ABSTRACT

Non-uniform basis functions for construction of interpolating and approximating spline curves and surfaces are presented. The construction is based on the theory of B-splines and enables a continuous change from interpolation to approximation of given data. It is also possible to change the tension of the curves and surfaces.

Keywords: B-spline curves, geometric modelling, interpolation, approximation, basis functions, tensor-product surfaces, spline-blended surfaces.

1. INTRODUCTION

The fundamental problem of geometric modelling is the construction of smooth parametric curves and surfaces to model real objects mainly in industrial design, for example a shoe or a body of a car. Many suitable methods were developed, for example approximating B-splines, see [1].

The problem of approximation of data set was sufficiently solved in the B-spline theory. C2-continuous cubic B-splines are probably the best for the practical use. The control points affect the shape of the curve or surface only locally, what is a common request for the practical use. Known interpolating curves are either only C1-continuous, what means not smooth enough, or the control points affect the curve or surface globally. Other common request is a possibility of a continuous change of the approximating character of the curve to interpolating. It is not always easy to decide whether it is better to use the interpolation or the approximation. The interpolation is more natural, but highlights a possible noise. The approximation is more suitable for noisy data, see Fig. 1,2,3. And finally, it is easier to manipulate data of a curve or surface than approximated control points.

In this paper we will construct an C2-continuous interpolating curve with local affect of control points. Then we will generalise it to a set of
approximating curves containing the B-spline curve. Finally we will generalise this construction for tensor-product and spline-blended surfaces.

Fig. 3
Approximating curve and noisy data

2. SPLINE CURVE AND SURFACE CONSTRUCTION METHODS

A spline curve can be defined using control vertices and basis functions. The curve approximates the shape of the control polygon. Let us denote $W_1, \ldots, W_n$ the control vertices of the curve and $u_1, \ldots, u_n$ its knot sequence. A spline curve is defined as follows:

$$ L(u) = \sum_{i=1}^{n} W_i L_i^n (u) $$

(1)

where $L_i^n (u)$ are the basis functions. The curve is correctly defined if the following equation is satisfied:

$$ \sum_{i=1}^{n} L_i^n (u) = 1, \quad \forall u \in [u_1, u_n] $$

The curve is interpolating if

$$ L_i^n (u_i) = 1, \quad i = 1, \ldots, n. $$

(2)

A tensor-product surface is defined as follows:

$$ S(u, v) = \sum_{i=1}^{m} \sum_{j=1}^{n} W_{ij} L_i^n (u) L_j^n (v) $$

(3)

where $W_{ij}, i = 1, \ldots, m, j = 1, \ldots, n$ are the control vertices, $L_i^n (u)$ are the basis functions and the surface approximates the shape of the control mesh.

A spline-blended surface is a surface interpolating a given network of curves defined as follows:

$$ G(u, v) = G_1(u, v) + G_2(u, v) - G_1(u, v) $$

$$ G_1(u, v) = \sum_{i=1}^{n} G(u, v) L_i^n (u) $$

$$ G_2(u, v) = \sum_{j=1}^{n} G(u, v) L_j^n (v) $$

(4)

where $G(u, v), j = 1, \ldots, n$ are $u$-curves and $G(u, v), i = 1, \ldots, m$ are $v$-curves and $L_i^n (u)$ are the basis functions called also blending functions, see Fig. 4.

