On the Expected Number of Common Edges in Delaunay and Greedy triangulation

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Abstract

So far some average-case properties in the Delaunay and greedy triangulation were given by complicated probabilistic analysis. In this paper, we present a rather simpler proof on that the expected number of common edges between Delaunay and Greedy triangulation is at least 40% when points are uniformly distributed, where n is the number points in a convex planar region.

Our analysis shows that the value c of $o(c \cdot n)$ expected number of common edges between two triangulations is greater than 1.26. That constant $c = 1.26$ implies that at least 40% of Delaunay edges are common to the edges of Greedy triangulation. Applying this property, we can easily find at least $1.26n$ greedy edges in linear time from a Delaunay triangulation, if points are uniformly distributed in a region.

Finally we give two experimental results showing that in practice c approaches up to 2.7, which means about 90% edges are common between two triangulations.

Keywords: computational geometry, triangulation, common edges.

1 Introduction

Computing a "good" triangulation is a long standing problem in computational geometry. For the optimal triangulation with various objective functions, lots of heuristic triangulation algorithms have been developed [4, 5, 8].

Among these various triangulations, there are two well-known triangulations, which are greedy triangulation and Delaunay triangulation. Since algorithms for those triangulations are relatively easy to understand and simple to implement, two triangulations are used as an initial solution to be refined for the final optimal minimum weight triangulation.

The greedy triangulation inserts edges one by one by the increasing length of the edge from a shortest one. If the inserted edge does not break the triangulation

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constraint, it is added in on-constructing triangulation. Otherwise it is discarded and the next longest edge is considered. And the Delaunay triangulation is generated from a dual of the Voronoi diagram for a given point set easily in $o(n \log n)$ time.

Till now a lot of works were done on these two triangulation in the aspect of geometric property, algorithm complexity, the approximation ratio to the minimum total edge weight and so on. Also some general and extremal stochastic properties on the Delaunay triangulation were given assuming that the points are distributed on the unit-intensity Poisson process [1, 6].

In this paper we will give a proof for that there are at least $1.26n$ common edges for two triangulations for a given uniformly distributed point set in the plane. The fact that $o(n)$ edges are common in two triangulations can be proved by using a rigorous probabilistic [1, 2].

The first result on the common edges in two different triangulations is found in [7]. He proved that if n points are chosen with a Poisson process distribution, then the expected fraction of *reciprocal* pairs is $6\pi/(8\pi + 3(3)^{1/2}) \approx 0.6215$. The reciprocal pair of point set is defined as a pair, (a, b), satisfying symmetrical nearest neighbor relation such that a and b is the nearest neighbor of b and a, respectively. Since if there is any segment which crosses (a, b) and that is shorter than the length of (a, b) then that point would be the nearest neighbor of a or b, it is easy to see the segment connecting a reciprocal pair points should be an edge of greedy triangulation with that point set. According to his analysis, we can say at least 10% $(= 0.32n)$ of Delaunay edges are common to greedy edges, since every edge connecting a reciprocal pair must be included in Delaunay edges.

At first we will show that at least 30% edges of Delaunay edges are expected to be common to the greedy edge set. The value 30 % implies that among $3n - 6$ edges about $1.0n$ edges are common in two triangulations. Applying this *strong commonness* between two triangulations, we can find $o(1.2n)$ greedy edges in linear time from a given Delaunay triangulation.

Since ref.[2] have given an algorithm finding Delaunay triangulation in linear time expected with uniformly distributed points in a unit square, combining this algorithm to our proposed algorithm makes it possible to get a linear time expected algorithm for greedy triangulation. Though there has been other linear time expected algorithm for greedy triangulation with uniform point set[3], we believe our algorithm is straightforward and easy to implement.

In final section we give two experimental results for the number of common edges between two triangulations with more than 100 points in a square. A little surprisingly about 90% of edges are common in two triangulation, which proposes an interesting conjecture, that is, Delaunay triangulation has at least n common edges to the greedy triangulation with every planar point set.

2 Expected number of common edges

Let S be a convex region and P be a point set uniformly distributed on S. Let the number of points in P be n, that is $|P| = n$. In the following we assume that points are uniformly distributed on that region S. Let $Ep(A)$ and $area(A)$ denote the expected number of points in region A and the area of region A, respectively.

