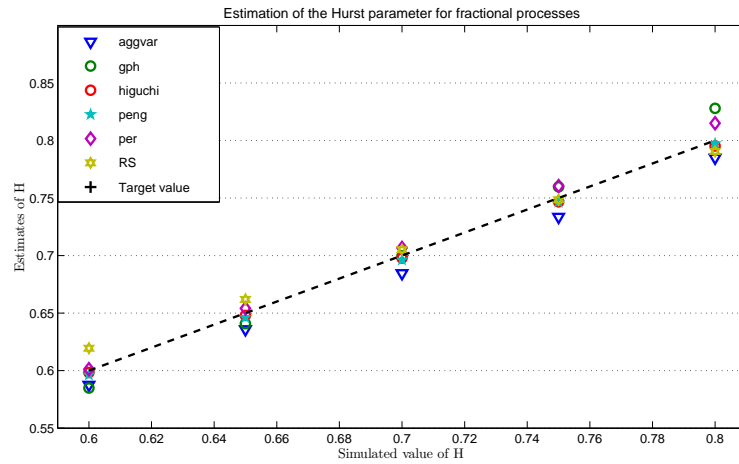
FIGURE 4.1: Estimates of the Hurst exponent.
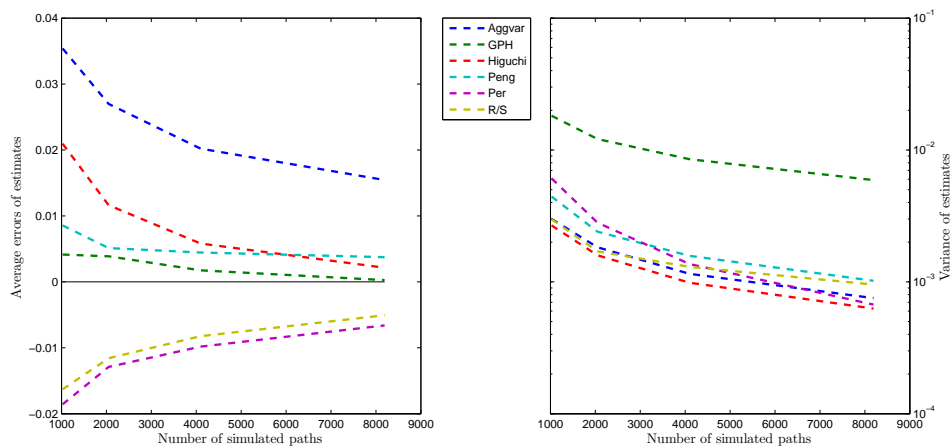
FIGURE 4.2: Estimate quality versus length of simulated trajectory ($H = 0.7$).

Results for $H = 0.7$ are pictured in Figure 4.2. Most consistent results throughout $H^{(sim)}$ were observed for Higuchi, Aggregate Variance and R/S methods. GPH estimator performed (in terms of the average error) very well for $H = 0.7$, unlike for $H = 0.6$ or $H = 0.8$. Sample variance of GPH estimates was higher than variances of the other approaches. More detailed analysis and discussion on LRD estimation can be found in the article by Rea et al [50].

4.3   Estimation on realized volatilities

In this section, we estimate the Hurst exponent for 30, 60 and 90 day realized volatilities of the FTSE 100 Index [21]. The data set consists of 2527 daily observations from 2nd January 2004 to 31st December 2013.

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[21]The underlying of our option price surface, see the following chapters.
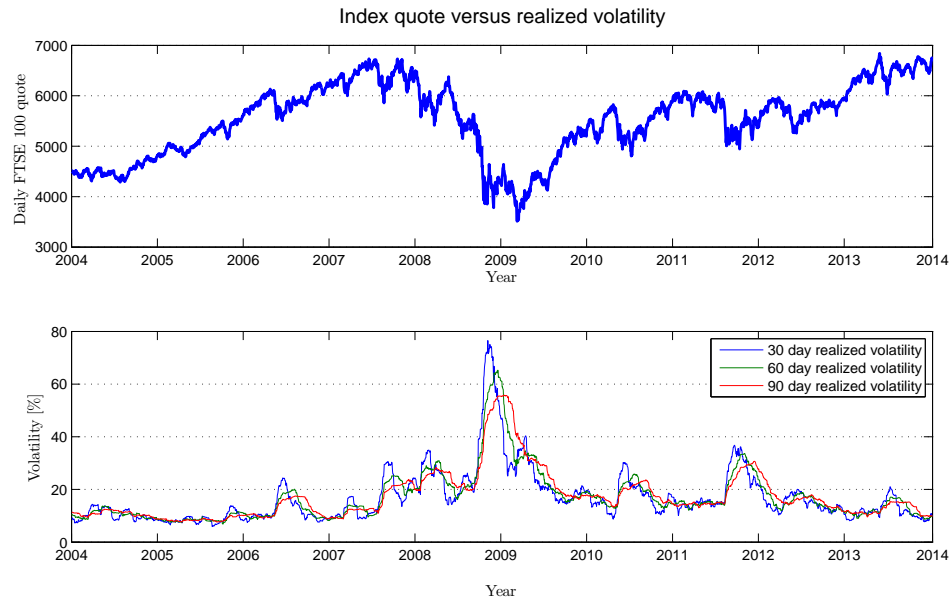
FIGURE   4.3:   Quotes   of   FTSE   100   Index   and   its   realized   volatilities.
[*source: Bloomberg terminal*, 24th *March* 2014]

Using the same methods as in the previous section, we obtained results summarized
in Table 4.4. For 60 and 90 day volatility, all applied estimators suggested long-range
dependence $\hat{H} > 0.5$. Apart from the GPH estimator and Periodogram analysis, all
methods used a linear regression with statistically significant parameters [22]. A coeffi-
cient of determination R^2, which describes the relative amount of data variation that is
explained by the regression, exceeds 88% all methods, except for GPH and Per estima-
tors.

For GPH and Per methods [23], we failed to explain the transformed data by a linear
regression and hence we conclude that these methods are not well suited for estimation
of the Hurst exponent with respect to the realized volatility data that we have used. We
illustrated the quality of regression lines (which effects the quality of Hurst parameter
estimates) for Aggregate Variance, Higuchi, Rescaled-Range and GPH methods by Fig-
ures 4.5 - 4.8. The left top plot of each sub figure describes the underlying regression,
where the transformed data are represented by blue dots. In case of Aggvar and R/S
graphs, we also included a blue line that would correspond to $\hat{H} = 0.5$. The bottom
right plot depicts a decay in auto-correlation for increasing lag.

Each figure can be reconstructed by the attached function *T_Hestimator.m*:

---

[22]We tested both adjoint (F-test) and individual statistical significance (T-test).

[23]In fact, both methods are similar and based on so-called periodograms, see [24].

```
1  load('Vol_data2013.mat');
2  [H,conf, stats]=T_Hestimator(I60D,'higuchi');
```

FIGURE 4.4: The Hurst parameter estimation for 60 day realized volatility by the Higuchi method (using *higuchi.m*).

TABLE 4.4: Estimates of Hurst exponents.

Time series	Aggvar	GPH	Higuchi	Peng	Per	R/S
30d Vol.	0.6586	(0.3440)	0.6559	0.8472	(0.7462)	0.6569
60d Vol.	0.7837	(0.9142)	0.8090	0.9674	(0.8636)	0.8003
90d Vol.	0.8246	(0.8334)	0.8457	0.9422	(0.8919)	0.8160

The results in this chapter are in-line with those in [43] (for S&P 500 realized volatilities) or [7] and provide us with a motivation for the fractional volatility models as introduced in Chapter 2.
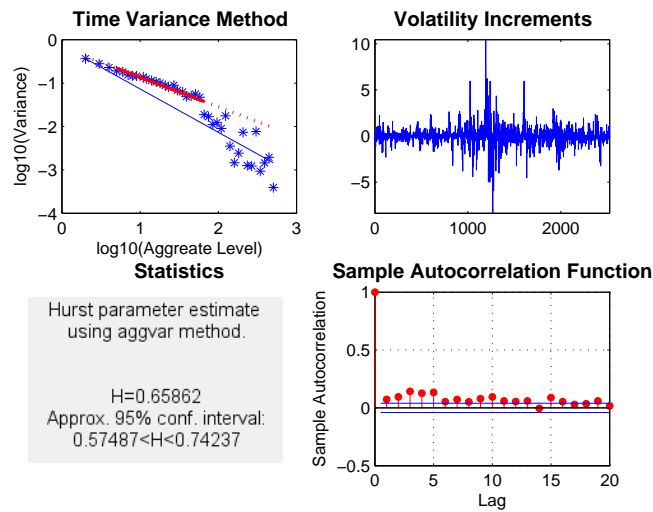
(A) 30 day volatility
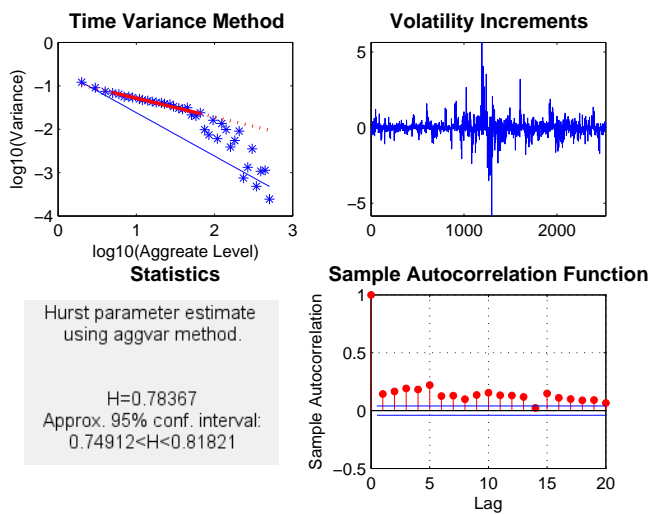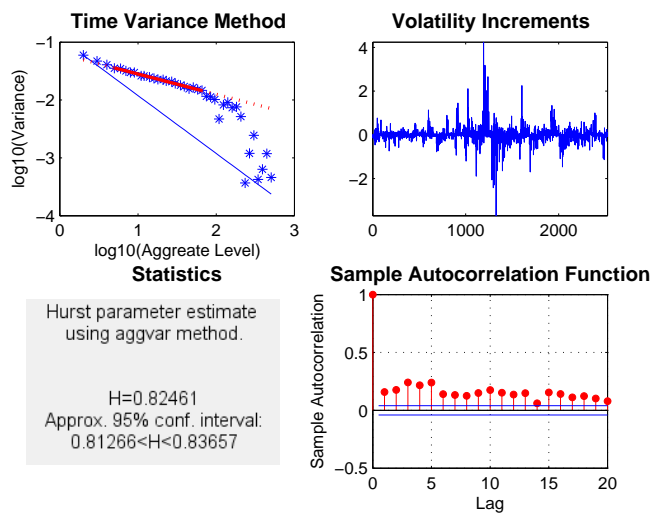
(B) 60 day volatility

