On Edges Elimination for the Shortest Mesh

Iveta Magová - Andrej Ferko - Ďudovít Niepel
Comenius University, 842 15 Bratislava, SLOVAKIA
kpgso@fmph.uniba.sk

Abstract. ***The construction of more representative wire frame models in 3D or the shortest mesh in the plane motivate the further research of the minimum weight triangulation problem. The paper gives the detailed state-of-the-art report on recent results.***

Keywords: ***optimal triangulation, NP-completeness, heuristics, mesh generation.***

Introduction

The unstructured mesh is a discretization of a geometric domain typically into a triangulation. The shortest mesh is a mesh with minimum total edge length. As pointed in [Chaz96], some mesh generation goals vary with the application, and there are a few public-domain codes (e.g. PLTMG, GEOMPACK), and some commercially available code (e.g. ICEM CFD). On the other hand, there is a list of open questions in mesh generation: heuristics for point-set triangulations with guaranteed behavior, design heuristics for constructing good meshes, volume mesh generation, point placement strategies, mesh partitioning, robustness, automatic remeshing, etc. As shown in [Boba94], very flexible practical solutions almost for each formalizable criterions could be obtained using the simulated annealing approach, e.g. the (sub)optimum for the shortest triangular mesh in the plane has been experimentally found in most cases in linear time by modification of the starting triangulation. Up to now, one of the open problems has attracted a lot of attention and remains open from 1979 [Gare79]. It is the Minimum Weight Triangulation (MWT) problem: for given set of points in the plane compute the triangulation with minimal total edge length. The solution of this problem is very interesting at least fromt from three different aspects: practical, theoretical, and perceptual ones. 1. The MWT mesh guarantees good behavior of rounding errors in FEM computation (where the MWT problem appeared first [Dupp70], and it could give better new heuristics for the solution of the Travelling Salesman Problem, [Anag94]. 2. The complexity status of the MWT-problem is unknown. 3. The shortest triangulation preserves the visual system of the operator providing the complete triangular visualization for a set of points using minimum light energy.

The MWT-problem is formulated as follows: **Let P be a given set of N points (sites) in the plane. The triangulation T(P) of P is a subdivision of the interior of convex hull of P into triangles with vertices from P. Compute the Minimum Weight Triangulation MWT(P) which minimizes the total edge length over all triangulations.**

Note that Euler's formula implies that each T(P) has 3N - 3 - K edges and 2N - 2 - K triangles where K is the number of CH(P) edges (CH stands for convex hull). Therefore the task means to identify either the MWT edges (among all N(N-1)/2 ones) or the MWT triangles (among all N(N-1)(N-2)/6 possible triangles). [Chiba85] showed that for N points and O(N) edges the maximum number of triangles is $O(N^{1.5})$.

The rest of the paper is organized as follows: 1 The detailed state-of-the-art report on recent MWT results, 2 Discussion from the point of view of eliminating edges, 3 some open questions.

1 Recent Results

The considerable progress in understanding of the nature of the MWT problem in the plane has been done recently. We can identify in the mewest results following four directions: 1 proving the NP-completeness for some generalizations or for close problems, 2 identifying the polynomially solvable instances of the problem, 3 computing exact solution, if possible, and 4 new heuristics. However, there are some more special research areas (e.g. MWT with Steiner points and their approximations, constrained triangulation, etc.) which we omit here.

1.1 NP-completeness

Let E(P) denotes all edges in the complete graph in the plane. [Lloyd77] proved the following problem related to MWT to be NP-complete. Triangulation existence: Given a set P of points in the plane, a set of line segments E' in E(P). Does E' contain a triangulation of P? [Ling83] showed that the problem of determining the minimum weight geometric triangulation of multiconnected polygons is NP-complete. [Heath94] has proved next two results. The crossing graph CG for the straight line drawing of a graph G represents all the crossings of the edges. The vertex in CG means the edge in G with the weight equal to the Euclidean edge length. The reformulated MWT problem (MWT, Optimization Version) is to find a maximal independent set of minimum total weight in CG. The first new problem (Restricted Maximal Independent Set) is the decision problem. For given rational positive number k is there a maximal independent set in CG such that the total weight is less or equal than k? The second new problem, Generalized MWT (GMWT) is formulated as follows: Given the set of points and the set of edges E' such that the set of edges contains a triangulation, and a positive rational number k. Is there a triangulation in E' whose weight is no greater than k? Both new problems were shown to be NP-hard. There is a claim in [Heath94] that the original MWT problem is NP-hard, too.