This assumption of the uniform distribution implies that the expected number of points located in any sampled region A over S is estimated as follows.

$$Ep(A) = n \cdot \frac{area(A \cap S)}{area(S)}$$

If the convex hull has $o(n)$ edges, then simply those convex hull edges are included in both greedy and Delaunay triangulations, which gives a trivial proof on $o(n)$ common edges. So we assume that the number of convex hull edges is constant or sub-linear. For the simplicity of the proof procedure we assume that the number of convex hull edges is constant. If the number of convex hull edges is sub-linear, then the proof for this case would be obtained in a similar way of the following procedure.

Let $DT(S)$ and $GT(S)$ denote the Delaunay triangulation and greedy triangulation, respectively from P on S. Also we denote each triangle as T_i in DT_S and its circumscribed circle as C_i. Consider one triangle T_i in $DT(S)$ in Fig.1. Let a b, c denote the three vertices of T_i. (a, b) and $|(a, b)|$ denotes the edge and its length, respectively. o_i is the center point of C_i. And let d_i denote the diameter of C_i, that is $d_i = |(o_i, a)|$. For the following proof procedure, we need to construct one extra *enlarged circle* from C_i, which will be denoted as EC_i. The center point of EC_i is o_i same to C_i and its diameter is extended as follows.

At first, we take the shortest edge in the three edges of T_i. Without loss of generality we assume that (a, b) is the shortest one in the Fig. 1. Then we move that line (a, b) in parallel to the outside of the circle C_i till the middle point of (a, b), that is m_i, just encounter with C_i. Let x and y be the end points of the line segment which was moved from (a, b) parallel to the direction $\overline{(o, m)}$.

Therefore we get another circle EC_i of a slightly extended diameter with length $|(o_i, x)|$ compared to $d_i = |(o_i, a)|$. Notice that $|(o_i, x)| = |(o_i, y)|$. And now we have another geometric object "annulus" between E_i and EC_i, denoted as $A_i = EC_i - C_i$.

Lemma 1 *If there is no point in A_i, then edge (a, b) is one edge of greedy triangulation with the point set P.*

Proof: Greedy triangulation procedure considers an edge at a time by examining each pair in order of length and adding or discarding it based on its compatibility with the edges already added[8, 5]. Thus for an edge (a, b), if there is no line segment which is shorter than (a, b) and crosses (a, b), then (a, b) should be included in greedy triangulation edges, since when (a, b) is considered by the greedy procedure, there would be no blocking line segment of (a, b).

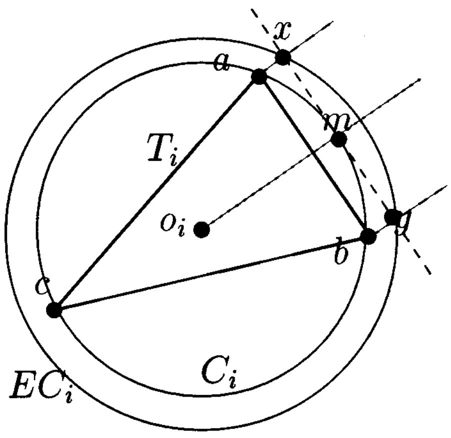

Fig.1: One Delaunay triangule $\triangle abc$ and its corresponding enlarged circle.

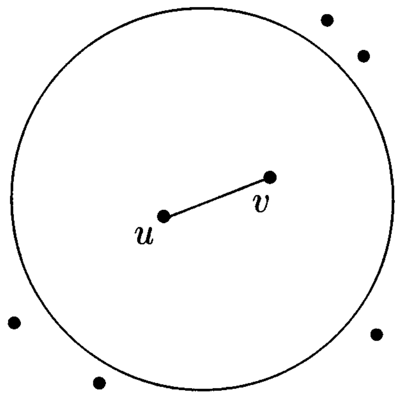

Fig.2: (u,v) is a greedy edge since no line segment shorter than (u,v) cuts this edge.

