# Improving the zonal method through the use of series developments to approximate volume/volume form factors

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Abstract

This paper introduces a new acceleration technique for the zonal method. We present mathematical developments which improve, in a considerable way, the time due to the form factor calculus. More precisely, we show that, under the assumption of classical modeling conditions, we can :

- simplify the mathematical expression of the volume/volume form factor,
- approximate this simplification by a series development of orthogonal polynomials with a complete error control. Analog results can be obtained for the other types of form factors.

Keywords : radiosity, participating medium, zonal method, form factor.

## 1. Introduction

The radiosity technique has its basis in the theory of heat transfer or interchange between surfaces [SH92]. A system of equations describes the interreflections in a closed environment. The zonal method extends this approach to participating medium by following the same discrete approach : surfaces are meshed into surface elements (patches) and the volume containing the medium is subdivided into discrete volume elements (voxels) across which the radiosity (i.e. the energy leaving an element) is assumed to be constant [SP94].

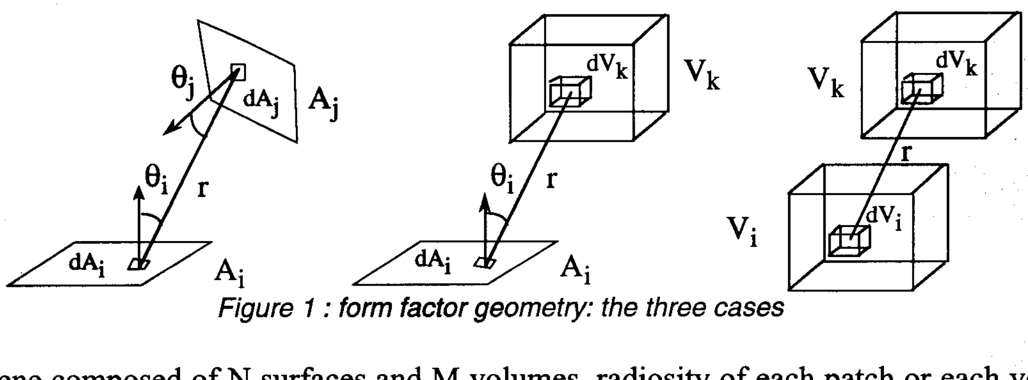



*Figure 1 : form factor geometry: the three cases*

For a scene composed of N surfaces and M volumes, radiosity of each patch or each volume is defined, with the notations of figure 1, by the couple of equation systems (1) and (2) :

- For a surface $A_i$ : $$B_i A_i = E_i A_i + \rho_i \left[ \sum_{j=1}^{N} B_j F_{A_j \to A_i} + \sum_{k=1}^{M} B_k F_{V_k \to A_i} \right] \qquad (1)$$

where $B_i, B_k$ : radiosity of the surface $A_i$ and the volume $V_k$,
$E_i, \rho_i$ : emissivity and reflectivity coefficient of $A_i$,

$$F_{A_j \to A_i} = F_{A_i \to A_j} = \iint_{A_j A_i} \tau(r) \frac{\cos\theta_i \cos\theta_j}{\pi r^2} dA_j dA_i$$ : form factor between $Aj$ and $Ai$,

$$F_{V_k \to A_i} = F_{A_i \to V_k} = \iint_{V_k A_i} \tau(r) \frac{K_t(k) \cos\theta_i}{\pi r^2} dV_k dA_i$$ : form factor between $Vk$ and $Ai$,

$K_t(k)$ : extinction coefficient of $Vk$.

- For a volume $Vi$ : $$4K_t(i) B_i V_i = 4K_a(i) E_i V_i + \Omega_i \left[ \sum_{j=1}^{N} B_j F_{A_j \to V_i} + \sum_{k=1}^{M} B_k F_{V_k \to V_i} \right] \quad (2)$$

with $\Omega_i, K_a(i)$ : scattering albedo (similar to the diffuse reflectivity term for a surface) and absorption coefficient of $Vi$,

$$F_{V_k \to V_i} = F_{V_i \to V_k} = \iint_{V_i V_k} \tau(r) \frac{K_t(i) K_t(k)}{\pi r^2} dV_i dV_k$$ : form factor between $Vk$ and $Vi$.

Form factor evaluation is the heart of this method. These various form factors are similar to those for conventional radiosity [CW93], except that they include the effect of attenuation through the medium. The function $\tau(r) = e^{-\int_{medium} K_t(x)dx}$ is called *transmittance* and expresses the fraction of light passing through the medium without being absorbed and scattered. Occlusion problems and transmittance term are solved simultaneously by a classical ray tracing algorithm between two voxels (resp. patches). This algorithm either detects intersections with another patch of the scene [AM95a] or compute the transmittance $\tau(r)$ by integrating along the considering path. More generally, form factors are computed with numerical methods, specially in [RT87], where H. Rushmeier and K. Torrance extend the hemi-cube principle [CG85] to calculate the surface/volume form factor. The value of the form factor between two volumes $Vi$ and $Vj$ is approximated by the following formula :

$$F_{V_i \to V_j} = \iint_{V_i V_j} \tau(r) \frac{K_t(i) K_t(j)}{\pi r^2} dV_i dV_j \approx \frac{\tau(r) K_t(i) K_t(j)}{\pi r^2} V_i V_j \quad (3)$$

where $r$ is the distance between the two volumes.

