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Various methods of numerical estimation of generalized stress intensity factors of bi-material notches

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Abstract

The study of bi-material notches becomes a topical problem as they can model efficiently geometrical or material discontinuities. When assessing crack initiation conditions in the bi-material notches, the generalized stress intensity factors H have to be calculated. Contrary to the determination of the K-factor for a crack in an isotropic homogeneous medium, for the ascertainment of the H-factor there is no procedure incorporated in the calculation systems. The calculation of these fracture parameters requires experience. Direct methods of estimation of H-factors need choosing usually length parameter entering into calculation. On the other hand the method combining the application of the reciprocal theorem (Ψ-integral) and FEM does not require entering any length parameter and is capable to extract the near-tip information directly from the far-field deformation.

Keywords: generalized stress intensity factor, fracture mechanics, bi-material notch, general singular stress concentrator

1. Introduction

The study of bi-material notches becomes a topical problem as they can model efficiently geometrical or material discontinuities. When assessing conditions for a crack initiation in the bi-material notches, the generalized stress intensity factors H are necessary to be calculated. In contrast to the determination of the K factor for a crack in an isotropic homogeneous medium, for the ascertainment of a generalized stress intensity factor (GSIF) there is no procedure incorporated in the calculation systems. The calculation of these fracture mechanics parameters is not trivial and requires certain experience. Nevertheless the accuracy of the H-factors calculation directly influences the reliability of assessment of the stress concentrators. Direct methods of estimation of H factors require choosing usually length parameter entering into calculation. On the other hand the method combining the application of the reciprocal theorem (Ψ-integral) and FEM does not require entering any length parameter and is capable to extract the near-tip information directly from the far-field deformation where the numerical fields are more accurate. The latter method can be readily applied to bi-materials composed of orthotropic materials components. In the paper various methods of calculation of the GSIFs are presented, tested and mutually compared. Recommendations for reliable evaluation of critical conditions of the bi-material notches are suggested.

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2. Bi-material notch

The model of a bi-material notch is suitable to simulate a number of construction points from which a failure is initiated. In the case of layered or fibre composite locations where the layers or fibres touch the surface of the composite body the singular stress concentrations occur. The singular stress distribution is derived on the basis of Airy stress functions in the form of Williams' expansion [10]. In most of the geometrical and material configurations of a bi-material notch there are two terms of the expansion with the real stress singularity exponents p_1 and p_2 in the interval $(0;1)$. Contrary to a crack in a homogeneous material, the exponents differ from 1/2 and, furthermore, each singular term includes both normal and shear mode of loading, see [4] for detail. Then the singular stress components can be written in the polar coordinates:

$$\sigma_{mij} = \frac{H_1}{\sqrt{2\pi}} r^{-p_1} F_{ij1m} + \frac{H_2}{\sqrt{2\pi}} r^{-p_2} F_{ij2m}\,, \tag{1}$$

where for $\{i,j\} = \{r,\theta\}$ and for each singular term $k = 1, 2$:

$$\begin{aligned}
F_{rrkm} &= (2-p_k)(-a_{mk}\sin((2-p_k)\theta) - b_{mk}\cos((2-p_k)\theta) + \\
&\quad +3c_{mk}\sin(-p_k\theta) + 3d_{mk}\cos(-p_k\theta)) \\
F_{\theta\theta km} &= (p_k^2 - 3p_k + 2)(a_{mk}\sin((2-p_k)\theta) + b_{mk}\cos((2-p_k)\theta) + \\
&\quad +c_{mk}\sin(-p_k\theta) + d_{mk}\cos(-p_k\theta)) \\
F_{r\theta km} &= (2-p_k)(-a_{mk}\cos((2-p_k)\theta) + b_{mk}\sin((2-p_k)\theta) + \\
&\quad +c_{mk}\cos(-p_k\theta) - d_{mk}\sin(-p_k\theta))
\end{aligned}$$

The subscript m differentiates the materials 1 and 2 where the stresses are determined. The values H_k are the generalized stress intensity factors that follow from the numerical solution of the studied geometry with given materials and boundary conditions [4, 3, 7]. The numerical calculation of the values H_k is necessary step for the final determination of the stress distribution. Numerical approaches to calculation of GSIFs have varying level of difficulty and accuracy. In the following paragraphs the direct methods and the method combining the application of the reciprocal theorem and FEM are described.

3. Direct methods of the generalized stress intensity factors H_k determination

Direct methods compare the results of some appropriate magnitude from a numerical solution with its analytical representation.