(C) 90 day volatility

FIGURE 4.5: Long-memory estimation using the Aggregate Variance method.
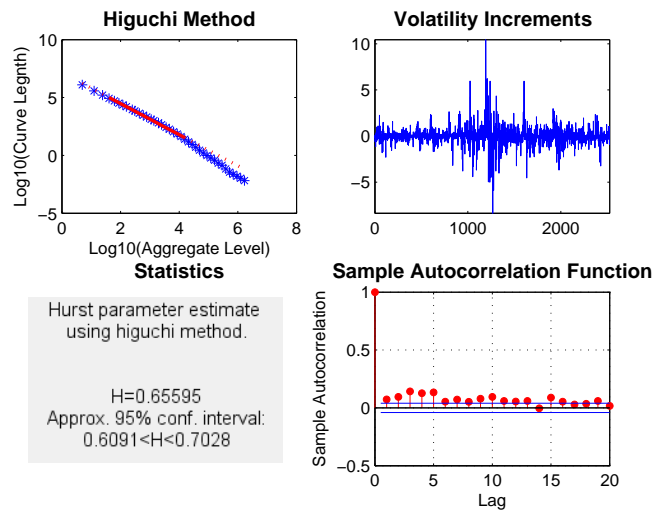
(A) 30 day volatility
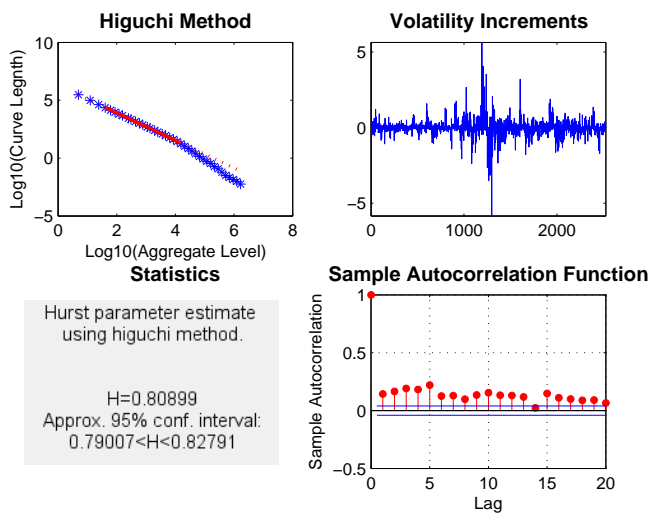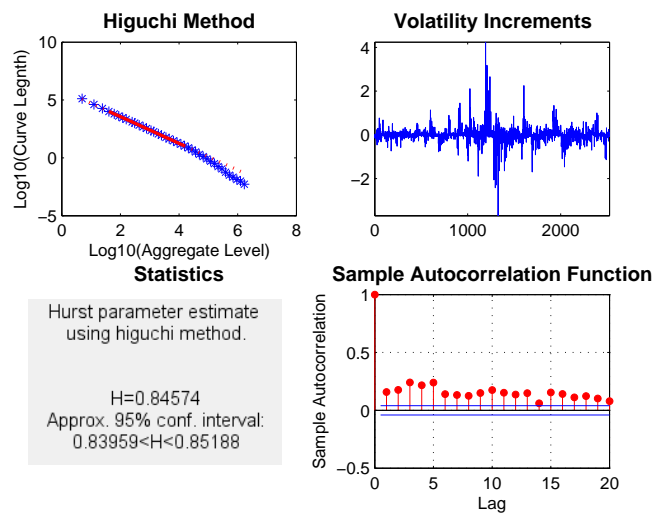
(B) 60 day volatility

(C) 90 day volatility

FIGURE 4.6: Long-memory estimation using the Higuchi method.
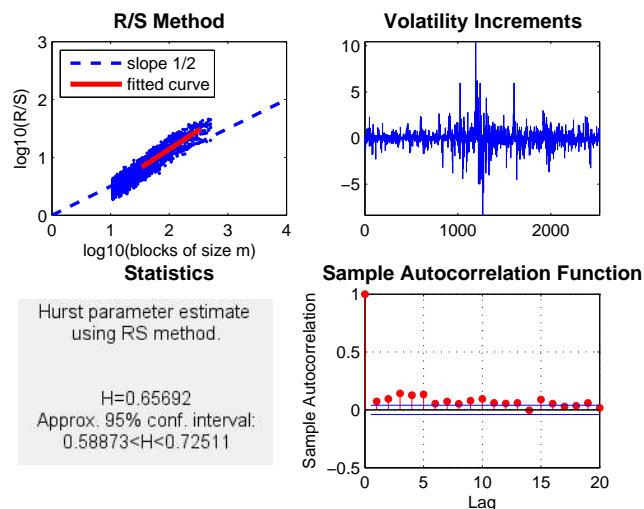
(A) 30 day volatility
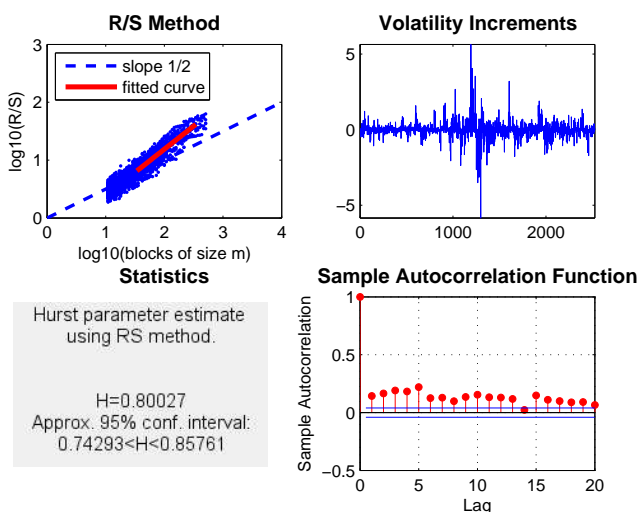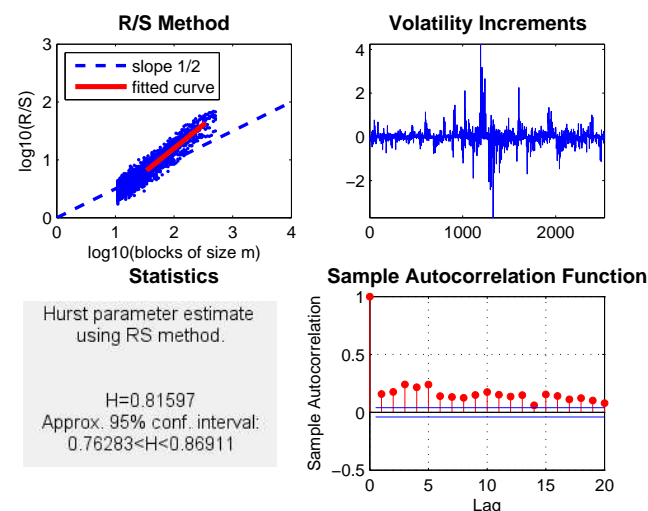
(B) 60 day volatility

(C) 90 day volatility

FIGURE 4.7: Long-memory estimation using the R/S method.
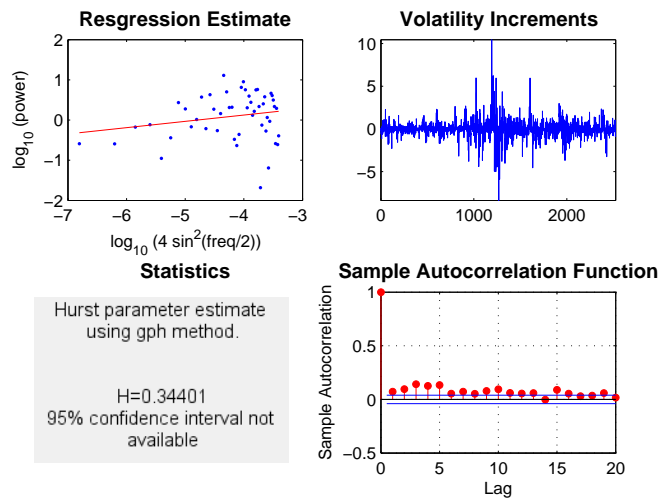
(A) 30 day volatility
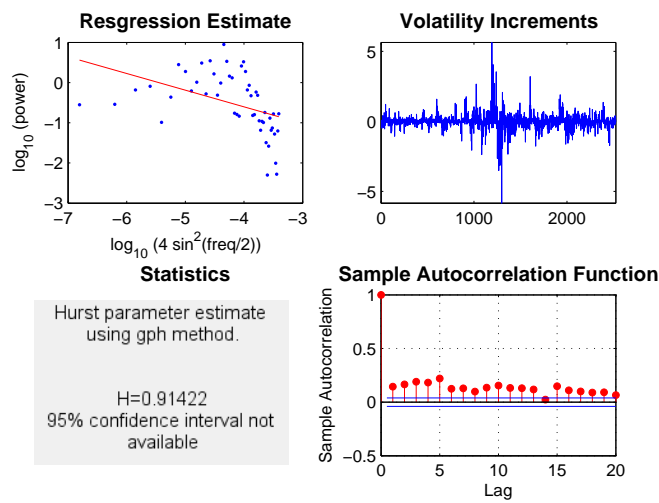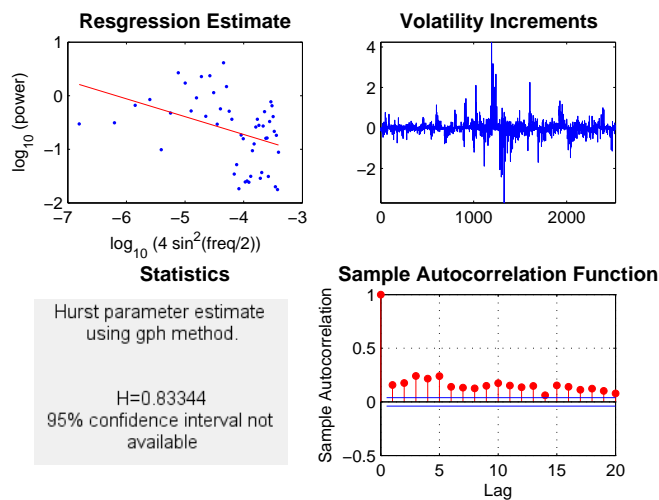
(B) 60 day volatility

(C) 90 day volatility

FIGURE 4.8: Long-memory estimation using the GPH estimator.

Chapter 5

Option pricing and market calibration

In this chapter we introduce the simplest financial derivatives with a non-linear pay-off function. These contracts are called options and later in the chapter we give some insight on how to calibrate pricing models from the real option markets.

5.1 Options

Before defining options, we will look at a generic financial derivative.

Definition 5.1 (Financial derivatives)**.** A financial contract between two parties (i.e. buyer and seller) with a value derived from another entity quote is called a (financial) derivative. As such entity one may consider assets (i.e. stocks), indices, interest rates or even different derivatives [24]. In literature, the entity is referred to as the underlying.

Further definition of a specific contract includes time and value of derived payments between involved parties. A set of all settlement dates is referred to as the exercise dates. In case the set consists of a single date, it is called the maturity of the contract.

Definition 5.2 (European call option)**.** A European call option is a derivative that gives the buyer a right, not an obligation, to buy the underlying for a fixed price K at the maturity $T > 0$. The seller grants the right for an option premium (i.e. option price).

---

[24]The most common example of this would be a swaption which is an option on entering the underlying swap.

Using the standard notation, K will be called the strike and S_t refers to a price of the underlying at $t \geq 0$. Similarly, we are able to define a European put option as a right to sell. If the strike price appears not to be in favour of the buyer at the maturity [25], the right is not exercised. Thus we are able to deduce a pay-off function as

$$F_{\text{call}} = (S_T - K)^+ , \tag{5.1}$$

for a call option and accordingly,

$$F_{\text{put}} = (K - S_T)^+ , \tag{5.2}$$

for a put. The main interest throughout the thesis will be laid on European calls, hence we only briefly describe some other types of financial options.
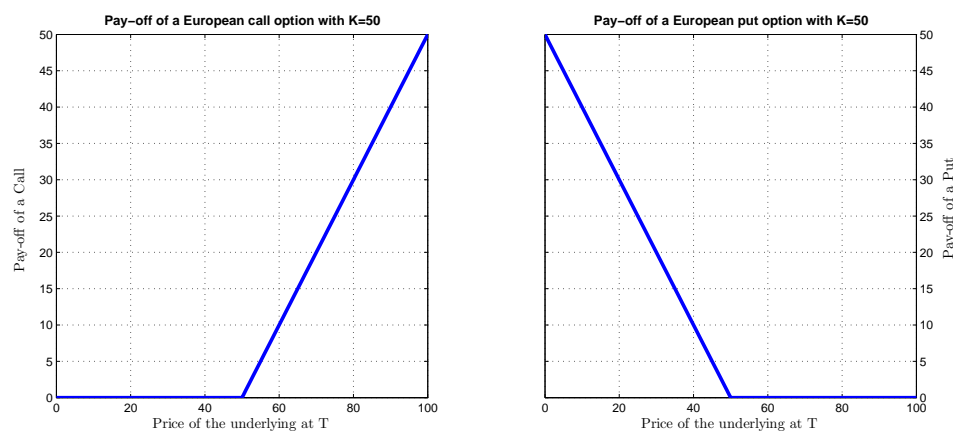
FIGURE 5.1: European option pay-offs.