3. CONSTRUCTION OF BASIS FUNCTIONS

We will construct the basis functions as B2-spline functions, which are similar to the B-spline functions. Instead of $2n+1$ B-spline control vertices $D_0, \ldots, D_{2n}$ a B2-spline is determined by even control vertices $D_0, D_2, \ldots, D_{2n}$. Odd control vertices $D_1, D_3, \ldots, D_{2n-1}$ are replaced by points $P_i, i = 1, \ldots, n$ joining the $(2i-1)$-th and the $2i$-th segment. Let us denote the knot sequence $u_0, \ldots, u_{2n+2}$. The joint $P_i$ is given by the equation:

$$ p_i = \frac{\Delta_{2i-1} \Delta_{2i} \Delta_{2i+1} \Delta_{2i+2}} {\Delta_{2i-1} + \Delta_{2i} + \Delta_{2i+1} + \Delta_{2i+2}} D_{2i+1} + $$

$$ + \frac{\Delta_{2i} \Delta_{2i+1} \Delta_{2i+2} \Delta_{2i+3}} {\Delta_{2i-1} + \Delta_{2i} + \Delta_{2i+1} + \Delta_{2i+2}} D_{2i+2} + $$

$$ + \frac{\Delta_{2i+1} \Delta_{2i+2} \Delta_{2i+3} \Delta_{2i+4}} {\Delta_{2i-1} + \Delta_{2i} + \Delta_{2i+1} + \Delta_{2i+2}} D_{2i+3} $$

(5)

where $\Delta_j = u_j - u_j$. 

Fig. 5
B2-spline curve
The equation for curves is the same as the equation for functions (2). To sum up, B₂-spline function or curve interpolates its joints. Even control vertices affect the shape of the function or curve, see Fig. 5. The following equation for enumeration of B₂-spline functions (or curves) can be derived:

$$Q_{2n} (t) = (t, t^2, t^3) \frac{1}{24} \begin{pmatrix} -7 & 18 & -16 & 6 & -1 \\ 18 & -36 & 18 & 0 & 0 \\ -12 & 0 & 12 & 0 & 0 \\ 0 & 24 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} D_{2i-2} \\ p_i \\ D_{2i} \\ p_{i+1} \\ D_{2i+2} \end{pmatrix}$$

$$Q_k (t) = (t, t^2, t^3) \frac{1}{24} \begin{pmatrix} 1 & -6 & 16 & -18 & 7 \\ -3 & 18 & -30 & 18 & -3 \\ 3 & -18 & 0 & 18 & -3 \\ -1 & 6 & 14 & 6 & -1 \end{pmatrix} \begin{pmatrix} D_{2i-2} \\ p_i \\ D_{2i} \\ p_{i+1} \\ D_{2i+2} \end{pmatrix}$$

where \( t = u - (2i - 1) \) or \( t = u - 2i \).

In [3] a spline curve has been constructed, and it has been determined by control vertices \( W_0,...,W_n \) and B₂-spline basis functions, defined by equation (1). In [2] a relationship between B₂-spline control vertices of basis functions and B₂-spline control vertices of a spline curve has been derived. Let \( P_i \) and \( D_j \) be joints and even control vertices of \( j \)-th basis function, respectively. Let \( C_{2k} \) and \( R_k \) be even control vertices and joints of B₂-spline curve, respectively. These vertices satisfy the equation:

$$R_k = \sum_{i=1}^{n} W_i P_i^k \quad C_{2k} = \sum_{i=1}^{n} W_i D_{2k} \quad (6)$$

In this equation the control vertices of the basis functions \( D_0,D_2,...,D_{2n} \) and \( P_1,...,P_n \) are considered to be only real numbers equal to the second coordinate of the points.

In [3] we have derived B₂-spline control vertices for the uniform interpolating and approximating curve with control vertices \( W_1,...,W_n \):

$$R_k = a \cdot W_{i+1} + (1 - 2a) \cdot W_i + a \cdot W_{i-1}$$

$$C_{2k} = p \cdot W_{i+1} + \left( \frac{1}{2} + p \right) \cdot W_i + \left( \frac{1}{2} + p \right) \cdot W_{i-1} - p \cdot W_{i+2}$$

where \( a \) is a parameter of approximation and \( p \) is a parameter of the tension. For \( a = 0 \) we get an interpolating curve and for \( a = 1/6 \) and \( p = 0 \) we get the uniform B₂-spline curve.