[Cheng94] has generalized the MWT to a constrained independence set where for the general minimization problem no polynomial-time algorithm is known but the paper exhibits (for the total edge length) the lower bound based on light edges. The light edge [Aich95] is not intersected by any shorter edge. MWT can consist of all light edges and their total length is the minimum. Another generalization of MWT-problem is mentioned in [Aich95]: the NP-complete assignment problem [Gare79] with certain constraint yields a solution to the original MWT-problem.

1.2 Polynomial-time Instances of the MWT-problem

Evidently, if the pointset admits a light triangulation, the MWT is polynomial (in this case MWT(P) incides with the greedy triangulation GT(P)). If the set of points forms a simple polygon then the MWT problem is solvable in cubic time. [Gilb79] and [Klin80] developed a dynamic programming algorithm. Their result (and cubic time) has been improved by [Heath94] who extended the case for a cell (a closed face in the planar graph without any unconneceted component inside) in [Heath92].In the identification of more and more general cases for which the MWT problem is polynomial, probably the most exciting result is given by [Anag93]: the pointsets with points placed on constant number of nested convex hulls (or on a constant number of parallel lines with one arbitrarily line) are polynomial instances of the problem. However, their dynamic programming algorithm is unfeasible, $O(N^D)$ where D is the depth of the pointset. The analogous case (points placed on the constant number of parallel lines) has been algorithmized in [Meij92]. Again, the dynamic programming polynomial algorithm is unfeasible, $O(N^L)$ where L is the number of lines. For regularly placed sites the result of [Yang94] applies. They showed for an MWT-edge a condition stronger than lightness in [Aich95] - all mutual neighbors in All Nearest Neighbor graph give the MWT edges. The application for regularly placed sites shows immediately the MWT. The weakest identification of a MWT-edge could be obtained using the result from [Gilb79] - the shortest edge(s) belong(s) to MWT.

1.3 Exact Solutions and Their Limitations

The trivial exact brute force algorithm has to test all triangulations. The number of them is exponential and therefore the results are obtainable for very small point sets. [Bart96] computed MWT for 15 points using a backtrack algorithm, [Heath94] employed in the backtrack procedure the prune and search paradigm which yielded optimum for sets up to 25 points. [Cheng95] developed a method with O(N^{U+2})) running time where U is the number of unconnected components (after the convex hull edges are added). Let us recall the Tree Representation of All Triangulations, [Boba93]. The root of this tree means DT(P), obtainable in O(NlogN) worst-case time (and linear memory), e.g. using the algorithm (and software) by Steven Fortune, based on [Fort87]. We can construct O(N) new triangulations using the Delaunay diagonal flip technique (in both directions) to exchange each edge which is not stable [Xu92] (i.e. unflippable). Assume now (with no loose of general validity) that each two edges in the given set are of different legth; this allows to sort them in an unique way, accordingly to the increasing edge length. Any new T(P) is generated by flipping of a particular edge. This gives an O(N) degree of the root node of the tree. Repeating this process with each new triangulation we obtain after quadratic number of iterations a tree representing all possible triangulations of given set P. Each no-leaf node is of degree O(N) or less, the leafs are of degree 1. The total depth of this tree is O(N^2), accordingly to [Fort93]. The total number of nodes is thus limited by O(N^{N*N})). Some node(s) of the tree represent(s) the desired MWT(P). The assumption that the root of this tree represents DT(P) is not necessary because using the result from [Fort93] there is a proof in [Boba93] that the quadratic number of Delaunay diagonal flips provides transformation of any triangulation T(P) to each different T'(P). (It is a path in the tree.)