American options. A contract with similar pay-off function as a corresponding European option. The right can be, however, exercised any time before the maturity. American options are popular for over-the-counter trades.

Asian (average) options. Instead of a plain price of the underlying, a pay-off depends on the average price over some fixed period of time.

Barrier options. The pay-off is conditional on the underlying hitting a certain pre-set price level which is commonly known as the barrier. Either the right is extinguished by hitting the barrier or conversely the right springs into existence.

---

[25]For a call option it would mean that the strike is above the underlying price at T.

Bermuda options. The contract has a similar terminal payments as American options, but it can be executed only at a finite set of times that are specified by the contract agreement.

Binary (digital) options. The option pays off either a fixed amount K or nothing, based on the terminal level of the underlying.

Lookback options. Owner of the option has a right to buy (or sell) the underlying for the lowest (highest) quoted price that was observed at a defined time period.

Option markets

First trades of option contracts are assumed to appear in ancient Greece [49]. However, it was not until 16th century when options became well defined and thus trades on the contracts started at market fairs and later at regular bourses. The first contracts were originally written on commodities such as wheat or sugar [10].

Nowadays, options in Europe are traded both over-the-counter and on exchanges. NYSE Euronext [26] became a pan-European trading place for futures and options after an acquisition of London International Financial Futures and Options Exchange (LIFFE) in 2002 [40]. Options on stocks and indices are quoted at Euronext exchanges in Amsterdam, Brussels and Paris. LIFFE offers also currency and commodity options [27]. European options on British stock index *FTSE 100* traded on LIFFE, will be consider in the next chapter for tasks of real market calibration.

The biggest option marketplace in the USA is the Chicago Mercantile Exchange (CME) [28]. The exchange was founded in 1898 as an agriculture commodity trading place. Even now, one can buy at the CME agriculture derivatives alongside commodity, currency, index, equity and US Treasury options [29].

Most of the option trading, however, is done over-the-counter (OTC). The contract is closed directly between the two trading parties which agreed on both premium and strike values. The price of an OTC option is typically not disclosed in public, but it should be in accordance with the known market prices to rule out arbitrages.

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[26]https://nyse.nyx.com
[27]Source: [20], March 2014
[28]http://www.cmegroup.com
[29]According to http://www.cmegroup.com/education/options.html, retrieved on 14th April 2014.

Market option prices

Market makers provide Ask and Bid quotes for each derivative. The Ask quote is a price for which they are prepared to sell the contract and, conversely, the Bid represents the amount of funds they are willing to pay for buying the contract. Having a set of N options, indexed by $i \leq N$, a Mid price of a ith call is obtained as

$$C_i = \frac{C_i^{ask} + C_i^{bid}}{2}.$$

Alternatively one can quote market prices using the Black-Scholes formula. The premium is then expressed in terms of the (Black-Scholes) implied volatility which was described by Rebonato in [51] as:

"The wrong number put in the wrong formula to obtain the right price."

To get the implied volatility we solve the following equation for v_i

$$BS(v_i) = C_i;$$

where $BS()$ stands for the Black-Scholes formula (as in the original paper [5], also displayed in the Appendix B). The root of the previous equation is denoted by v_i^{imp} and we refer to it as the implied volatility of the ith call option. To find v_i^{imp}, one can use, for instance, a standard Newton root-finding algorithm.

In the text we also use a classification of options with respect to suitability of exercising the right.

- In The Money (ITM) option - The option right is worth exercising at the current time, i.e. for a call $K < S_t$, conversely for a put $K > S_t$ at the current time t.

- At The Money (ATM) option - The strike equals to the underlying price at t [30].

- Out of The Money (OTM) option - The exercise right would not be used, i.e. for a call $K > S_t$, conversely for a put $K < S_t$.

---

[30]Or more commonly, the strike is close to the underlying price at t, $K \approx S_t$.
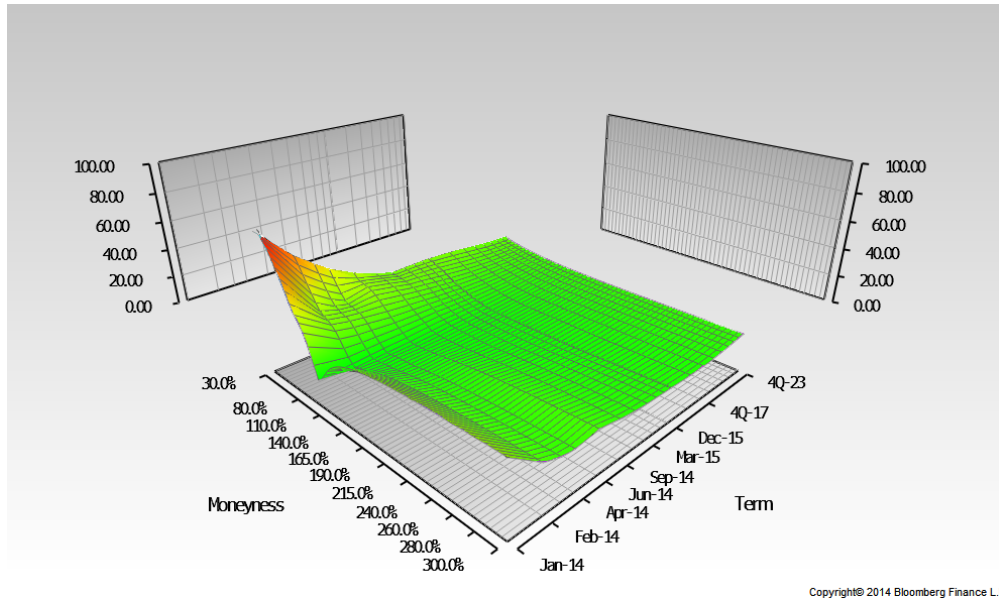
FIGURE 5.2: The implied volatility surface of FTSE 100 index call options from 8th January 2014, [*source: Bloomberg option monitor*].

5.2 Calibration from the option market

A task of finding a set of model parameters that can, to a certain degree, explain observed market prices is commonly referred to as the model calibration. More precisely, we consider an option pricing model with set of parameters $\Theta \in \mathsf{A}$ where A is a space of all admissible model parameters. We also consider a financial market with N quoted prices of European call options. In this text we try to derive a set of model parameters $\Theta^\star$ such that

$$\Theta^\star = \arg \inf_{\Theta \in \mathsf{A}} G(\Theta); \tag{5.3}$$

$$G(\Theta) = \sum_{i=1}^{N} w_i \left| C(S_0, K_i, T_i, r) - C^{model}(S_0, K_i, T_i, r, \Theta) \right|^p. \tag{5.4}$$

Intuitive choices of $w_i = 1/N$, for $i = 1, 2, ..., N$ and $p = 1$, might lead to unsatisfactory results. For $p = 2$, we get a classical problem of finding "least squares" between the observed and modelled prices. Again, this approach commonly does not provide very good results for the real option market calibration (see [46] and [45]). On this note, we comment that a well suited choice of the utility function $G(\Theta)$ is vital for obtaining calibrated parameters $\Theta^\star$.

In practise, it is important to fit prices of liquid contracts more precisely than the deep-out-of-the money counterparts. Having this in our mind, we set weights as a function

of the price spread [31]. For synthetic data we minimised the criteria using the following weight functions:

$$w_i^{(A)} = \frac{1}{|C_i^{ask} - C_i^{bid}|}, \tag{5.5}$$

$$w_i^{(B)} = \frac{1}{\sqrt{|C_i^{ask} - C_i^{bid}|}}, \tag{5.6}$$

$$w_i^{(C)} = \frac{1}{|C_i^{ask} - C_i^{bid}|^2}, \tag{5.7}$$

where C_i^{ask}, C_i^{bid} denote the ask price of the ith call option and its bid price respectively. We employed all weights in synthetic data test, for the real market calibration, we chose $w_i^{(B)}$ for all approaches.

Because of non-convexity of the optimisation problem (5.3), there might be more sets of parameters that can describe market data equally well. Also the structure of $G(\Theta)$ is highly non-linear (and model dependent). Hence, using traditional local optimisation methods without a good initial starting point might be useless (the method might end up in the nearest local minimum). To retrieve the initial guess, we employed a genetic algorithm and simulated annealing method from the Global optimisation toolbox in Matlab.

5.3   Global optimisation procedures

Genetic algorithm

The Genetic algorithm solver in Matlab is suitable for mixed-integer and for continuous variable optimisation. The latter will be of our main interest in the thesis. Also one can solve both constrained and unconstrained minimisation problems. For our purpose, a simple bound constraint is used, i.e. we set up the upper and lower thresholds for each model parameter. According to the Matlab documentation [32] :

"You can apply the genetic algorithm to solve a variety of optimization problems that are not well suited for standard optimization algorithms, including problems in which the objective function is discontinuous, nondifferentiable, stochastic, or highly nonlinear. The genetic algorithm can address problems of mixed integer programming, where

---

[31]Option spread is a distance between quoted ask and bid price. The closer both quotes are, the more efficiently is the given contract priced.

[32]See http://www.mathworks.co.uk/help/gads/what-is-the-genetic-algorithm.html.

some components are restricted to be integer-valued."

The optimisation procedure was fine-tuned for market calibration using *gaoptimset* function, as displayed in Figure 5.3.

```
1  function [params,fval,exitflag] = T_FSV_calib_ga(s0,C,K,T,A,B,r,eps)
2  % parameters: [v0 kappa vmean xi rho lambda alphaJ gammaJ H]
3  % lower bound for the optimisation
4   lower=[0, 0, 0, 0, -0.99 0 -10 0 0.51];
5  % upper bound for the optimisation
6   upper=[1,100,1,1, 0.99 10 10 10 0.99];
7  options = gaoptimset('Generations',500,'PopulationSize',60,'TolFun',1e-10,...
8      'UseParallel','always','StallGenLimit',100);
9  [params,fval,exitflag] =ga(@(params) cost_fun(params,s0,C,K,T,...
10     A,B,r),9,[],[],[],[],lower,upper,[],options);
11 end
```

FIGURE 5.3: Calibration of the FSV model using Genetic algorithm routine *ga*.

Simulated annealing

Simulated annealing routine can be used for various optimisation problems. However, only bound constraints are allowed and neither integer nor mixed-integer variables are supported. These restrictions do not affect our calibration process. As constraints we use upper and lower bounds and parameters of models can take arbitrary values within the bounds. The algorithm is inspired by the annealing process in metallurgy and, in a nutshell, it can be described as follows:

"At each iteration of the simulated annealing algorithm, a new point is randomly generated. The distance of the new point from the current point, or the extent of the search, is based on a probability distribution with a scale proportional to the temperature. The algorithm accepts all new points that lower the objective, but also, with a certain probability, points that raise the objective. By accepting points that raise the objective, the algorithm avoids being trapped in local minima, and is able to explore globally for more possible solutions. An annealing schedule is selected to systematically decrease the temperature as the algorithm proceeds. As the temperature decreases, the algorithm reduces the extent of its search to converge to a minimum." [33]

After the global optimisation was performed, we also used a traditional local search method to improve obtained results. The reason why not to employ a local method

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[33]Sourcing Matlab documentation - http://www.mathworks.co.uk/help/gads/what-is-simulated-annealing.html.

```matlab
1  function [params,fval,exitflag,info] = T_Bates_calib_sa(s0,C,K,T,A,B,r,
       init_param)
2  % parameters: [v0 kappa vmean xi rho lambda alphaJ gammaJ]
3  % lower bound for the least square optimisation
4   lower=[0 0 0 0 -0.99 0 -10 0];
5  % upper bound for the least square optimisation
6   upper=[1 100 1 1 0.99 10 10 10];
7
8   options = saoptimset('PlotFcns',{@saplotbestx, @saplotbestf, ...
9          @saplotstopping, @saplottemperature},'MaxFunEvals',100000);
10 [params,fval,exitflag,info]=simulannealbnd(@(params) cost_fun(params,s0,C,K,T,...
11     A,B,r),init_param,lower,upper,options);
12
13 end
```

FIGURE 5.4: Calibration of the Bates model using Simulated annealing.

beforehand is caused by the nature of our optimisation problem (5.3) together with non-linearity of considered models. The optimisation problem is not convex and has many local minima. To improve calibration by a local search method, we also reset the weights: $w_i = 1$, for all i. This helps us enhance non-weighted error measures, provided we obtained a good starting point from the weighted and globally optimised criteria.