This simple expression is efficient when both volumes are wide apart (relatively to their sizes), however it produces an important error when they are close.

In this paper, we present a new method that approximates the values of the volume/volume form factors, by developing the form factor integral into Gegenbauer polynomial series, with the control of the generated error.

## 2. Simplifying hypothesis

In a previous paper [AM95b], we demonstrated that the exit of the radiometric characteristics ($\tau(r), K_t(m)$ and $K_t(k)$) from the integrals defining the form factors, leads to a small error (<1%) under classical modeling hypothesis. This assertion is trivial when the two volumes are faraway, but we will demonstrate that the generated error is small also when volumes are close.

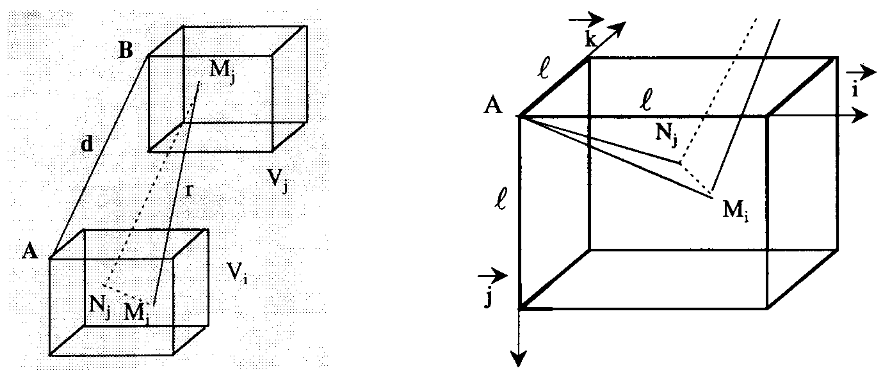
**Proposition 1**
For two volumes $Vi$ and $Vj$ of size $\ell$ (the size of an edge), the following approximation

$$F_{V_i \to V_j} = \iint_{V_i V_j} \tau(r) \frac{K_t(i) K_t(j)}{\pi r^2} dV_i dV_j \approx \tau(d) K_t^2 \iint_{V_i V_j} \frac{1}{\pi r^2} dV_i dV_j \qquad (4)$$

generates a relative error of the order of $C(\vec{u}) K_t \ell$ where $C(\vec{u})$ only depends on the relative position of the two volumes in the same medium.

**Demonstration**

Let us consider two volumes $V_i$ and $V_j$ in the same medium characterized by $K_t$. Moreover $V_i$ and $V_j$ own to the same regular subdivision of the space (voxmap).




*Figures 2 and 3 : Hypothesis and notations*

Since $K_t$ is constant, the transmittance function becomes

$\tau(r) = exp(-\int_{medium} K_t(x) dx) = exp(-K_t r)$ where $r = \left| \overrightarrow{M_i M_j} \right|$ (according to figure 2).

We want to evaluate the error between $\iint_{V_i V_j} \tau(r) \frac{K_t(i) K_t(j)}{\pi r^2} dV_i dV_j$ and the approximation $\tau(d) K_t^2 \iint_{V_i V_j} \frac{1}{\pi r^2} dV_i dV_j$ where $\tau(r) = exp(-K_t d)$, with $d$ the distance between $A$ and $B$ (figure 2).

This error is expressed by $e = \dfrac{\iint_{V_i V_j} \frac{exp(-K_t r) - exp(-K_t d)}{r^2} dV_i dV_j}{\iint_{V_i V_j} \frac{exp(-K_t r)}{r^2} dV_i dV_j}$ and we are interesting in its rough estimate when $\ell \to 0$.

Consider $N_j$ the point of $V_i$ translated of $M_j$ by $\overrightarrow{BA}$.(figure2). We have the relation $\overrightarrow{M_i M_j} = \overrightarrow{M_i N_j} + \overrightarrow{AB}$. Let us now consider $\ell$ the length of one edge of $V_i$. We can replace $\overrightarrow{AM_i}$ by $\ell \overrightarrow{Am_i}$ and $\overrightarrow{AN_j}$ by $\ell \overrightarrow{An_j}$, where $m_i$ and $n_j$ own to the local coordinate system $(A, \vec{i}, \vec{j}, \vec{k})$ of $V_i$ (figure 3). Let $\vec{u}$ defined by $\overrightarrow{BA} = \ell \vec{u}$ . $\vec{u}$ characterizes (with integer coordinates) the relative position of the two volumes in the voxmap. For example, if $\vec{u} = \pm \vec{i}$ or $\vec{u} = \pm \vec{j}$ or $\vec{u} = \pm \vec{k}$, the couple of voxels have a common face. In this case