In [Yang94] (and independently in [Bart96]) there is shown that all mutual neighbors in All Nearest Neighbor graph are the MWT edges. This means that the union of two surrounding circles (centered at the endpoints of the edge with radii equal to the edge legth) contains no next point of P. This generalizes the result from [Gilb79] - the shortest edge(s) belong(s) to MWT and to ANN, too. If the triangulation constists of equilateral triangles, this property gives the MWT. For general pointset the set of ANN double edges may have only one element - the shortest edge. Moreover, this edge could belong to the convex hull giving no contribution for the construction of MWT. [Keil94] computed a subgraph of MWT named b-skeleton, for $b=\sqrt{2}$. B-skeletons are polynomially computable Euclidean graphs introduced by [Kirk85]. Similarly to ANN double edges it is possible to give the edge characterization in terms of empty region. There are two variants of b-skeletons, based on lune like neighbourhood and disk-based one. For b > 1 the forbidden neighbourhood for edge endpoints x and y is defined to be the union of the two disks of radius b*d(x,y)/2 that pass through both x and y. The $\sqrt{2}$-skeleton can be computed in O(NlogN) time. The result of [Keil94] has been improved by [Yang95] for b = csc(2p/7) = 1.17682. There is little room to improve this beta because it is known that for b < 1/sin(p/3) or approximately b < 1.1547 there exists a four point counter-example in [Keil94].

An important progress has been done by [Dick96]. They discovered the LMT-skeleton as both theoretical and practical tool for identifying MWT edges using the local optimality of an edge in all pairs of possible empty adjacent triangles. Their algorithm is biquadratic, with cubic memory consumption. This can be improved by more sophisticated implementation. LMT-skeleton yields in many cases connected subgraph of MWT and the remaining small areas can be triangulated using dynamic programming algorithm by [Cheng95]. Previously impossible number of 250 points are exactly globally optimally triangulated. The faster implementation promises successful processing of maybe 500 points. The disadvantage of the algorithm is that the connectivity of LMT-skeleton is not guaranteed although it works on many general distributions [Dick96]. The extended LMT-skeleton uses the diamond property test. This provided better results and computing MWT for 400 points in 17 minutes, the expected time is quadratic. However, there is a proof in [Ferk96b] that there is O(N) possible unconnected components after LMT-skeleton is done. It seems to be a disadvantage of the LMT-skeleton that it gives no characterization of a MWT-edge in the terms of empty region. We can say that this is a statistical characterization instead of the geometric one which yields more efficient algorithmization. The LMT-skeleton idea has been generalized in [Ferk96a], as shown below.

Probably the most pesimistic result is given by [Ferk96b] where the unstability of MWT edges has been shown. The situation is shown in Fig. 1. Small difference in point placement gives completely different MWT. This leads to the - not new - claim that MWT problem belongs to NP.

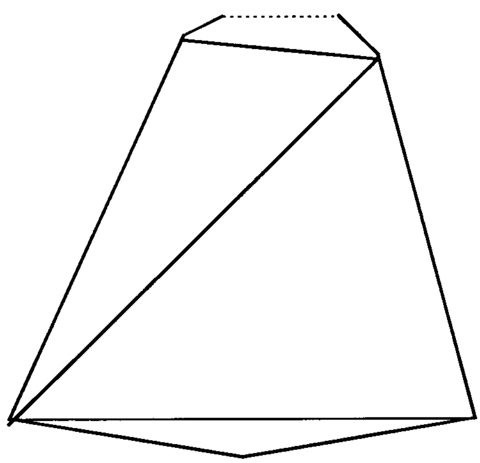
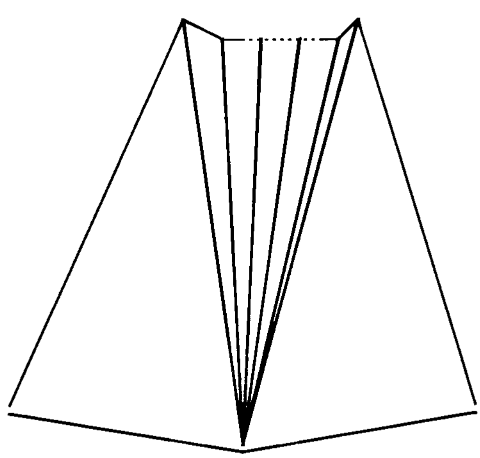