Chapter 6

Numerical experiments

In this chapter we comment on several issues that are related to the process of model implementation. As previously, we developed all codes in the MATLAB environment. After data sets are introduced, we compare results of the standard stochastic volatility models and the FSV model.

6.1 On implementation of the pricing models

As we have already established in previous chapters, to obtain the price of a European call contract under one of the introduced models, it is sufficient to compute inverse Fourier transform (3.20) with appropriate characteristic functions and input P_1, P_2 into (3.17). The transform integral, however, needs to be numerically computed. For this purpose, we employ a MATLAB function *integral()* that uses an adaptive Simpson's rule. As shown by Kahl in [39] and Kilin in [38], the integrand of the Heston model decays exponentially and when considering alternative form of characteristic functions (see Appendix A.1) the integrand is continuous in τ. In case of the FSV model, we have to set the approximation factor ε beforehand. Since the derivation of the standard stochastic volatility dynamics poses no restriction on how small ε should be, we tried several choices, namely $\varepsilon_1 = 1/100$, $\varepsilon_2 = 1/1000$ and a higher order one $\varepsilon_3 = 10^{-10}$. In fact, we were able to calibrate the model in a similar way for all three choices and only ε_3 might effect stability of the price computation for several parameter sets [34]. All presented result are computed for a safe choice ε_1. Integrands with respect to the FSV model have not yet been analysed, but for all obtained sets of parameters (in both real and synthetic calibrations) we did not encounter any integration problem for a wide range of strikes and maturities.

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[34]We divide by ε in $C(\tau, \phi)$.
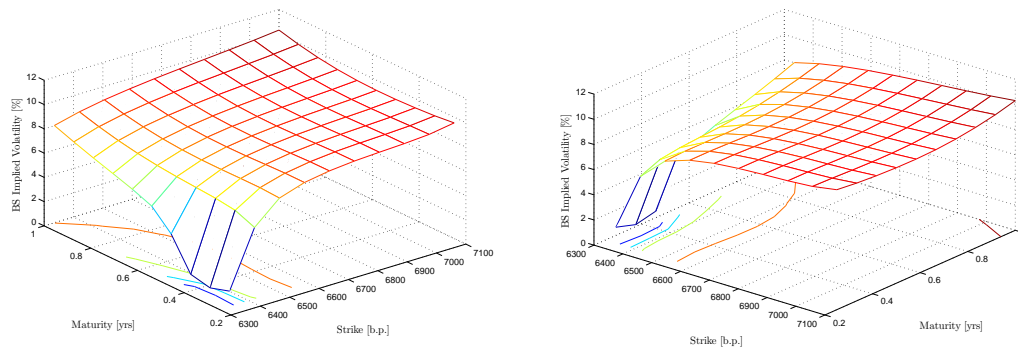
FIGURE 6.1: Option prices in terms of implied volatilities for the synthetic data test.

6.2   Data sets

As the calibration data, we used both market and artificially generated option prices. Using Bloomberg's Option Monitor, we obtained 82 contracts traded on 8th January 2014. The strikes, maturities and ask, bid, mid prices are stored in the MATLAB MAT-file *FTSE_EUc8-1-2014.mat* which is included in the thesis attachment.

For a synthetic data generation, we used the Bates model with the following set of parameters:

v0	kappa	vmean	xi	rho	lambda	alphaJ	gammaJ
0.0060	4.6772	0.0387	0.1214	0.9900	0.6622	-0.3871	0.4062

The parameters were set to create the implied volatility surface (Figure 6.1) that is consistent with the known market data, only for a more dense and regular set of options. The data includes 100 options and it is used for testing different approaches to calibrate each model. In the next section we will see if we are able to calibrate the Bates model perfectly [35], i.e. if we obtain the set of previously displayed parameters by any optimisation routine.

---

[35] The data was created using the Bates model with a known set of parameters, hence we have at our disposal a global minimum of the calibration problem with respect to the model. In the tests, however, we will act as if we did not know the right values of model parameters.

6.3 Measured errors

We evaluate these calibration errors:

$$AAE(\Theta) = \frac{1}{N} \sum_{i=1}^{N} \left| C_i - C^{model}(\Theta) \right|; \qquad (6.1)$$

$$AARE(\Theta) = \frac{1}{N} \sum_{i=1}^{N} \frac{\left| C_i - C^{model}(\Theta) \right|}{C_i}; \qquad (6.2)$$

$$MAE(\Theta) = \max_{i=1,2,\ldots,N} \left| C_i - C^{model}(\Theta) \right|. \qquad (6.3)$$

Both the synthetic and real market data are of varying price levels (i.e. they consist of ITM, ATM and OTM options) - the FTSE call option premia in our data sample range from £17.5 to £514.5. Hence, as the most interesting error measure we consider $AARE(\Theta)$ which reflects the average with respect to the absolute values of relative errors. We also might want to fit the calibrated surface with errors that are lower than a pre-set level for any option. The minimal level that will do for each calibration trial is represented by $MAE(\Theta)$.

6.4 Calibration from synthetic data

Firstly, we recall the calibration process - we have at our disposal two (heuristic) global optimisers and we combine each with a local search method (denoted by LSQ) [36]. Furthermore, we perform these trials for different calibration weights; $w^{(A)}$ defined by (5.5), $w^{(B)}$ as in (5.6) and respectively $w^{(C)}$ as in (5.7).

The calibration results on synthetic data are presented in Table 6.1. According to the most of our numerical trials, it is worth using local search methods on top of the global optimisation routines. When combining the Genetic algorithm (GA) alongside LSQ, we were able to "perfectly" fit the surface by the Bates model for weights $w^{(B)}$ and $w^{(C)}$. Of course, the quotation marks are in place here, due to the numerical evaluation of the prices, the residual sums of (unweighted) squares are of the order 10^{-7} (for both weights). The Genetic algorithm, in several experiments, provided us with inferior results compared to the other approaches, but the set of likewise obtained parameters was either close to the global minimum or close to a very suitable local minimum. On that note, we comment that combination of GA and C^{model} proved to be very useful for

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[36]This is done using MATLAB's *lsqnonlin()* routine that employs either *Trust-Region-Reflective* or *Levenberg-Marquardt* method with initial parameter $\lambda_{LM} = 0.01$. For more details see http://www.mathworks.co.uk/help/optim/ug/lsqnonlin.html .

TABLE 6.1: Calibration results for synthetic data. Best values of error measures for each weight are typed in bold.

weights	Model	Error meas.	GA	GA+LSQ	SA	SA+LSQ
$w^{(A)}$	Bates model	AARE [%]	4.36	1.58	3.59	3.28
		AAE [£]	7.53	1.81	6.34	5.77
		MAE [£]	32.92	8.09	20.10	18.04
	Heston model	AARE [%]	2.95	2.99	4.74	4.76
		AAE [£]	5.00	4.98	7.79	7.75
		MAE [£]	14.87	14.05	24.18	24.39
	FSV model	AARE [%]	4.32	1.72	1.99	**0.93**
		AAE [£]	6.57	2.00	2.89	**1.35**
		MAE [£]	21.72	9.69	6.85	**4.94**
$w^{(B)}$	Bates model	AARE [%]	4.68	$\mathbf{3.2 \times 10^{-5}}$	3.51	2.99
		AAE [£]	5.70	$\mathbf{3.5 \times 10^{-5}}$	6.46	4.98
		MAE [£]	17.90	$\mathbf{1.8 \times 10^{-4}}$	21.43	14.05
	Heston model	AARE [%]	3.51	2.99	3.42	3.36
		AAE [£]	5.77	4.98	6.37	6.36
		MAE [£]	17.48	14.04	23.91	24.06
	FSV model	AARE [%]	4.51	1.58	6.95	6.95
		AAE [£]	7.12	1.81	11.56	11.56
		MAE [£]	23.49	8.13	34.22	34.22
$w^{(C)}$	Bates model	AARE [%]	6.25	$\mathbf{4.4 \times 10^{-5}}$	3.19	1.12
		AAE [£]	6.44	$\mathbf{4.9 \times 10^{-5}}$	6.18	1.68
		MAE [£]	26.19	$\mathbf{2.5 \times 10^{-4}}$	22.43	6.76
	Heston model	AARE [%]	3.12	3.41	3.10	2.99
		AAE [£]	5.83	5.27	5.86	4.98
		MAE [£]	18.96	15.42	20.94	14.04
	FSV model	AARE [%]	8.54	1.58	11.46	2.65
		AAE [£]	13.49	1.81	10.67	4.04
		MAE [£]	47.81	8.10	34.09	12.14

the synthetic data tests for all models.

We also managed to calibrate the FSV model from the data, in most of the trials, with slightly lower error measures than in case of the Heston model. For $w^{(A)}$ we obtained the best fit (in terms of all measured errors) for the FSV model [37], which we did not expect considering the origin of the synthetic data. The choice of a specific calibration procedure is a complex problem and might depend on which data we would like to mimic in a better way. To observe absolute relative errors for different options, see Figures 6.2 - 6.4.