3.1. Tangential stress

The tangential stress $\sigma_{\theta\theta}$ is used here as the appropriate magnitude for the comparison. If the stress distribution is described by a combination of H_1 and H_2, it is necessary to solve the system of two equations. To achieve this, the values of $\sigma_{\theta\theta}$ following from the finite element method are determined for two different directions θ_1, θ_2. Then knowing the analytical relations e.g. for $\sigma_{\theta\theta}$ (1) we solve the system of equations for H_1 and H_2:

$$\begin{bmatrix} r^{-p_1}F_{\theta\theta 1m}(\theta=\theta_1) & r^{-p_2}F_{\theta\theta 2m}(\theta=\theta_1) \\ r^{-p_1}F_{\theta\theta 1m}(\theta=\theta_2) & r^{-p_2}F_{\theta\theta 2m}(\theta=\theta_2) \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \begin{bmatrix} \sigma_{m\theta\theta}(r,\theta_1) \\ \sigma_{m\theta\theta}(r,\theta_2) \end{bmatrix}, \tag{2}$$

The valid values of H_1, H_2 are then determined by an extrapolation of the solutions (2) into $r = 0$. The dependence of GSIF H_k on the polar coordinate r and the extrapolation into the notch vertex is shown in fig. 1.

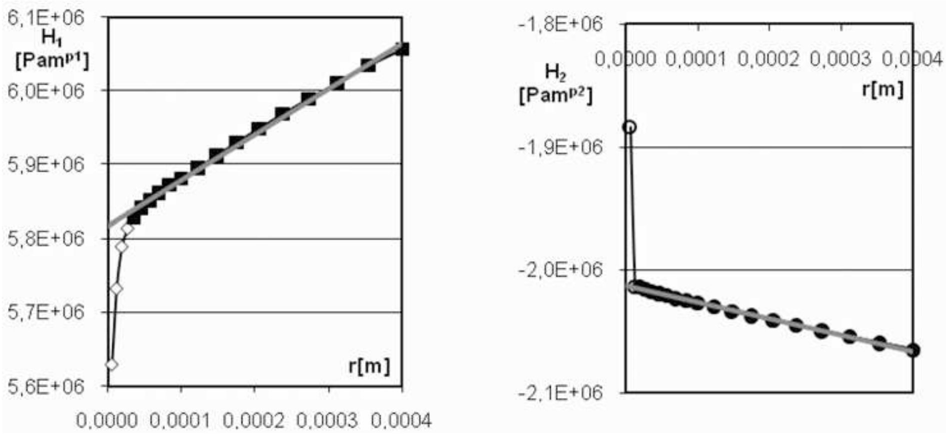

Fig. 1. Extrapolation of H_1 and H_2 values into the notch tip ($r = 0$)

3.2. *Generalized strain energy density factor*

For combined mode of loading it is suitable to use strain energy density (SED) approach to describe crack behaviour and to estimate the GSIFs as well. In the early 70s Sih [9] showed that damaging of a material could be estimated using strain energy density factor S that is defined by following equation:

$$S = r \cdot \mathrm{d}W/\mathrm{d}V = r \int_0^{\varepsilon} \sigma \, \mathrm{d}\varepsilon \,, \tag{3}$$

In the same way the generalized strain energy density factor (SEDF) Σ is defined for the case of a bi-material notch. Limiting only to plane problems we can obtain the relation for the distribution of the SEDF Σ that in contrast to a crack, depends on the radial distance r. Because of the dependency of Σ on r, mean value over a certain distance d will be considered in the following [4]. For material m it is:

$$\begin{aligned} \overline{\Sigma_m} &= \frac{1}{d}\int_0^d \Sigma_m \mathrm{d}r \\ &= \frac{H_1^2}{16 G_m \pi}\left(\frac{d^{1-2p_1}}{2-2p_1}U_{1m} + \frac{d^{1-2p_2}}{2-2p_2}h_{21}^2 U_{2m} + \frac{d^{1-p_1-p_2}}{2-p_1-p_2}2h_{21}U_{12m}\right) , \end{aligned} \tag{4}$$

where

$U_1 = [(F_{rr1m}^2 + F_{\theta\theta 1m}^2)(k_m+1) + 4F_{r\theta 1m}^2 + 2F_{\theta\theta 1m}F_{rr1m}(k_m-1)]$