Fig. 1a): Convex case: the horizontal edge belongs to the MWT while in the concave case not.



b): Concave case

1.4 A Survey of New Heuristics

Up to now, there is no heuristics which guarantees the constant factor approximation of MWT. The best known one is that by [Plai87] providing the O(logN) approximation with $O(N^2 \log N)$ time. They use the EMST (Euclidean Minimum Spanning Tree) and CH edges to decompose the problem into the sequence of convex polygons which are then triangulated by ring heuristics. It makes no difference for the O(logN) approximation if the second step is completed using the optimal dynamic programming algorithm by [Gilb79]. Their Ring Heuristics used in the second step simply creates the edges which incrementally "cut the ears" from the convex polygon. The authors suppose that their approach has better (constant) approximation factor as it has been proven. The same is claimed by [Heath94]. This heuristics improved the idea by [Ling87] who computed the Delaunay triangulation (DT) and the minimum spanning tree to divide the input set into the cells which are triangulated by dynamic programming approach. [Heath94] improvement has been done in the first step where the greedy triangulation is involved instead of DT. The experimental results seem excellent but the proof of the approximation factor is missing. The disadvantage of the method is the cubic time which disables processing of bigger datasets [Dick95]. The other ideas in new heuristics use frequently the greedy strategy, combined with fixed radius search, two or three edges identification, employing the area to perimeter ratio or other measure for the added triangle, using the iterated ANN double edges, etc. ([Dick95], [Bart96]). In previous research it has been claimed that good approximations are DT(P) or GT(P), but there are examples which show that DT(P) can be the $\Theta(N)$ approximation [Levc87] and GT(P) can be the $\Omega(N^{0.5})$ approximation of MWT [Kirk80]. The same observation is given in [Mana79]. Thus both easy computable unique triangulations can be arbitrarily bad MWT approximations although they work well in the average case [Ling86].

2 Eliminating of Edges

The emphasis in a prospective future approach seems to be put on eliminating edges instead of seaking for MWT-edges. This shift in emphasis could be motivated by a simple observation. The edge eliminated simplifies the situation in recognizing the MWT-edges or MWT-triangles and finally this leads to the same result. Thus we have the task to eliminate $N(N-1)/2 - 3N + 3 + K = N(N - 7)/2 + K + 3$ edges what gives the halting criterion for any method. There are some known subsets of MWT edges. It should be focused on those which can be employed in edges elimination. There are two edge subsets which cannot help in other edges elimination and they must be in MWT (and in any T(P)). They are the convex hull edges and the stable edges introduced by [Xu92]. The stable edges are the intersection of all triangulations or in other words they are not intersected by any other edge from E(P). Note that the stable edges have been discovered independently by [Bart95] where is the proof that the number of stable edges (named there mandatory edges) can be linear. On the other hand, the set of stable edges (without convex hull edges) can be empty. The necessary but not sufficient condition for an edge to be in MWT is the diamond property. [Das89] proved that for every edge in MWT there are two triangular regions defined on both sides of the edge with base angle p/8 and at least one of the triangles contains no other point in P. [Dick96] observed experimentally that this pretest eliminates the quadratic number of edges. For a uniform distribution the expected number of edges that satisfy the diamond property is O(N), too. The proof for general pointset is missing.