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[37] Using the Simulated annealing alongside LSQ.
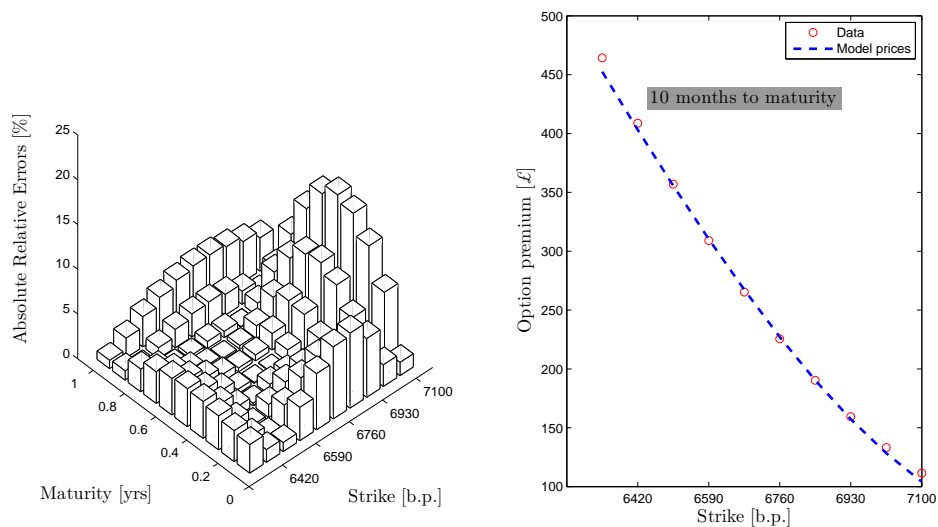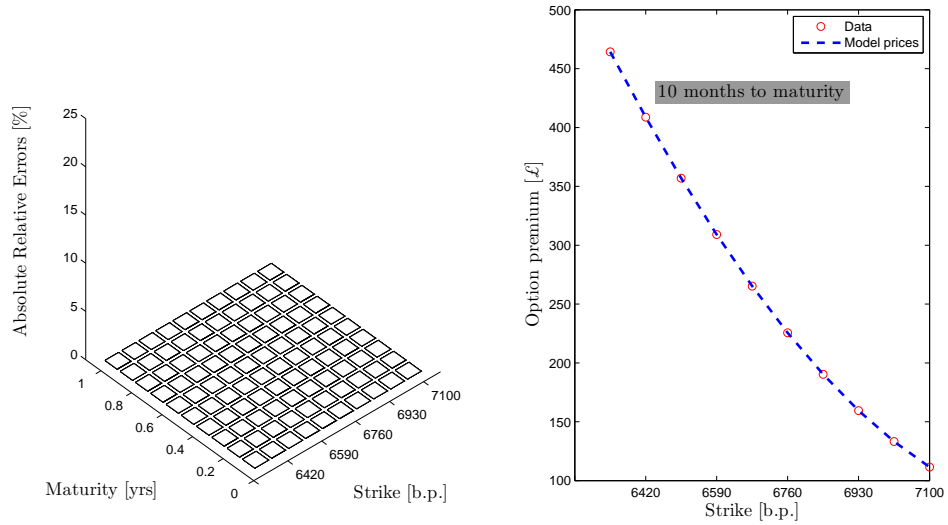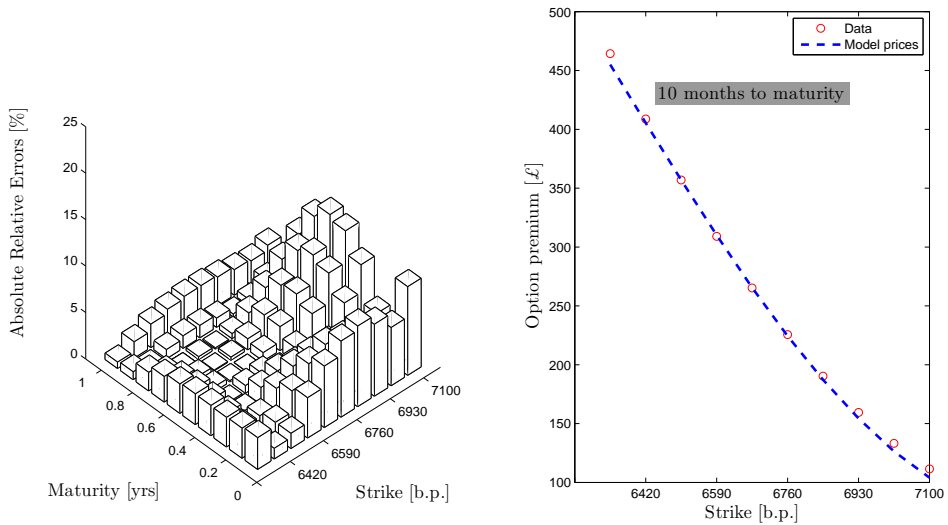
(A) Bates model
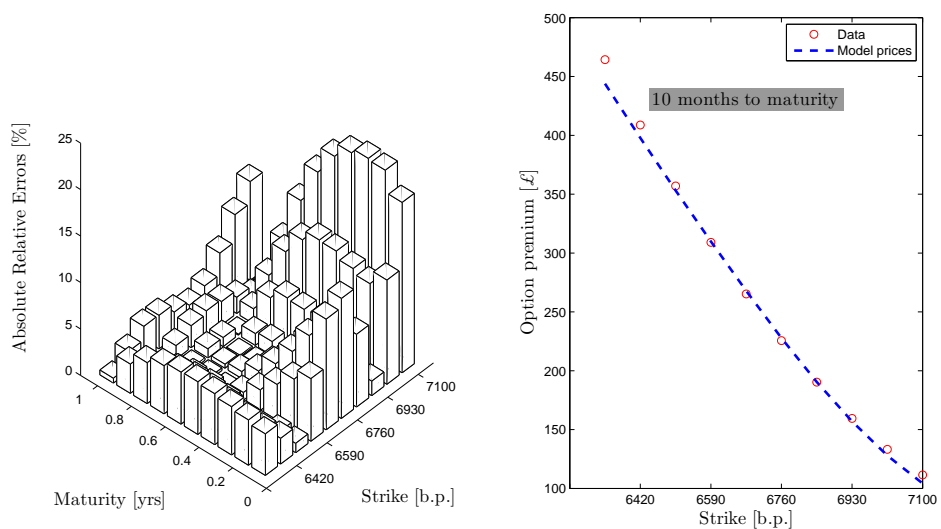
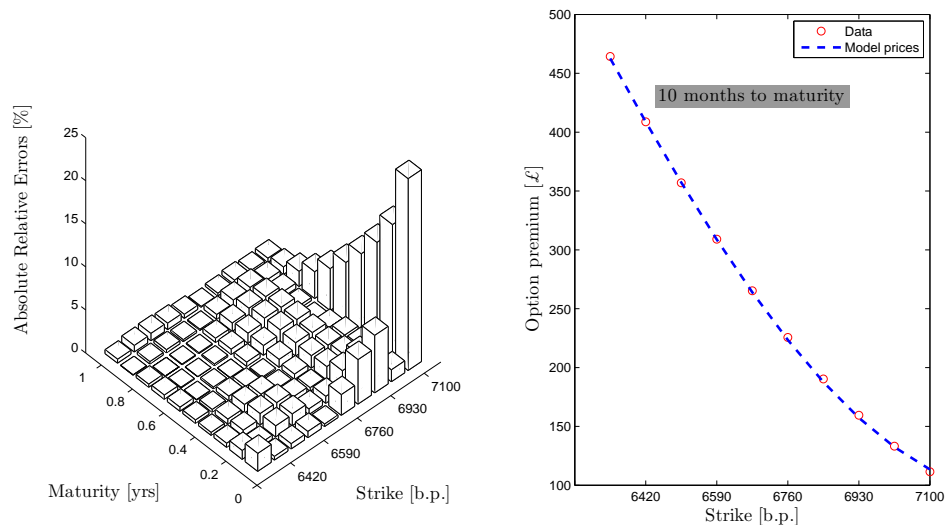
(B) Heston model
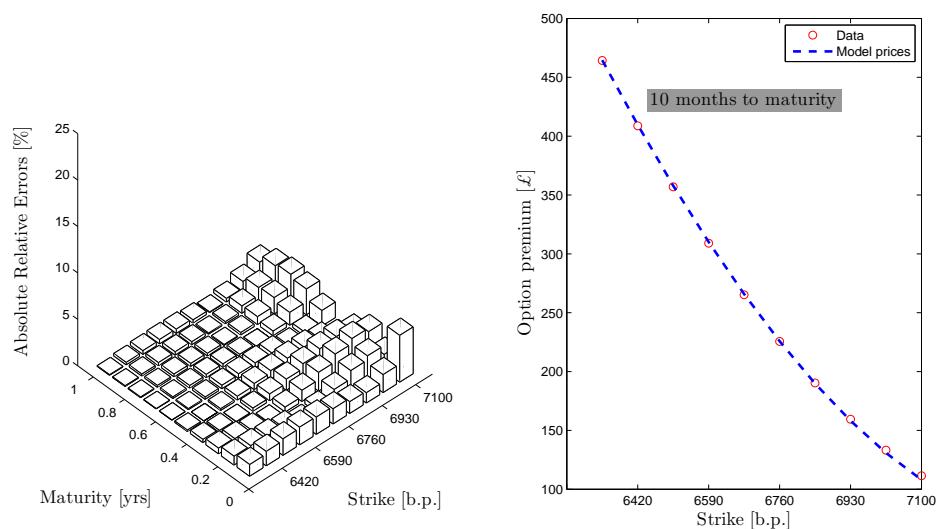
(C) FSV model

FIGURE 6.2: Calibration from synthetic data using SA + LSQ and weights $w_i^{(A)}$.
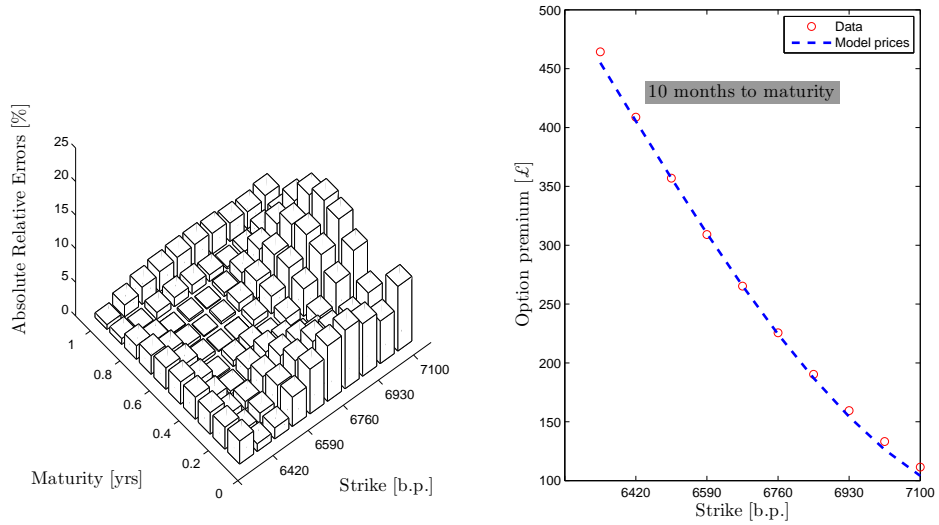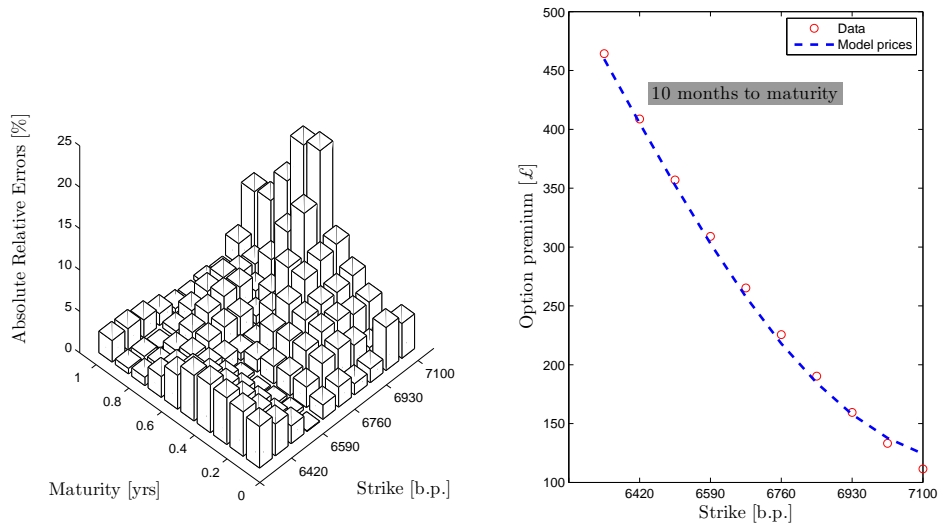
(A) Bates model