There would be two cases that some line segments crosses (a, b) in Fig. 1. One case is that two points outside A_i blocks (a, b), and another case is that one point is of inside EC_i and the other point is of outside EC_i. Then by the property of EC_i which was enlarged from C_i, the shortest length of the line segment whose end points are outside of EC_i and which crosses over any points in C_i, that is the edge (a, b), should be greater than $|(a, b)|$.

Let's consider the other case. Assume that there is an edge (x, y), where x is outside of EC_i and y is in C_i, and it crosses over edge (a, b). Since (a, b) is the shortest edge in three edges of T_i, y must be point c in C_i. Also if (c, x) intersects (a, b) for x outside EC_i, then $|(c, x)| \geq |(a, b)|$, since $|(c, a)| \geq |(a, b)|$ and $|(c, b)| \geq |(a, b)|$.

Therefore there is no shorter line segment than $|(a, b)|$, and which crosses over (a, b). This completes the proof of this lemma. □

Therefore if we find that the region A_i is empty, then (a, b) must be one of greedy edges. However we do not have to examine the whole region of A_i to assure that (a, b) is a greedy edge. For this we define another subregion, called "forbidden region" to guarantee (a, b) as one Delaunay edge.

For an Delaunay edge (u, v), if some associated region which surrounds (u, v) is large enough, then we can expect that (u, v) also could be an greedy edge, since the abundant surrounding region to (u, v) may exclude all crossing line segments which could be shorter than $|(u, v)|$.

As was shown in Fig.2, if there is no point in the circle containing edge (u, v) except u, v, then we can easily see that (u, v) must be included in the greedy edge set. Here now we raise one question, that is how we can find the empty region to guarantee one greedy edge and how much that empty region is needed to do so. We call such an empty surrounding region to guarantee (a, b) to be a greedy edge as *forbidden region* of edge (a, b). So the main point of our proof is reduced to find the smallest *forbidden region* for (a,b) in general case. Though we could not give the minimal(optimal) forbidden region for (a,b) in this paper, we can give one relatively small forbidden region with respect to the area of T_i.

Definition 2.1 *Let T_i be one of triangles in a Delaunay triangulation and (a,b) be the shortest edge in T_i and m is the mid point of (a,b). And let point p and q be on C_i with $|(p,a)| = |(q,b)| = l$ and $p \neq b$, $q \neq a$ and $p \neq q$. Then one forbidden region of (a,b) in T_i, denoted as FR_i is the angular sector of annulus $A_i(= EC_i - C_i)$ with angle $\angle po_iq = 3 \cdot \angle ao_ib$.*

In Fig.3 our forbidden region FR_i is illustrated as slashed region. Now we give one corollary on a property of FR_i by Lemma 1.

Corollary 1 *If there is no point in the forbidden region FR_i of T_i, the shortest edge (a,b) of T_i must be included in greedy edges. And if C_i is a unit circle and $\angle ao_ib = 2\theta$, then we get the function of $area(FR_i)$ in terms of the half of central angle $\angle ao_im = \theta$ as follows.*

$$f_{FR}(\theta) = area(FR_i) = area(A_i) \cdot \frac{6\theta}{2\pi} = 3 \cdot \theta \cdot sin^2\theta \qquad \square$$

The proof of this corollary is similar to the proof of Lemma 1. Suppose that there is an edge (x,y) which is shorter than (a,b) and it crosses (a,b). Then its one vertex, namely x, must be in A_i and is within the distance $|(a,b)|$ from a or b. So if we want to search such x in A_i, we do not have to examine the whole region of A_i. Thus it is enough to search only some partial region of A_i.

For $f_{FR}(\theta)$, let $\angle ao_im = \angle mo_ib = \theta$, then examining $6 \cdot \theta/(2 \cdot \pi)$ part of $area(A_i)$ is enough region to search x. Since $|(p,a)| = |(a,b)| = |(b,q)|$, it is easy to see $area(FR_i) = area(A_i) \cdot 3\theta/\pi$, which can be verified easily in Fig. 3. However one more careful observation would show that we can get smaller "forbidden region" than $area(FR_i)$. That smaller region's angular sector is denoted as δ in Fig. 3, which is smaller angle than 3θ.