The light edge [Aich95] is not intersected by any shorter edge. If the cardinality of the set of light edges equals to 3N - 3 - K then all MWT edges are light. Unfortunately, there exist light

The light edge [Aich95] is not intersected by any shorter edge. If the cardinality of the set of light edges equals to 3N - 3 - K then all MWT edges are light. Unfortunately, there exist light edges which are not in MWT. This can be easily proven by a 5-point example, e.g. in [Aich95] and [Bart96]. We recall this example for other reason below in Fig. 2, the light edge is edge CE. Therefore the light edge cannot eliminate in general the edges, but this is provided by ANN double edges and by the LMT-skeleton edges.

It is possible to obtain more MWT edges using Generalized LMT-skeleton [Ferk96]. A pentagon has two diagonals and three triangles. The local minimality has been generalized as follows. Instead of one one-flip operation in a quadrilateral we assume more two-flips operations in a pentagon. If an edge (together with some other diagonal in the pentagon: these are the edges of a quadrilateral where the actual edge is a diagonal) is minimal within all possible triplets of triangles, then the edge belongs to any LMT, and therefore into the MWT. This must be done for successful computing the following five-points example: In the pentagon (A(0,0) B(1,0) C(1,1) D(0.5,1.002) E(0,1)) the shortest diagonal ((1,1) (0,1)) gives the GT different from MWT. MWT is given by other edges ((0,0) (0.5,1.002)) and ((1,0) (0.5,1.002)) with the total "diagonal" sum 2.24. (GT has the same value equal to 2.414...). In this point set, the MWT cannot be obtained using quadrilaterals because all "global" information is necessary.

The newest MWT oriented paper seems to be [Levc97].

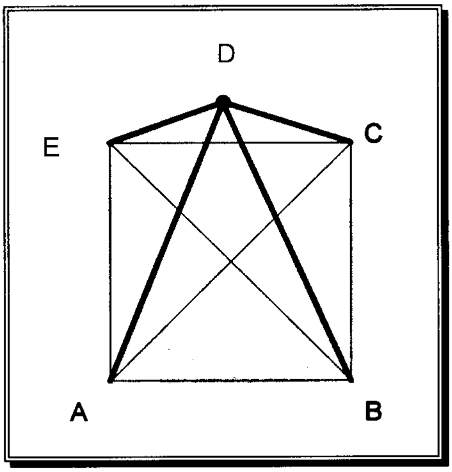


Fig. 2: MWT (heavy lines) cannot be obtained using quadrilateral based definition of local minimality.

3 Open Questions

Up to now nobody gave the proof for the complexity status of the MWT problem. The other open questions extend the problem to 2..5D and 3D where is the key for construction of arbitrarily "good" terrains, wire frame models or meshes with better understanding of the nature of optimal triangulation eventually combined with better solutions in point placement strategies.

References

[Aich95] Aichholzer, O. et al. Triangulations Intersect Nicely 11. ACM Symp. on Comp. Geom. pp. 238-247, Vancouver 1995

[Anag93] Anagnostou, E. - Corneil, D. Polynomial-time instances of the MWT problem Computational Geometry, Theory and Applications 3 (1993) 247-259

[Bart96] Bartanus, M. et al. New Heuristics for MWT WSCG96, Pilsen 1996

[Boba94] Bobáková, G. et al. On Minimum Weight Triangulation SCCG95, Bratislava 1995

[Cern82] Cerny, V. 1982 A Thermodynamical Approach to the Travelling Salesman Problem: An Efficient Simulation Algorithm, J. Opt. Th. and Appl.

[Chaz96] Chazelle, B. et al. Application Challenges to Computational Geometry, http://www.cs.princeton.edu/~Chazelle/taskforce/CGreport.ps, TR-521-96, Princeton University, April 1996

[Cheng95] Cheng, S. et al Expected Case Analysis of b-skeletons with Applications to the Construction of Minimum Weight Triangulation, CCCG Conf. Proc. pp. 279-284, Quebec City 1995

[Chiba85] Chiba, N. - Nishizeki, T. Arboricity and subgraph listing algorithms SIAM J. Computing 14, pp. 210-223, 1985