(B) Heston model
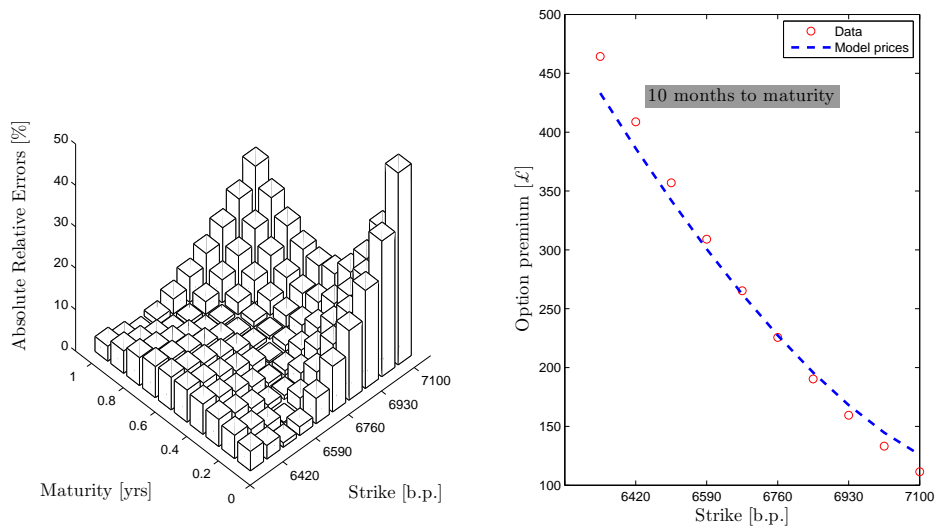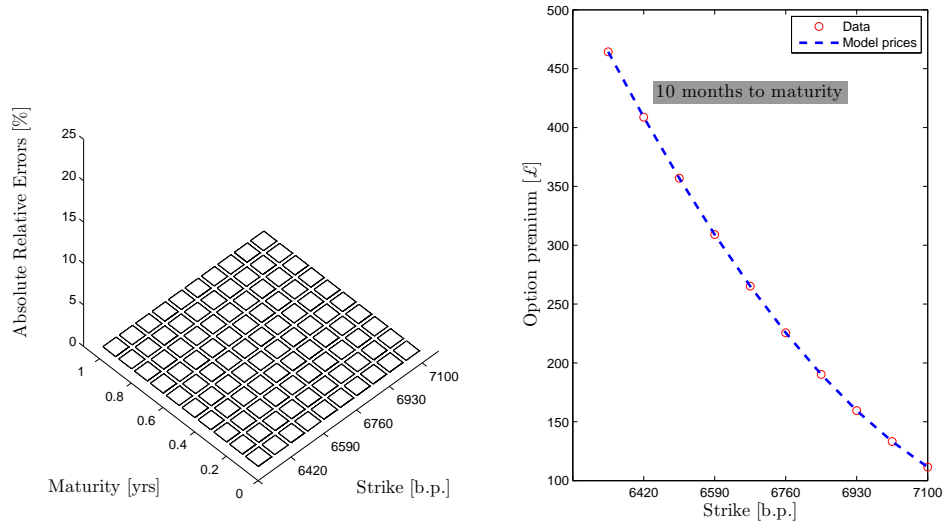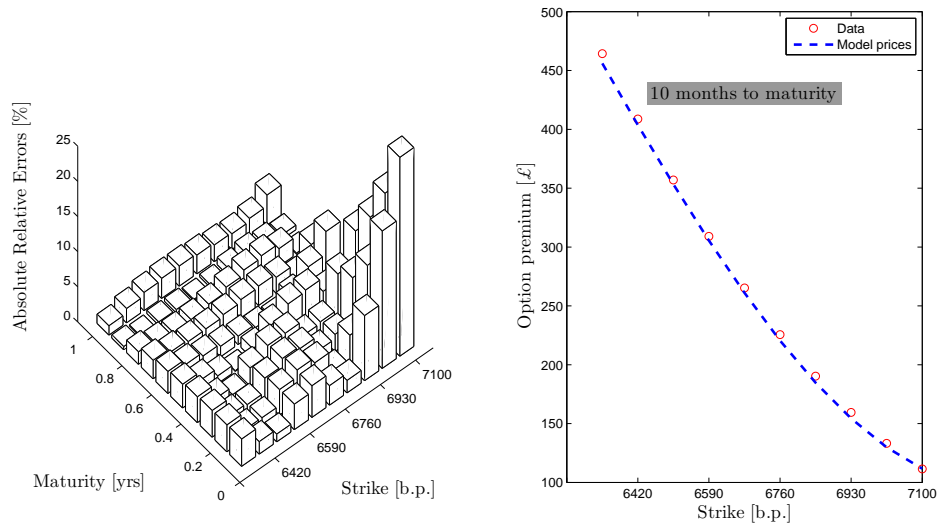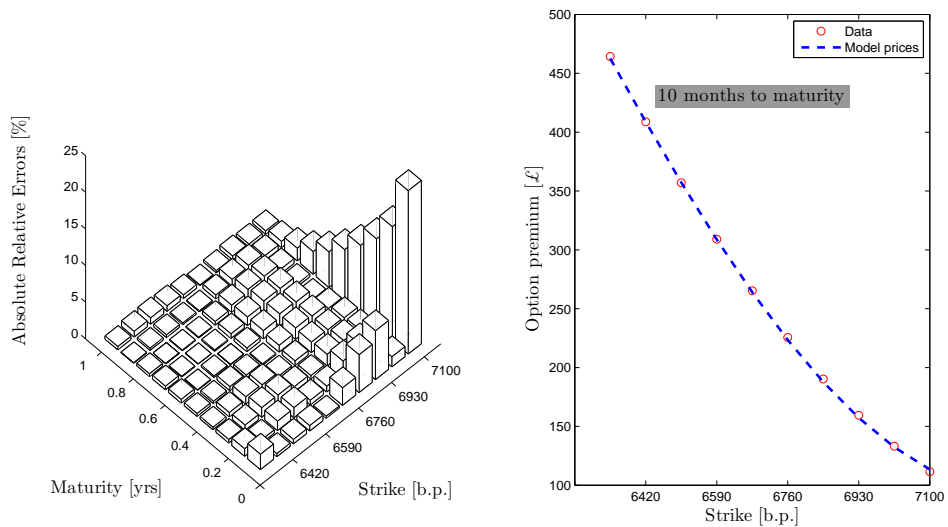

(C) FSV model

FIGURE 6.3: Calibration from synthetic data using GA + LSQ and weights $w_i^{(B)}$.

(A) Bates model

(B) Heston model

(C) FSV model

FIGURE 6.4: Calibration from synthetic data using SA + LSQ and weights $w_i^{(B)}$.

(A) Bates model

(B) Heston model

(C) FSV model

FIGURE 6.5: Calibration from synthetic data using GA + LSQ and weights $w_i^{(C)}$.

6.5 Calibration from the real market data

To calibrate models from the FTSE 100 option market, we utilise weights $w^{(B)}$ only. The best calibration result, with respect to all measures, was obtained from the Bates model and SA+LSQ calibration routine. The following parameters were found:

TABLE 6.2: Calibrated parameters of the Bates model. Calibrated using SA + LSQ.

v0	kappa	vmean	xi	rho	lambda	alphaJ	gammaJ
0.0067	4.4171	0.0355	0.0753	0.9899	0.6181	-0.3881	0.3715

The second best result is due to the newly proposed FSV model. The average absolute relative errors for both GA+LSQ and SA+LSQ routines read still very acceptable 2.34%.

TABLE 6.3: Calibrated parameters of the FSV model. Calibrated using GA + LSQ (the first line), SA + LSQ respectively.

v0	kappa	vmean	xi	rho	lambda	alphaJ	gammaJ	H
0.0091	0.8570	0.1546	-1.7539	0.9900	0.2190	1.3661	0.0000	0.5935
0.0091	0.8571	0.1546	-2.4423	0.9900	0.2190	1.3661	0.0000	0.6654

Both sets of parameters are almost equally good in terms of all measured calibration errors. The second set, however, includes higher value of H and thus the model calibrated with these parameters should attain stronger correlation in the variance process.

We did not manage to fit the market data with Heston model such that we would observe better error measures. This follows our intuition; the Heston model is the simplest out of the three, having the least degrees of freedom to fit the data. On the other hand, the calibration process took the least amount of time compared with other models for all calibration routines. Relative errors of the models are displayed in Figures 6.7 - 6.10.

TABLE 6.4: Calibrated parameters of the Heston model. Calibrated using the Genetic algorithm.

v0	kappa	vmean	xi	rho
0.0074	0.0463	0.0387	0.0718	0.9899

TABLE 6.5: Calibration from the real market data.

Model	Error meas.	GA	GA+LSQ	SA	SA+LSQ
Bates model	AARE [%]	2.04	2.34	1.81	1.51
	AAE [£]	3.94	3.27	2.68	2.44
	MAE [£]	23.36	17.13	11.01	11.70
Heston model	AARE [%]	3.10	3.35	3.78	3.52
	AAE [£]	6.05	5.85	6.68	5.90
	MAE [£]	30.84	30.69	31.09	30.68
FSV model	AARE [%]	4.61	2.34	3.01	2.34
	AAE [£]	7.57	3.27	5.04	3.27
	MAE [£]	35.74	17.13	25.84	17.13

6.6   Option price & the Hurst exponent

Inspired by having a very similar market fit for two different values of the Hurst exponent, we illustrate how value of H effects a modelled European call price.

In Figure 6.6 we compared modelled prices of ITM, nearly ATM and OTM options with the real market prices. For the illustration, parameters obtained by GA+LSQ (except for H) are used. When employing the second set of well calibrated parameters and also pricing different market options, we observed that the value of H, needed to precisely fit a given market option, is higher for the In-The-Money contracts than for ATM and OTM options. However, this is just an observation which has to be more thoroughly analysed. The analysis does not fit into the scope of this thesis.
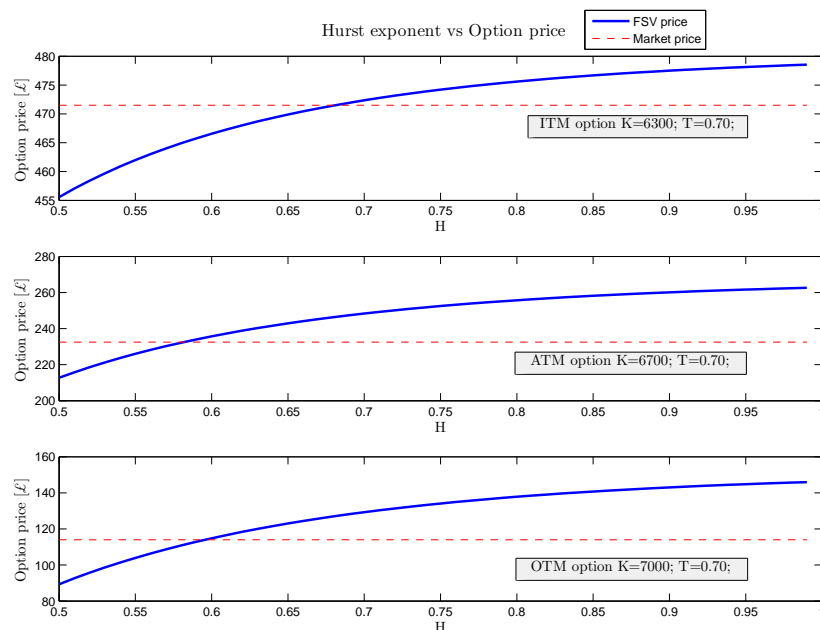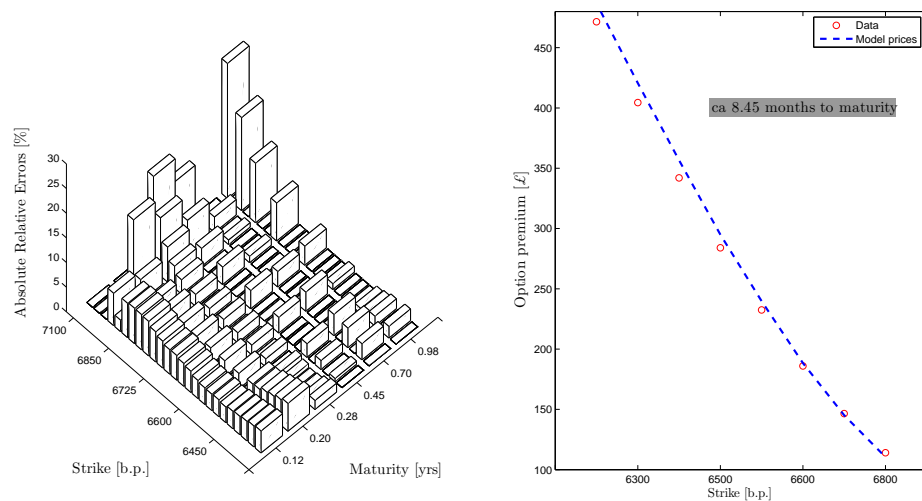