It is easy to see that (u_1,u_2) is the shortest line segment which connects two points outside EC_i with crossing (a,b). Thus as was illustrated in Fig. 3, the angular sector A_i with angle of $2 \cdot \angle u_2o_ia + \theta$ also can be used another forbidden region which has smaller area than $area(FR_i)$. However, we do not consider this smaller forbidden region for simplicity of analysis. We hope if more complicated analysis is possible to find smaller forbidden region, then we can see more common edges in two triangulations.

This concept of forbidden region raises one interesting question, "what is the optimal(=minimal area) forbidden region to guarantee (a,b) to be included in greedy edge ?" If we move EC_i slightly with parallel to (a,b), then we find another smaller, but irregular shaped forbidden region than our FR_i with the 3δ angular sector of

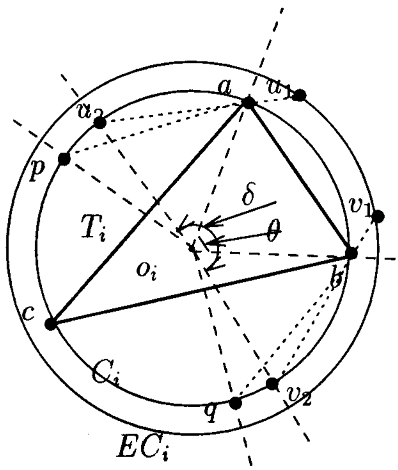

Fig. 3. The forbidden region FR_i of T_i

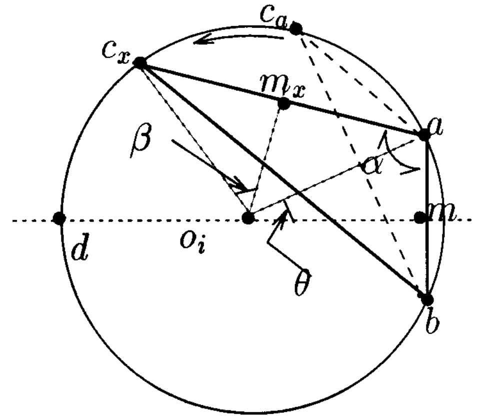

Fig. 4. Computing average value of r_i

EC_i. We hope more careful analysis would get the minimal area for the forbidden region of (a, b), which would be helpful to expect more higher fraction of common edges in two triangulations.

Since the number of common edges in greedy and Delaunay triangulation is related to the number of points in each forbidden region, now we are required to estimate the expected number of points in each forbidden region. For this we consider firstly the total sum of the each area of forbidden regions FR_i of T_i, formally saying $\cup_i FR_i$. We hope to compute the expected ratio of $area(FR_i)$ to $area(T_i)$. But here we can not estimate $area(T_i)$ exactly, since we only know about the smallest angle. Thus we substitute the $area(T_i)$ as the smallest area in all possible T_i with preserving that (a, b) is the smallest edge of T_i. Thus as shown in Fig. 4, c_x only moves on circle C_i from c_a to d. Minimal area, $min(T_i)$, is defined as following.

$$\min(T_i) = 2sin^2\theta \cdot sin(2\theta)$$

So let us define one another variable $r_i = area(FR_i)/min(T_i)$ for all i. To calculate the average of r_i we have to integrate r_i assuming that $\angle acb$ is minimum, which is one of conditional probability.

If points are uniformly distributed in P, it is known already that any two angles of T_i has a joint distribution function [7]. That probability density function of two arbitrary angles α, β in an arbitrary triangle of a Delaunay triangulation is

$$f(\alpha, \beta) = \frac{8}{3\pi} sin\alpha \cdot \sin\beta \cdot \ sin(\alpha + \beta), \quad \alpha \geq 0,\ \beta \geq 0, \quad \alpha + \beta \leq \pi.$$

So the density function of a random Delaunay angle can be obtained by integrating $f(\alpha, \beta)$ over β then it is

$$f(\alpha) = \frac{4sin\alpha}{3\pi}(sin\alpha + (\pi - \alpha)cos\alpha), \qquad 0 \leq \alpha \leq \pi.$$

By the following lemma we compute the average value of r_i.

Lemma 2 *If points are uniformly distributed in any 2-dimensional convex region, then the expected value of ratio* $R_i = area(FR_i)/area(T_i)$ *is greater than* 1.112.