[Das89] Das, G. - Joseph, D. Which Triangulations Approximate the Complete Graph, Lecture Notes in Computer Science 401, pp. 168-192, Springer 1989

[Dick95] Dickerson, M.T. et al. New Algorithms and Empirical Findings on MWT Heuristics 11. ACM Symp. on Comp. Geom. pp. 278-284, Vancouver 1995

[Dick96] Dickerson, M.T. - Montague, M.H. A (Usually?) Connected Subgraph of the Minimum Weight Triangulation ACM Symp. on Comp. Geom., 1996

[Dupp70] Duppe, R.D. - Gottschalk, H.H. Automatische Interpolation von Isolinien bei willkurlichen Stutzpunkten, Allgemeine Vermessungsberichten 77, pp. 423-426, 1970

[Ferk96] Ferko, A. More MWT Edges using Generalized LMT-skeleton, technical report of Department of Computer Graphics and Image Processing No. 96/1) Comenius University, January 26, 1996

[Ferk96b] Ferko, A. Criticism of Hunting MWT Edges, SCCG96, pp. 259-264, Budmerice 1996

[Fort93] Fortune, S. 1993 A note of Delaunay flips, Pattern Recognition Letters Vol. 14 No. 9, North-Holland pp. 723-726

[Gabo72] Gabow, H. An efficient implementation of Edmond's maximum matching algortihm, Tech. Rep. 31, Computer Science Department, Stanford Univ., 1972

[Gare79] Garey, M.R. and Johnson, D.S. Computers and Intractability, Freeman, San Francisco, 1979

[Kirk80] Kirkpatrick, D. G. 1980 A note on Delaunay and optimal triangulations, Inf. Process. Lett. 10, pp. 127-128

[Heath92] Heath, L.S. - Pemmaraju, S.V. New Results for the MWT Problem Report No. TR 92-30, Dept. of Comp. Science, Virginia Polytechnic and State University 1992

[Heath94] Heath, L.S. - Pemmaraju, S.V. New Results for the MWT Problem Algorithmica (1994) 12: 533-552

[Kirk82] Kirkpatrick, S. - Gelatt Jr., C. D. - and Vecchi, M. P. 1982 Optimization by Simulated Annealing, IBM Tech. Report, also: Science 220 (1983) pp. 671-680

[Kirk85] Kirkpatrick, D.G. - Radke, J.D. A Framework for computational morphology, pp. 217-248 in: Toussaint, G.T., ed., Computational Geometry, Elsevier 1985

[Levc87] Levcopoulos, C. An O(N^0.5) lower bound for the non-optimality of the greedy triangulation, Inf. Process. Lett. 25, pp. 247-251

[Levc97] Levcopoulos, C. - Krznaric, D. A Near-Optimal Heuristic for MWT of Convex Polygons, SIAM Symposium on Discerete Algorithms, January 1997 (to appear)

[Ling83] Lingas, A. The greedy and Delaunay triangulations are not bad in the average case and MWT of multiconnected polygons is NP-complete, Lecture Notes in Computer Science 158, pp. 270-284, Springer 1983

[Ling86] Lingas, A. 1986 The greedy and Delaunay triangulations are not bad in the average case, Inf. Process. Lett. 22, pp. 25-31

[Mana79] Manacher, G.K. - Zobrist, A.L. Neither the greedy nor the Delaunay triangulation of a planar point set approximates the optimal triangulation, Inf. Process. Lett. 9, pp. 31-34

[Meij92] Meijer, H. - Rappaport, D. Computing the MWT of a set of linearly ordered points Information Processing Letters 42 (1992) 35-38

[Plai87] Plaisted, D.A. - Hong, J. A Heuristic Triangulation Algorithm J. Algorithms 8, pp. 405-437, Academic Press 1987

[Prep85] Preparata, F. P. - Shamos, M. I. Computational Geometry, An Introduction, Springer 1985

[Sham75] Shamos, M.I. - Hoey, D. 1975 Closest Points Problems. In Proceedings of the 16th Annual IEEE Symposium on FOCS, pp. 151-162.