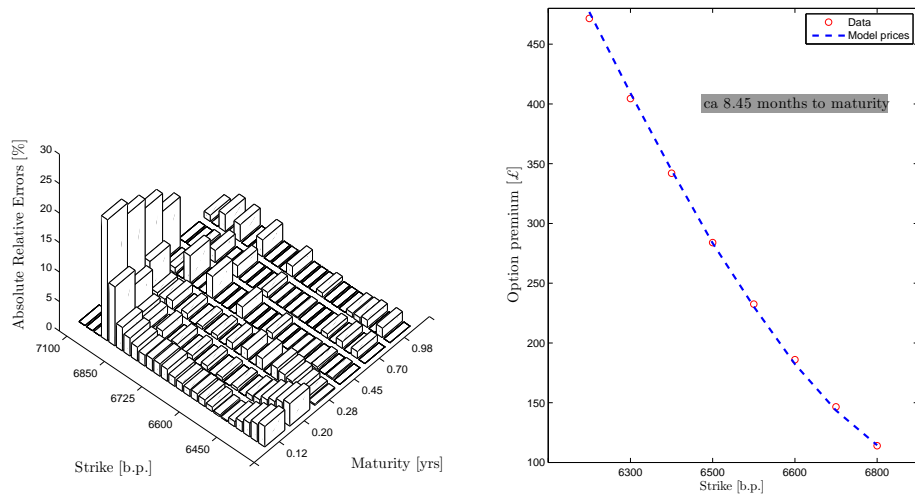FIGURE 6.6: FSV model option prices for different values of H.
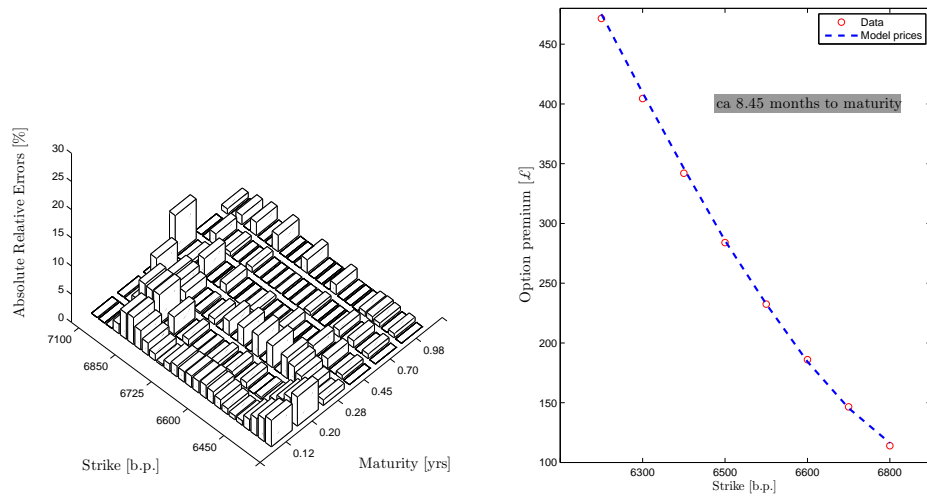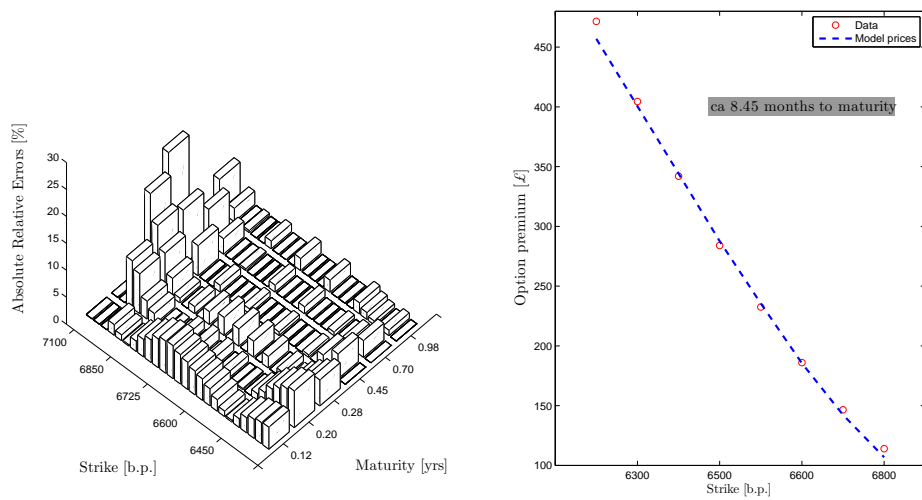
(A) Bates model
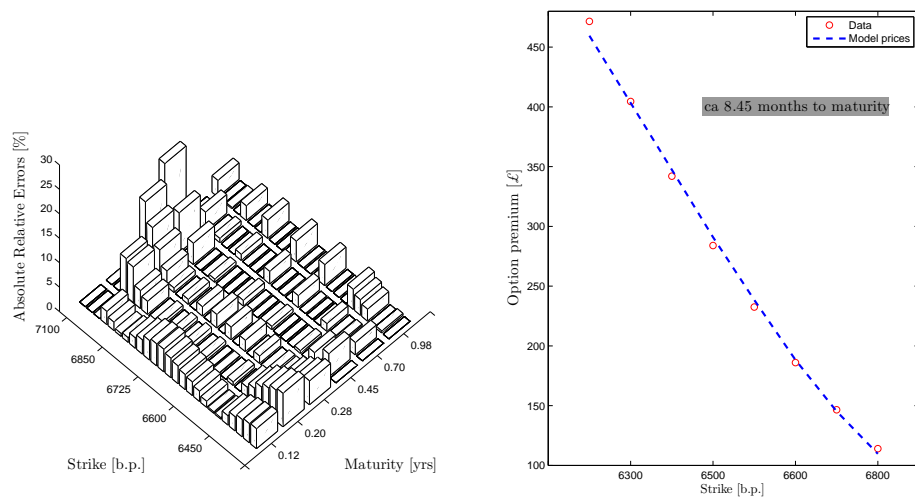
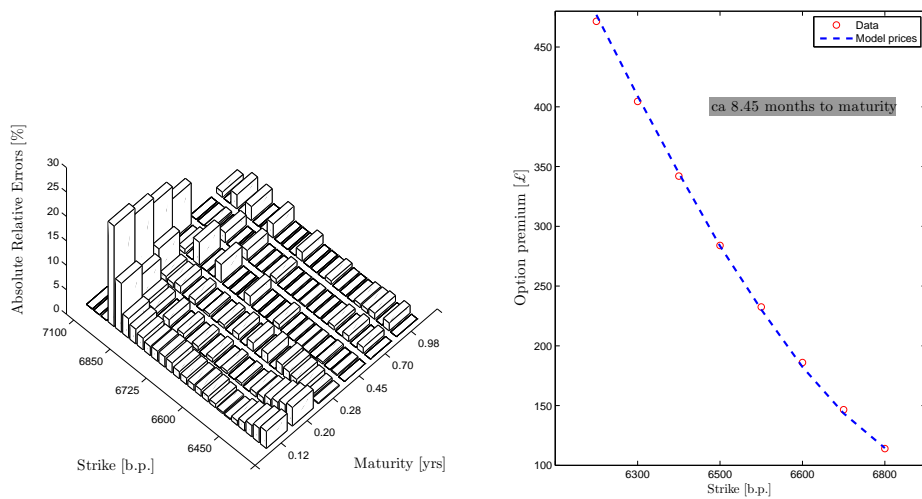
(B) Heston model

(C) FSV model

FIGURE 6.7: Calibration from FTSE 100 call option market using GA - 8th January 2014 [*Data source: Bloomberg terminal*].
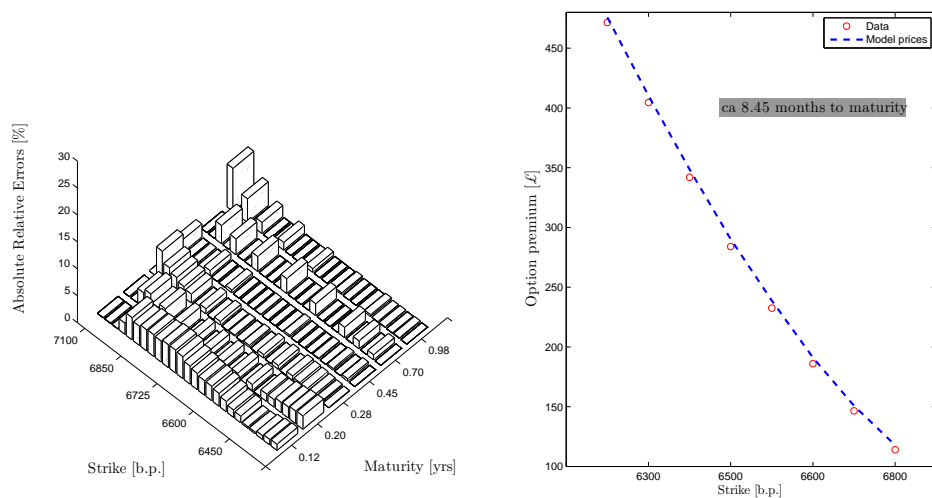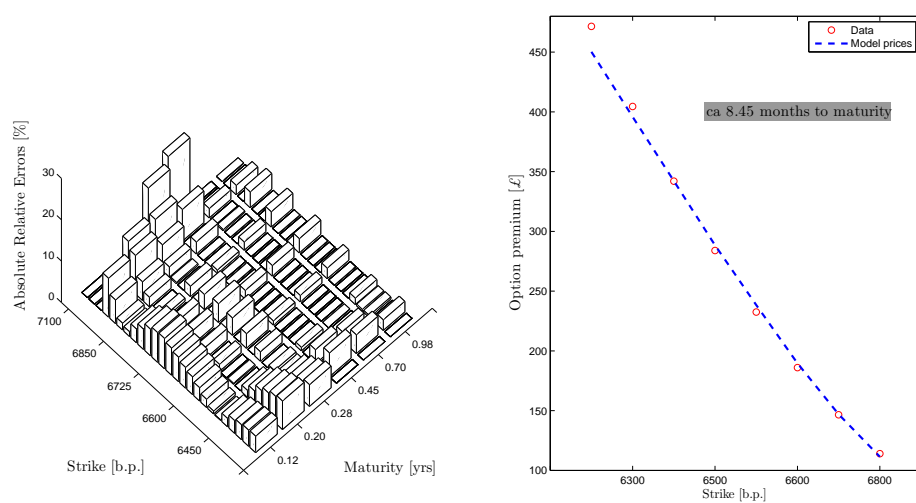
(A) Bates model

(B) Heston model

(C) FSV model

FIGURE 6.8: Calibration from FTSE 100 call option market using GA + LSQ - 8th January 2014 [*Data source: Bloomberg terminal*].

(A) Bates model

(B) Heston model

(C) FSV model

FIGURE 6.9: Calibration from FTSE 100 call option market using SA - 8th January 2014 [*Data source: Bloomberg terminal*].