This construction gives good results only for approximately equidistant data points, see Fig. 6,7,8.
Let us consider a B-spline curve (see Fig. 9) consisting of 3 segments, interpolating joints \( W_i = (0,0), W_2=(0,C), W_3=(B,C), W_4=(B,0) \) with knot sequence \( u_0,\ldots,u_7 \) where \( u_0=0, \Delta_1=A, \Delta_2=C, \Delta_3=B, \Delta_4=C, \Delta_5=A, \Delta_6=u_{i+1}-u_i \), \( u_i \) and \( u_7 \) will be chosen later. Now it is possible to find the B-spline control points \( C_0,\ldots,C_5 \) of the curve, where \( C_0, C_4 \) are B-spline control points of the original curve and \( C_1, C_2, C_3 \) are B-spline control points of the curve with split segments, the control vertices satisfy these equations:

\[
C_{2i} = \frac{\Delta_i/2 + \Delta_{i+1}}{\Delta_{i+1} + \Delta_i + \Delta_{i+1}} \cdot C_i + \frac{\Delta_{i+1} + \Delta_i/2}{\Delta_{i+1} + \Delta_i + \Delta_{i+1}} \cdot C_{i+1},
\]

\[
C_{2i+1} = x'_{i} \cdot C_{i+1} + x'_{i} \cdot C_{i+1} + x'_{i+1} \cdot C_{i+1}.
\]

\[
x'_{i} = \frac{1}{4} \left( \frac{2\Delta_{i+1} + \Delta_i}{\Delta_{i+1} + \Delta_i + \Delta_{i+1}} \cdot \Delta_{i+1} \right)
\]

\[
x'_{i+1} = \frac{1}{4} \left( \frac{2\Delta_{i+1} + \Delta_i}{\Delta_{i+1} + \Delta_i + \Delta_{i+1}} \cdot \Delta_{i+1} \right)
\]

\[
x'_2 = 1 - x'_1 - x'_3.
\]

We will investigate the y-coordinate of the point \( C_i=(B/2,y) \). This point satisfies the equation

\[
C_i = -p \cdot W_i + \left( \frac{1}{2} + p \right) \cdot W_2 + \left( \frac{1}{2} + p \right) \cdot W_3 - p \cdot W_4.
\]

If we consider only the y-coordinate of this equation we get \( y=(1+2p)C \), and we get the equation \( p=(y/C-1)/2 \). Using the equations (7) a (5) we get

\[
P = \frac{B^2}{A(B+C) + 2BC + C^2}.
\]

Now we can choose a value of \( A \). Let us suppose \( B=C \). Let us choose \( \frac{1}{4}B \) to be a value of the y-coordinate of the control vertex \( C_4 \). This condition gives \( A = \frac{1}{4}B \) and

\[
P = \frac{B^2}{B^2 + 5BC + C^2}
\]

This is a symmetrical curve. Now we can generalise the result for a non-symmetrical curve:

\[
C_{2i} = -p_{i-1} \cdot W_{i-1} + \left( \frac{1}{2} + p_{i-1} \right) \cdot W_i + \left( \frac{1}{2} + p_{i-1} \right) \cdot W_{i+1} - p_{i-1} \cdot W_{i+2}
\]

\[
p_i^- = \frac{\Delta_i^2}{\Delta_i^2 + 5\Delta_i \Delta_{i+1} + \Delta_{i+1}^2}
\]

\[
p_i^+ = \frac{\Delta_i^2}{\Delta_i^2 + 5\Delta_i \Delta_{i+1} + \Delta_{i+1}^2}
\]

We have constructed an interpolating curve not causing problems shown in Fig. 8 as it can be seen in Fig. 10.

The curve is suitable to model objects determined by few points. But if we are given a lot of data for a simple object, the curve does not seem to be very suitable, see Fig. 11.