$U_2 = [(F_{rr2m}^2 + F_{\theta\theta 2m}^2)(k_m+1) + 4F_{r\theta 2m}^2 + 2F_{\theta\theta 2m}F_{rr2m}(k_m-1)]$

$U_{12} = [(F_{rr1m}F_{rr2m} + F_{\theta\theta 1m}F_{\theta\theta 2m})(k_m+1) + 4F_{r\theta 1m}F_{r\theta 2m} +$
$\quad + (F_{\theta\theta 1m}F_{rr2m} + F_{\theta\theta 2m}F_{rr1m})(k_m-1)]$

and $k_m = (1-\nu_m)/(1+\nu_m)$ for plane stress and $k_m = (1-2\nu_m)$ for plane strain, G_m is shear modulus and ν_m is the Poisson ratio. $h_{21} = H_2/H_1$ denotes ratio of the generalized stress intensity factors.

The integration distance d enters the calculations as a structural parameter or a parameter related to the mechanism of rupture.

The values H_1 and H_2 can be determined from the mean value of the SEDF. Here the two unknown parameters are calculated from the two following conditions. First the ratio h_{21} is

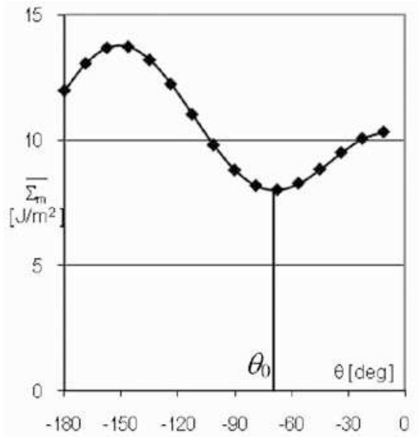

Fig. 2. Numerically gained distribution of the mean value of the generalized strain energy density factor with its minimum in the direction θ_0

gained from the numerically ascertained angle θ_0 of minimum of the mean value of the generalized SEDF $\overline{\Sigma_m}$, see fig. 2. For the minimum of $\overline{\Sigma_m}$ it must be satisfied the condition of the first derivation equal to zero:

$$\left(\frac{d^{2-2p_1}}{2-2p_1}\frac{\partial U_{1m}}{\partial\theta}+\frac{d^{2-2p_2}}{2-2p_2}h_{21}^2\frac{\partial U_{2m}}{\partial\theta}+\frac{d^{2-p_1-p_2}}{2-p_1-p_2}2h_{21}\frac{\partial U_{12m}}{\partial\theta}\right)_{\theta_0}=0\,, \tag{5}$$

From two possible solutions h_{21} to the quadratic equation (5) we take only one that satisfies positive the second derivation, and thus it implies the minimum.

Finally knowing the ratio $h_{21} = H_2/H_1$ the value of GSIF H_1 is determined as:

$$H_1=\left[\frac{1}{\overline{\Sigma_m}8dG_m}\left(\frac{d^{2-2p_1}}{2-2p_1}U_{1m}+\frac{d^{2-2p_2}}{2-2p_2}h_{21}^2U_{2m}+\frac{d^{2-p_1-p_2}}{2-p_1-p_2}2h_{21}U_{12m}\right)\right]^{\frac{-1}{2}}\,. \tag{6}$$

The value $\overline{\Sigma_m}$ in (6) results from the numerical solution; d is a distance at which the value of SEDF is approximated. The value of d has to be chosen considering the failure mechanism. For the brittle fracture d relates to a grain size and thus it can express the increment of a crack. On the other hand for the fatigue loading d can be chosen as the plastic zone size. The value of the coefficient H_2 is determined reciprocally from: $H_2 = h_{21}H_1$.

4. Ψ-integral method

GSIF can also be determined using the so-called Ψ-integral [2]. This method is an implication of the Betti's reciprocity theorem which in the absence of the body forces states that the following integral is path independent:

$$\Psi(\mathbf{u},\mathbf{v})=\int_\Gamma\left[\sigma_{kl}(\mathbf{u})n_kv_l-\sigma_{kl}(\mathbf{v})n_ku_l\right]\,\mathrm{d}s\,,\quad k,l=1,2. \tag{7}$$

The contour Γ surrounds the notch tip and $\mathbf{u},\mathbf{v}$ are two admissible displacement fields. Major advantage of the integral (7) is its path independency for the case of the multimaterial wedges.