(A) Bates model
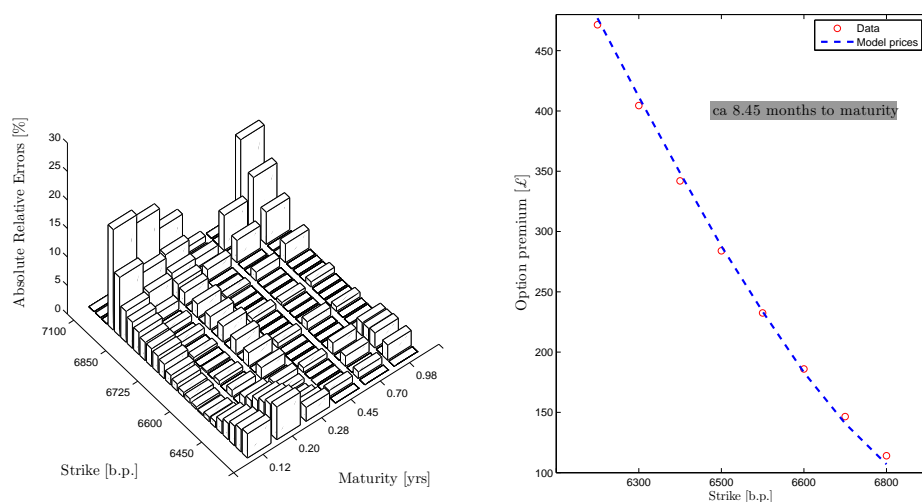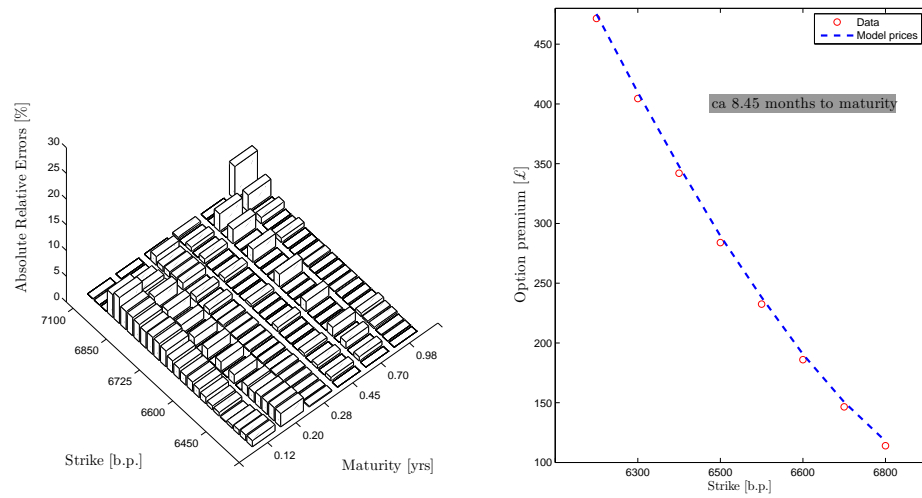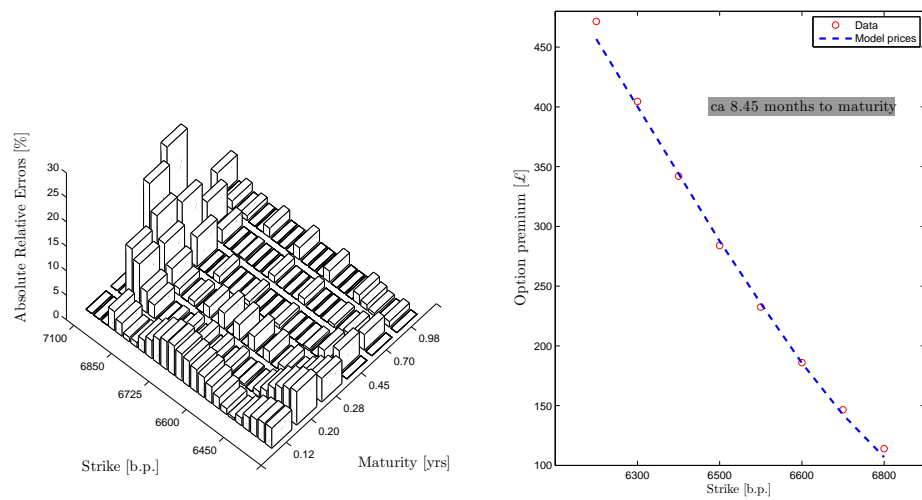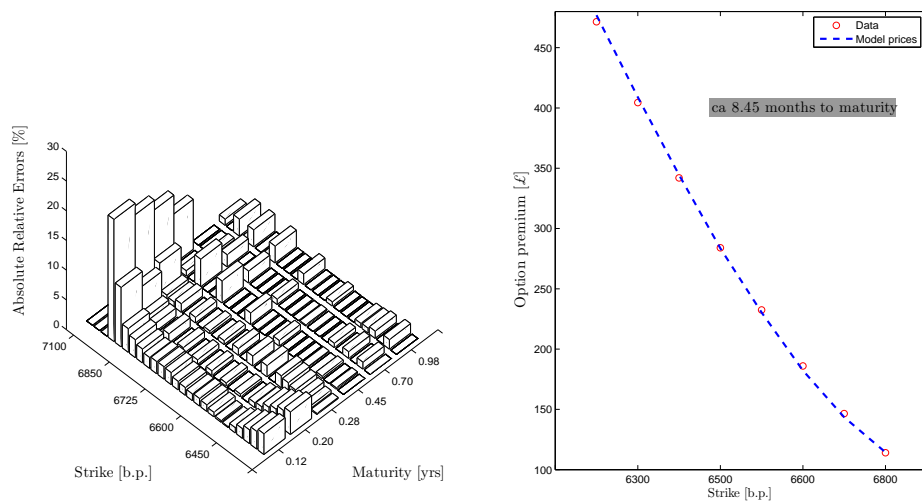

(B) Heston model

(C) FSV model

FIGURE 6.10: Calibration from FTSE 100 call option market using SA + LSQ - 8th January 2014 [*Data source: Bloomberg terminal*].

Chapter 7

Conclusion

After we presented an up-to-date overview of standard and fractional stochastic volatility models (Chapter 2), we showed several arguments on so-called approximative fractional approach. Using this approach, we implemented the FSV model, firstly proposed by Intarasit and Sattayatham in 2011 [35]. Theoretical part of the implementation involved derivation of an explicit formula for pricing European call options (Chapter 3). As a motivation for the FSV model, we presented an empirical evidence that realized volatility of financial returns might be long-range dependent.

Due to the existence of a semi-closed form option pricing solution, the FSV model should be tractable for market calibration. This hypothesis has not been verified in any previous work that is known to us[38]. To asses the issue, we employed both synthetic data and European call options on *FTSE 100* index. We also included a comparison with popular standard stochastic volatility models, namely the Heston and Bates model.

The calibration process can be understood as an optimisation problem of minimising model discrepancies. The problem, however, is a rather complex one and hence a suitable choice of utility functions and optimisation procedures is vital for obtaining a good market fit. For the synthetic data tests (option prices simulated using the Bates model), we compared different approaches of model calibration. We suggest, backed up by the synthetic data results (Chapter 6), that a combination of global and local optimisation routines can provide reasonable calibration errors.

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[38]This is to our best knowledge on 22nd May 2014.

The best results of each model were compared in the second part of Chapter 6. Market fit obtained by the Bates model was superior to the others ($AARE(\Theta) = 1.51\%$), closely followed by the results from the newly proposed FSV model ($AARE(\Theta) = 2.34\%$). The simplest, but widely popular, Heston's approach was able to fit market option prices with the average absolute relative error reading 3.10%.

Hence we conclude that the FSV model is suitable for calibration tasks, however, before it can be used in practise, it should be more thoroughly analysed. At the moment, it lacks a rigours analysis of the hedging performance, economic interpretation of calibrated parameters and analysis of the Hurst exponent effect with respect to option prices and hedging ratios. The optimal choice of ε was left for further research, we only discussed that for several values of ε, the model can fit the market data in the presented fashion.

Appendix A

Characteristic functions

A.1 Characteristic functions of the Heston model

$$f_j^{(Heston)}(\tau, \phi) = \exp\left\{C_j(\tau, \phi) + D_j(\tau, \phi)v_0 + i\phi x\right\},$$

where for $j = 1, 2$:

$$C_j(\tau, \phi) = r\phi i\tau + \frac{\theta}{\xi^2}\left[(b_j - \rho\xi\phi i + d_j)\tau - 2\ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right],$$

$$D_j(\tau, \phi) = \frac{b_j - \rho\xi\phi i + d_j}{\xi^2}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right),$$

$$g_j = \frac{b_j - \rho\xi\phi i + d_j}{b_j - \rho\xi\phi i - d_j},$$

$$d_j = \sqrt{(\rho\xi\phi i - b_j)^2 - \xi^2(2u_j\phi i - \phi^2)},$$

$$u_1 = 1/2, \quad u_2 = -1/2, \quad \theta = \kappa\bar{v}, \quad b_1 = \kappa - \rho\xi, \quad b_2 = \kappa.$$

Notation as in Chapter 2 and Chapter 3.

A.2 Characteristic functions of the Bates model

$$f_j^{(Bates)}(\tau, \phi) = \exp\left\{C_j(\tau, \phi) + D_j(\tau, \phi)v_0 + i\phi x + \psi(\phi)\tau\right\},$$

where for $j = 1, 2$:

$$C_j(\tau, \phi) = r\phi i\tau + \frac{\theta}{\xi^2}\left[(b_j - \rho\xi\phi i + d_j)\tau - 2\ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right],$$

$$D_j(\tau, \phi) = \frac{b_j - \rho\xi\phi i + d_j}{\xi^2}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right),$$

$$\psi = -\lambda i\phi\left(e^{\alpha + \gamma^2/2} - 1\right) + \lambda\left(e^{i\phi\alpha - \phi^2\gamma^2/2} - 1\right)$$

$$g_j = \frac{b_j - \rho\xi\phi i + d_j}{b_j - \rho\xi\phi i - d_j},$$

$$d_j = \sqrt{(\rho\xi\phi i - b_j)^2 - \xi^2(2u_j\phi i - \phi^2)},$$

$$u_1 = 1/2, \quad u_2 = -1/2, \quad \theta = \kappa\bar{v}, \quad b_1 = \kappa - \rho\xi, \quad b_2 = \kappa.$$

Notation as in Chapter 2 and Chapter 3 and λ is the intensity of a compensated, compound Poisson process.

A.3 Characteristic functions of the FSV model

$$f_j^{(FSV)}(\tau, \phi) = \exp\left\{C_j(\tau, \phi) + D_j(\tau, \phi)v_0 + i\phi x + \psi(\phi)\tau\right\},$$

where for $j = 1, 2$ and $\tau = T - t$:

$$C_j(\tau, \phi) = r\phi i\tau + \theta Y_j\tau - \frac{2\theta}{\beta^2}\ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right),$$

$$D_j(\tau, \phi) = Y_j\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right),$$

$$\psi = -\lambda i\phi\left(e^{\alpha + \gamma^2/2} - 1\right) + \lambda\left(e^{i\phi\alpha - \phi^2\gamma^2/2} - 1\right)$$

$$Y_j = \frac{b_j - \rho\beta\phi i + d_j}{\beta^2}$$

$$g_j = \frac{b_j - \rho\beta\phi i + d_j}{b_j - \rho\beta\phi i - d_j},$$

$$d_j = \sqrt{(\rho\beta\phi i - b_j)^2 - \beta^2(2u_j\phi i - \phi^2)},$$

$$\beta = \xi\varepsilon^{H-1/2}\sqrt{v_t}$$

$$u_1 = 1/2, \quad u_2 = -1/2, \quad \theta = \kappa\bar{v}, \quad b_1 = \kappa - (H - 1/2)\xi\varphi_t^\varepsilon - \rho\beta, \quad b_2 = \kappa - (H - 1/2)\xi\varphi_t^\varepsilon.$$

Notation as in Chapter 2 and Chapter 3 and λ is the intensity of a compensated, compound Poisson process. Integral φ_t^ε is defined in Chapter 2.

Appendix B

The Black-Scholes formula

A European call option price $C(K, T)$ under the classical Black-Scholes model (BSM), which assume a geometrical Brownian motion, is equal to [63]:

$$C(K, T) = S_t N(d_1) - K e^{-r(T-t)} N(d_2), \qquad (B.1)$$

where $N(x)$ is a cumulative normal distribution function and d_1, d_2 are defined as

$$d_1 = \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = \frac{\ln(S_t/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

where σ is the volatility parameter of the model. The notation as in Chapter 5. Since we are using the formula to obtained implied volatility surface and we link the reader to the original paper [5] and or to [63] for a complete definition of the model.

Appendix C

Thesis attachment

On the attached CD we created a root file structure:

- Calibration/

- Data/

- LRD_Estimation/

- readme.txt

- Sobotka_DP_2014.pdf

The file Sobotka_DP_2014.pdf is this thesis. In the folders *Calibration/* and *LRD_Estimation/* we included all *.m* and *.mat* files necessary for the market calibration and the long-range dependency estimation respectively. A documentation of attached MATLAB functions can be accessed by typing `help function_name.m` in the MATLAB console. In the folder *Data* we stored the data sets (in *.mat* format) used throughout the thesis.

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