We will consider the parameter of tension \( p \) again:

\[
p_i^- = \frac{6 \cdot p \cdot \Delta_i^2}{\Delta_i^2 + 5\Delta_i \Delta_{i+1} + \Delta_{i+1}^2}
\]

\[
p_i^+ = \frac{6 \cdot p \cdot \Delta_i^2}{\Delta_i^2 + 5\Delta_i \Delta_{i+1} + \Delta_{i+1}^2}
\]

If we use a tension \( p=0.13 \) for the object in Fig. 11, we get a good result (Fig. 12).
\[ p_i^* = \frac{6 \cdot p_i \cdot \Delta_i^2}{\Delta_i^2 + 5 \Delta_i \Delta_{i-1} + \Delta_{i-1}^2} \]

(9)

Let us summarise the definition of the interpolating curve. Given:
- control points \( W_1, \ldots, W_{n+2} \)
- a knot sequence \( u_{-1}, \ldots, u_n \)
- a parameter of tension sequence \( p_1, \ldots, p_{n+1} \)

Interpolating non-uniform curve defined on \([u_{i-1}, u_i]\), consisting of \(n-1\) segments, interpolates control points \( W_1, \ldots, W_n \), the \( i \)-th segment is defined on \([u_i, u_{i+1}]\) and parameter of tension \( p_i \) belongs to this segment. The curve is the same as B-spline curve consisting of \(2n-2\) segments determined by control vertices \( C_0, \ldots, C_{2n} \) satisfying the equations (5),(8),(9) and the knot sequence \( u'_0, \ldots, u'_{2n+2} \) is defined as follows:

\[ u'_i = \frac{u_i + u_{i+1}}{2}, \quad i = 0, \ldots, n; \]
\[ u'_i = (u_i + K u_{i+1})/(K+2), \quad i = 0, \ldots, n+1; \]

where \( K \) is big enough (for example \( K=20 \)).

To finish this section, we suggest a way how to determine suitable values of parameters of tension \( p_1, \ldots, p_{n+1} \). Our equation has been constructed to satisfy these conditions:
- If the direction of the control polygon edges is changed only a little, the tension is close to zero, see Fig. 15.

- If the direction is changed almost opposite way, the tension is close to zero again.

- If the direction is changed quickly, the tension rises, see Fig. 16.

We have suggested the following equation:

\[ p_i = p \cdot \frac{\cos \alpha_i + \cos \alpha_{i+1}}{2} \]

\[ \alpha_i = \angle W_{i-1} W_i W_{i+1} \]

where \( p \) is the parameter of tension. Suitable default value is \( p=0.2 \). Fig. 19 shows that this method is much better in detail (compare Fig. 1).
5. APPROXIMATING CURVE

In this section we will generalise the interpolating curve to a system of approximating curves with the parameter of approximation \(a\).

For \(a=0\) we get the interpolating curve and for \(a=1/6\) and \(p_i=0\), \(i=1, \ldots, n-1\) it is supposed to be the B-spline curve. The approximating curves for other values of parameter \(a\) will be constructed as linear interpolation of the interpolating and the B-spline curve. Advantage of this construction is that we can interpolate the control points of the interpolating and the B-spline curve and get the same result.

Let us have a B-spline curve \(L^B(u)\) with B-spline control vertices \(W_0, \ldots, W_{n+1}\) and a knot sequence \(u_1, \ldots, u_{n+2}\) (the B-spline curve does not depend on vertices \(W_1\) and \(W_{n+2}\)). Let us denote its B-spline control vertices: the joints are \(W_1^B, \ldots, W_n^B\) and the even control vertices are \(C_n^B, C_{n-1}^B, \ldots, C_2^B\). Its parameter of tension sequence is \(p_i^B=0\), \(i=1, \ldots, n-1\). We can get these vertices from given points \(W_1, \ldots, W_{n+2}\) using the equations (7) and (5). The control points of the interpolating curve \(L(u)\) are denoted in the same way as in previous Section: \(W_1^I, \ldots, W_{n+2}^I\) are the joints, \(C_0^I, \ldots, C_n^I\) are the even control vertices and \(p_1^I, \ldots, p_{n+1}^I\) is the parameter of tension sequence.