To apply the Ψ-integral, it is convenient to derive the displacement field $\mathbf{u}$ and $\mathbf{v}$ using the Lekhnickii-Eshelby-Stroh (L.E.S.) formalism which allows expressing displacements and resultant forces $\mathbf{T}$ along the material interfaces from the complex potential theory. For the case of the orthotropic bi-material notch, one can write

$$u_{m,i} = 2\mathrm{Re}\left\{\sum_{j=1}^{2} A_{m,ij} f_{m,j}(z_{m,j})\right\}, \quad T_{m,i} = -2\mathrm{Re}\left\{\sum_{j=1}^{2} B_{m,ij} f_{m,j}(z_{m,j})\right\}, \tag{8}$$

where

$$\begin{aligned} \mathbf{A}_m &= \begin{bmatrix} s_{m,11}\mu_{m,1}^2 + s_{m,12} & s_{m,11}\mu_{m,2}^2 + s_{m,12} \\ s_{m,12}\mu_{m,1} + s_{m,22}/\mu_{m,1} & s_{m,12}\mu_{m,2} + s_{m,22}/\mu_{m,2} \end{bmatrix}, \\ \mathbf{B}_m &= \begin{bmatrix} -\mu_{m,1} & -\mu_{m,2} \\ 1 & 1 \end{bmatrix}, \end{aligned} \tag{9}$$

$s_{m,ij}$ are elements of the compliance matrix, $\mu_{m,i}$ are the material eigenvalues and m differentiates the materials of the notch. The potentials $f_{m,j}(z_{m,j})$ are considered in the form

$$f_{m,j}(z_{m,j}) = \phi_{m,j} z_{m,j}^{1-p}, \quad j, m = 1, 2, \tag{10}$$

where p is the stress singularity exponent and $z_{m,j} = x + \mu_{m,j} y$. The compatibility equations are automatically satisfied by (8) and the application of the boundary conditions of the joint, i.e. the stress free conditions along the notch faces and the displacement and stress continuity conditions along the bimaterial interface, leads to the eigenvalue problem whose solution are eigenvectors $\phi_{m,i}$ corresponding to the eigenvalue (exponent) p. It can be proved [8], that if $p, \phi_{m,i}$ is the solution of the eigenvalue problem mentioned above, it exists the so-called auxiliary solution $p^*, \phi_{m,j}^*$ of the same eigenvalue problem, where $p^* = 2 - p$. If the displacements $\mathbf{u}$ and $\mathbf{v}$ in (7) are chosen so that e.g. $\mathbf{u}$ corresponds to the regular solution $p, \phi_{m,i}$ and $\mathbf{v}$ to the auxiliary solution $p^*, \phi_{m,j}^*$ and vice versa, the Ψ-integral is nonzero. The other combination of the solutions $\mathbf{u}$ and $\mathbf{v}$ gives zero value of the Ψ-integral.

The isotropic materials are from L.E.S. point of view degenerated because the complex numbers $\mu_{m,1} = \mu_{m,2} = i$ are double roots of the characteristic equation of each material m. The complex coordinates $z_{m \cdot i} = x + \mu_{m,i} y$ reduce to the single value $z = x + iy$ and the matrices $\mathbf{A}_m$ and $\mathbf{B}_m$ are singular so that L.E.S. representation is unable to define the various fields. In this circumstance, it is useful to introduce Mushkhelishvili's complex potentials $\varphi(z)$ and $\psi(z)$ which allow to express the displacement field $\mathbf{u}$ and resultant forces $\mathbf{T}$ as follows

$$\begin{aligned} -2iG_m(u_{m,1} + iu_{m,2}) &= \kappa_m \varphi_m(z) - (z - \overline{z})\overline{\varphi_m'(z)} - \overline{\psi_m(z)} \\ T_{m,1} + iT_{m,2} &= \varphi_m(z) - (z - \overline{z})\overline{\varphi_m'(z)} - \overline{\psi_m(z)}, \end{aligned} \tag{11}$$

where $\kappa_m = 3 - 4\nu_m$ for plane strain and $(3 - \nu_m)/(1 + \nu_m)$ for plane stress, ν_m and G_m are Poisson's ratio and shear modulus of material m, respectively. With a view to relate the potentials $\varphi(z)$ and $\psi(z)$ in (11) with $f_{m,j}(z_{m,j})$ in (8), the equation (11) can be rewritten into the form (8), [1], where

$$\mathbf{A}_m = \frac{1}{4G_m i}\begin{bmatrix} \kappa_m i & -i \\ \kappa_m & 1 \end{bmatrix}, \quad \mathbf{B}_m = \frac{1}{2}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}, \tag{12}$$

and

$$\begin{aligned} f_{m,1}(z) &= \varphi(z) = \phi_{m,1} z^{1-p}, \\ f_{m,2}(z) &= \psi(z) + (\overline{z} + z)\varphi'(z) \\ &= \phi_{m,2} z^{1-p} + (\overline{z} + z)(1 - p)\phi_{m,1} z^{-p} \quad m = 1, 2. \end{aligned} \tag{13}$$