Proof: This lemma implies that it holds $area(T_i) \geq 1.112 \cdot area(FR_i)$ for average case. We can compute the average value of r_i for all i since it is easy to see

$$R_i = \frac{area(FR_i)}{area(T_i)} \leq \frac{area(FR_i)}{min(T_i)} = r_i$$

Thus the average value of r_i is calculated as followings.

$$\begin{aligned}
r^* &= \left(\frac{1}{\int_0^{\pi/3} f(\theta)\, d\theta}\right) \cdot \int_0^{\pi/3} \frac{area(FR_i)}{min(T_i)} \cdot f(\theta) d\theta \\
&= \frac{1}{k_0} \cdot \int_0^{\pi/3} \frac{f_{FR}(\theta)}{2sin^2\theta \cdot sin(2\theta)} \cdot f(\theta)\ d\theta \\
&= \frac{1}{k_0} \cdot \int_0^{\pi/3} \frac{3\theta}{2 \cdot sin(2\theta)} \cdot f(\theta)\ d\theta = \left(\frac{1}{0.5288}\right)(0.5879) \\
&\approx 1.112
\end{aligned}$$

, where constant k_0 is calculated as follows.

$$\begin{aligned}
k_0 &= \int_0^{\pi/3} f(\theta)\ d\theta \\
&= \left[\frac{1}{6\pi}(2\pi + 4t - 2\pi cos(2t) + 2tcos(2t) - 3sin(2t))\right]_{t=0}^{t=\pi/3} \\
&= \frac{2}{3} - \frac{\sqrt{3}}{4\pi} \approx 0.5288
\end{aligned}$$

Thus by Lemma 2, we can compute the expected number of points located in each forbidden region, and the expected number of empty forbidden region.

Lemma 3 *If points are uniformly distributed in the 2 dimensional convex region, then the expected number of points located in* FR_i *is less than* 0.555.

Proof: Let S be point set on P convex polygon and k be the number of triangles in $DT(S)$. We get the sum of all $area(FR_i)$ for all i to estimate the number of points located in all FR_i region.

If the number of convex hull edges is c_v, then we prepare one extra point to each convex hull edge to make *pseudo*-face of $DT(S)$. In adding one extra point for each edge, we add it so far away for each convex hull edge to get $area(FR_i)$ of that pseudo face approaches 0 since each convex edge ought to be included in greedy edges. By this adding procedure, we can get exactly $2n$ number of faces of $DT(S)$ consisting of *real*-faces and *pseudo*-faces.

If FR_i is not enclosed we only need to consider the intersected area between FR_i and $Conv(P)$, since the outside of $Conv(P)$ has no points which do not effect the following analysis. Lemma 2 shows that the average ratio of $area(FR_i)$ to $area(T_i)$ is less than 1.112, so we get the following.

$$\begin{aligned} A_s &= \sum_{i=1}^{k} area(Conv(P) \cap FR_i) \leq \sum_{i=1}^{k} area(FR_i) \\ &\leq \sum_{i=1}^{k} 1.112 \cdot area(T_i) \qquad by\ Lemma\ 3 \\ &\leq 1.112 \cdot \left(\sum_{i=1}^{k} area(T_i) = area(Conv(P)) \right) \end{aligned}$$

Above calculation shows that the total sum of $area(FR_i)$ is less than $1.112 \cdot area(S)$. Note that each FR_i is not mutually disjoint. Thus the expected total number of points located in each FR_i is estimated $1.112 \cdot n$ by the uniformity of point distribution. Therefore $Ep(FR_i)$, the expected number of points in each FR_i is less than

$$\begin{aligned} Ep(FR_i) &= \frac{1.112 \cdot n}{The\ number\ of\ faces\ in\ DT(S)} \\ &\leq \frac{1.112}{The\ number\ of\ real\ faces} \\ &\leq 0.555 \quad \square \end{aligned}$$

Lemma 3 implies that there must be at least 0.899 empty FR_i by the *pigeon-hall principle.* If the number of points which are located in each FR_i is nearly equal to each other, we can expect the number of empty FR_i is at least 0.899. Thus we can say the probability of empty FR_i is at least 0.445.