Let us denote \(L^a(u)\) an approximating curve with parameter of approximation \(a\). Let us have \(L^a(u)=L(u)\) and \(L^{1/6}(u)=L^B(U)\). \(L^a(u)\) is defined as follows:

\[
L^a(u) = 6 \cdot a \cdot L^B(u) + (1 - 6 \cdot a) \cdot L(u)
\]

The B-spline control vertices of the approximating curve \(L^a(u)\) satisfy:

\[
W_i^a = 6 \cdot a \cdot W_i^B + (1 - 6 \cdot a) \cdot W_i, \quad i=1, \ldots, n
\]

\[
C_i^a = 6 \cdot a \cdot C_i^B + (1 - 6 \cdot a) \cdot C_i, \quad i=0, \ldots, n
\]

6. TENSOR-PRODUCT SURFACES

It is quite easy to generalise the definition of the approximating curve for tensor-product surfaces:

Given:
- control points \(W_{ij}, i=1, \ldots, m+2, j=1, \ldots, n+2\)
- knot sequences \(u_1, \ldots, u_{m+2}, v_1, \ldots, v_{n+2}\)
- parameter of tension sequences \(p_1, \ldots, p_{m+1}, q_1, \ldots, q_{n+1}\)

The curve defined in previous Section is denoted as \(C(v)\) and the tensor-product surface is denoted as \(S(u,v)\).

According to the equations (1) and (3) we can write

\[
C(v) = \sum_{i=0}^{n} W_i L^B_i(v)
\]

\[
S(u,v) = \sum_{i=1}^{m} \left[ \sum_{j=1}^{n} W_{ij} L^B_{ij}(v) \right] L^B_i(u) = \sum_{i=1}^{m} C(v) L^B_i(u)
\]

where \(C(v)\) is the approximating curve defined in the previous Section by the control vertices \(W_{ij}, i=1, \ldots, m+2, j=1, \ldots, n+2\) and the knot sequence \(u_1, \ldots, u_{m+2}\). We can also write \(S(u,v)=C(u)\), where \(C(u)\) is the approximating curve defined in the previous Section by the control vertices \(C'\), \ldots, \(C'^{m+2}\) and the knot sequence \(u_1, \ldots, u_{m+2}\).

It is not possible to generalise suggestion of the knot sequences and the parameter of tension sequences because we have to suggest only one knot sequence and parameter of tension sequence for all \(u\)-curves and a second for \(v\)-curves. This is the reason why we do not give any suggestions for tensor product surfaces. Both knot sequences and parameter of tension sequences must be suggested separately for specific needs of each application.
7. SPLINE-BLENDED SURFACE

Given:
- knot sequences $u_{-1}, \ldots, u_{m+2}, v_{-1}, \ldots, v_{n+2}$
- a network of curves $G(u_i, v_j), i = -1, \ldots, m+2,$ $G(u, v), j = -1, \ldots, n+2$
- parameter of tension sequences $p_1, \ldots, p_{m-1}, q_1, \ldots, q_{n-1}$

The points of the spline-blended surface defined in (4) can be enumerated in the similar way as in previous Section. The surface $G_{12}$ from definition (4) is a tensor-product surface. If we fix the parameter $v$ in the surface $G_1$ and the parameter $u$ in the surface $G_2$, we get the definition of a curve according to (1). So we can use algorithm for curves.

8. CONCLUSION

We have constructed an appropriate non-uniform system of spline curves containing an interpolating curves and the B-spline curve. The interpolating curve is special case of the B2-spline curve, but we have found appropriate even control points. User does not have to take care of them, but they can also be changed using the parameter of tension sequence. The results of this paper have been implemented and we represent them in several pictures shown in this paper. We think it would be useful to generalise these results to a system of non-uniform rational curves, containing the most used NURBS and try other parameterisation methods as well.

REFERENCES

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