Following (7) the displacement field anywhere around the notch tip can be written as

$$\mathbf{u}_m(r,\theta) = \mathbf{u}_m(0) + H_1 r^{1-p_1}\mathbf{u}_{m,1}(\theta) + H_2 r^{1-p_2}\mathbf{u}_{m,2}(\theta) + \ldots, \tag{14}$$

where $\mathbf{u}_m(0)$ is the rigid body motion, dots represent non-singular terms of the expansion, r and θ are polar coordinates and $r^{1-p_1}\mathbf{u}_{m,1}(\theta)$ and $r^{1-p_2}\mathbf{u}_{m,2}(\theta)$ are basis functions corresponding to the coefficients H_1 and H_2, respectively, derived from (10) for the case of orthotropic materials or from (13) for the case of isotropic materials. Due to the orthogonality conditions that satisfy regular and auxiliary solutions the GSIFs H_1 and H_2 can be computed as follows

$$H_1 = \frac{\Psi(\mathbf{u}_m, r^{2-p_1^*}\mathbf{u}_1^*)}{\Psi(r^{2-p_1}\mathbf{u}_1, r^{2-p_1^*}\mathbf{u}_1^*)}, \quad H_2 = \frac{\Psi(\mathbf{u}_m, r^{2-p_2^*}\mathbf{u}_2^*)}{\Psi(r^{2-p_2}\mathbf{u}_2, r^{2-p_2^*}\mathbf{u}_2^*)}. \tag{15}$$

Since the exact solution $\mathbf{u}_m(r,\theta)$ in (14) is not known, a finite element solution can be used as an approximation for $\mathbf{u}_m(r,\theta)$ to obtain an approximation for H_1 and H_2.

5. Stability criteria suggestion

For the final assessment of a construction with a bi-material notch it is necessary to check the stability of the notch. Determination of the stability conditions of notches means to find the external loading under which a crack is initiated in the notch tip. The classic fracture mechanics approach of comparison of the stress intensity factor K_I with its critical value $K_{I\text{crit}}$ (represented by fracture toughness K_{IC} or by the fatigue threshold value K_{Ith}) is generalized to the following relation:

$$H_k(\sigma_{\text{appl}}) < H_{k\text{crit}}(M_m)\,. \tag{16}$$

The value $H_k(\sigma_{\text{appl}})$ follows from the numerical solution and its determination is described in the previous paragraphs. The critical value $H_{k\text{crit}}$ depends on the critical material characteristic K_{IC} or K_{Ith} and has to be deduced with help of a controlling variable L, see [6]. The detail of derivation of critical values of the H-factor can be found in [4].

Then the critical applied stress is gained from the critical value of $H_{k\text{crit}}$:

$$\sigma_{\text{crit}} = \sigma_{\text{appl}}\frac{H_{1\text{crit}}}{H_1(\sigma_{\text{appl}})}\,. \tag{17}$$

Where σ_{appl} is the external loading stress applied in the numerical solution for the value H_1. The crack will not be initiated in the bi-material wedge tip if the applied stress is lower than the critical stress:

$$\sigma_{\text{appl}} < \sigma_{\text{crit}}\,. \tag{18}$$

6. Numerical example

The numerical study is performed on the rectangular bi-material notch loaded as shown in figure 3.

Within the numerical study the methods of calculation of GSIF were tested for varying combination of the material components expressed by Young's moduli. The results of both presented direct methods are compared. Fig. 4 shows the dependence of the values of GSIFs H_1 and H_2 on the ratio of moduli $E_1/E_2 \in \langle 0.012\,5; 10\rangle$.