But in practice some bigger FR_i is expected to have more points than a smaller FR_i, so the expected number of empty FR_i would increase. If we assume that each FR_i is nearly same size and 1.112 points are uniformly distributed and $20 \geq n\infty$ then we know $Prob[Ep(FR_i) = 0]$, the probability that a FR_i excludes point is given.

$$0.5697 \leq Prob[Ep(FR_i) = 0] = (1 - \frac{1}{2n})^{1.112n} \approx \left(\frac{1}{e}\right)^{0.555} \approx 0.574$$

So the total expected number of empty FR_i would be $2 \cdot 0.569 = 1.138$. However since above analysis is too crude, so some more exact analysis is needed to get a more accurate value for the expected number of empty FR_i.

Now we give another procedure for finding the common edge, also we will show that procedure will give about $1.26n$ common edges. This argument is based on the angle of each T_i.

Definition 2.2 *Let S be a point set in 2 dimensional plane P. Let (x, y) be an edge which is shared by two different, adjacent triangles $\triangle xyu$ and $\triangle xyv$. Then we call $\angle xuy$ is a facing angle of $\angle xvy$ and vice versa. And those two angles are called a pair of facing angles with (x, y). And (x, y) is called the common edge of the facing $\angle xuy$ and $\angle xvy$.*

Intuitively we can see that if an Delaunay edge is relatively small, it is more likely to be included a greedy edge. Thus we give one characterization on that if the common edge of a pair of facing angles could be an greedy edge in a Delaunay triangulation.

Lemma 4 *Suppose that p is a facing angle of q with respect to edge (x, y) in $DT(S)$. If $p \leq \pi/3$ and $q \leq \pi - p/2$ then (x, y) is one of greedy edge. If $\pi/3 \leq p \leq \pi/2$ and $q \leq 2\pi - 2p$ then (x, y) is also a greedy edge.*

Here we give our first result on the number of common edges between $GT(S)$ and $DT(S)$ by the moving the shortest edge of T_i explained above.

Theorem 1 *If points S are uniformly distributed in any convex region, then at least $1.26n$ of Delaunay edges are expected to be common to the edges greedy triangulation with the same point set.*

Proof:(sketch) We take the shortest edge of each T_i, which is called (a_i, b_i). And move it to the direction of outside till it just pass away C_i in parallel, as explained previously. Since (a_i, b_i) is the shortest edge in T_i, it is easy to see $\angle a_i c_i b_i \leq \pi/3$. Let $p \leq \pi/3$ be one angle of a Delaunay triangle. Then we can compute the probability that the facing angle of p is less than $2\pi - 2p$, since we know the probability distribution function for Delaunay angles[1]. This calculation shows that 1.26 is the lower bound of expected number of edges of common in two trianulations. □

3 Experiment and Conclusion

We conducted one experiment to find the number of common edges for five randomly generated points set. Table 1,2 show that about 90% of the edges in the Delaunay triangulation are also common to the greedy triangulation from uniformly distributed points set. We think that it is an interesting problem to find the greedy triangulation in $o(n)$ time from a given Delaunay triangulation. Finally we give one combinatorial conjecture.

Conjecture: More than $|S|$ edges are common between Delaunay and Greedy triangulation for any 2-D points set S.

size	ex1	ex2	ex3	ex4	ex5
100	0.919298	0.915194	0.886525	0.907801	0.929329
200	0.880342	0.901893	0.905822	0.914089	0.912220
300	0.891033	0.909194	0.914966	0.921857	0.925170
400	0.923077	0.922232	0.910246	0.912088	0.912860
500	0.899932	0.912897	0.894381	0.908232	0.914750

Table 1: Ratio of common edges in rectangle 2000×2000

size	ex1	ex2	ex3	ex4	ex5
20	0.959184	0.959184	0.958333	0.960784	0.958333
50	0.920290	0.941606	0.933333	0.918519	0.903704
100	0.904930	0.903915	0.922261	0.956522	0.910714
200	0.899306	0.915371	0.924007	0.915371	0.909722
300	0.907429	0.919134	0.922018	0.927024	0.903448

Table 2: Ratio of common edges in circle with diameter 3000

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