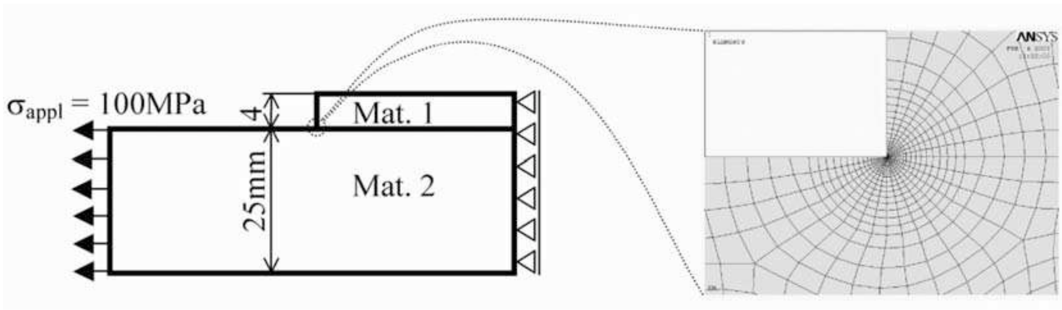

Fig. 3. Rectangular bi-material wedge used in the numerical example, a detail of a FEM mesh

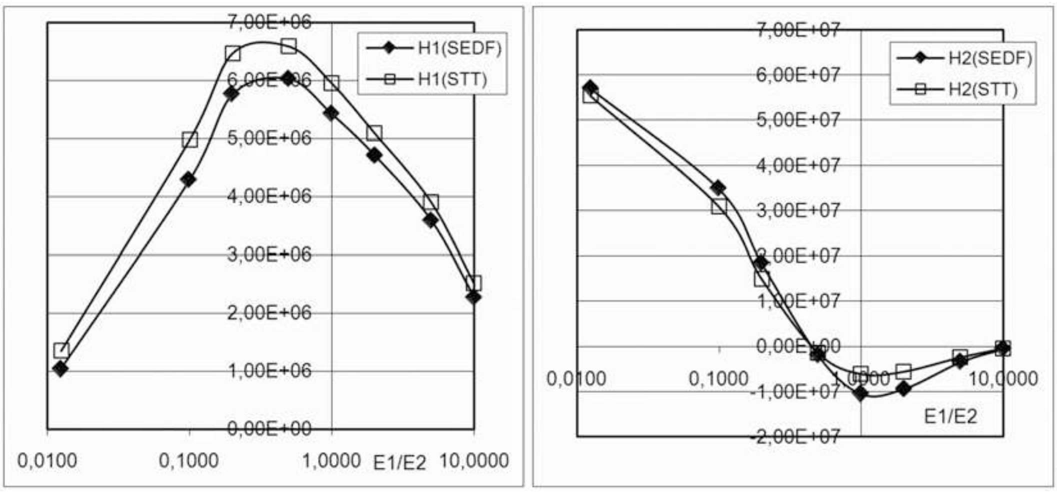

Fig. 4. Values of generalized stress intensity factors H_1, H_2 gained from the mean value of the generalized strain energy density factor (SEDF) and from the tangential stress (STT)

7. Conclusions

The results of GSIFs ascertained by means of two presented direct methods on the basis of two variables match well each other. Both methods give reliable input parameters to assessment of composite structures. The advantage of the direct methods is that they can be easily used with standard finite element calculation systems. The method gaining GSIFs in the two steps – from the supposed angle of potential crack initiation and from the mean value of the particular variable – can be easily programmed and thus automated. The two-step approach described here by means of mean value of generalized SEDF can be analogically derived for the mean value of tangential stress as well.

On the other hand the direct methods require choice of usually length parameter entering into calculation. It can be the choice of the region of the linear part suitable for the extrapolation of H-factor values (section 3.1) or the choice of the region for averaging the SEDF values. This handicap can be exploited for the optimization of the numerical processes by allowing entering structural parameters (e.g. grain size) or parameters connected with the loading type (e.g. plastic zone size, a crack increment) into calculations.

The method combining the application of the reciprocal theorem (Ψ-integral) and FEM is capable to extract the near-tip information directly from the far-field deformation where the numerical fields are more accurate. Thus the Ψ-integral method can help with the option of the unknown length parameter. Further the latter method can be readily applied to bi-materials composed of orthotropic materials components.

Note that for the final evaluation of the bi-material notch it is further necessary to determine critical applied stress on the basis of critical values of the generalized stress intensity factors. These approaches are described within suggestion of stability criteria, see e.g. [4]. When the critical applied loading is determined it is suitable to keep the same controlling variable for the GSIFs estimation as well as for the stability criterion suggestion.

Acknowledgements

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