$\overrightarrow{M_iM_j} = \overrightarrow{M_iN_j} + \overrightarrow{AB}$ is developed in $\overrightarrow{M_iM_j} = \ell\left(\overrightarrow{m_in_j} + \vec{u}\right)$ and $r = \ell\left|\vec{u} + \overrightarrow{m_in_j}\right|$

and error becomes
$$e = \frac{\ell^6 \iint\limits_{[0,1]^3[0,1]^3} \frac{exp\left(-K_t\ell\left|\vec{u}+\overrightarrow{m_in_j}\right|\right) - exp(-K_t\ell|\vec{u}|)}{\ell^2\left|\vec{u}+\overrightarrow{m_in_j}\right|^2} dm_i dn_j}{\ell^6 \iint\limits_{[0,1]^3[0,1]^3} \frac{exp\left(-K_t\ell\left|\vec{u}+\overrightarrow{m_in_j}\right|\right)}{\ell^2\left|\vec{u}+\overrightarrow{m_in_j}\right|^2} dm_i dn_j}.$$

When $\ell$ tends towards 0, we obtain the following equivalence :

$$e \underset{\ell\to 0}{\approx} -K_t\ell \frac{\iint\limits_{[0,1]^3[0,1]^3} \frac{\left|\vec{u}+\overrightarrow{m_in_j}\right| - |\vec{u}|}{\left|\vec{u}+\overrightarrow{m_in_j}\right|^2} dm_i dn_j}{\iint\limits_{[0,1]^3[0,1]^3} \frac{1}{\left|\vec{u}+\overrightarrow{m_in_j}\right|^2} dm_i dn_j} = K_t\ell\left(|\vec{u}| - \frac{\iint\limits_{[0,1]^3[0,1]^3} \frac{1}{\left|\vec{u}+\overrightarrow{m_in_j}\right|} dm_i dn_j}{\iint\limits_{[0,1]^3[0,1]^3} \frac{1}{\left|\vec{u}+\overrightarrow{m_in_j}\right|^2} dm_i dn_j}\right)$$

And denoting $C(\vec{u}) = \left(|\vec{u}| - \frac{\iint\limits_{[0,1]^3[0,1]^3} \frac{1}{\left|\vec{u}+\overrightarrow{m_in_j}\right|} dm_i dn_j}{\iint\limits_{[0,1]^3[0,1]^3} \frac{1}{\left|\vec{u}+\overrightarrow{m_in_j}\right|^2} dm_i dn_j}\right)$, we demonstrate the proposition 1.

In practice, if $L$ denotes the voxmap size and $N$ the number of voxels in a direction, $K_t$ is of the order of $1/L$ [RT87] and $\ell$ is equal $L/N$. Finally, under these assumptions we obtain $e \approx K_t\ell C(\vec{u}) = \dfrac{C(\vec{u})}{N}$.

If we compute numerically some values of $C$, we have for example :

- $C(\vec{i}) = -0.100917$ : the maximum value (modulus) of $C(\vec{u})$.
- $C(100\vec{i}) = 0.005045$.

For $N=30$ (a typical modeling value) and for two voxels with a common face, the error $e$ is around 0.3%.

By now, the mathematical hypothesis (4) allows us to only take into account the geometrical term : $\mathcal{F}_{V_i \to V_j} = \iint\limits_{V_iV_j} \frac{1}{\pi r^2} dV_i dV_j$.

### 3. Proximity development

#### 3.1. Main principle

In a previous paper [AM95a], we developed a similar algorithm for classical radiosity. We consider the calculus of the form factors between patch $B$ and the patches belonging to the mesh of a $C^{\infty}$ regular surface (figure 4).

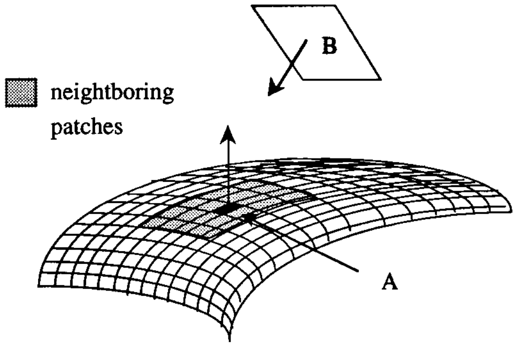


*Figure 4 : Neighboring patches*

When knowing the form factor between $B$ and $A$ is known, we can efficiently approximate form factors between $B$ and the neighboring patches of $A$ , instead of computing independent values. As a consequence, we only need to choose a set of reference patches, well chosen on the surface, and we can obtain all the form factor values by simple approximations.

## 3.2. Form factor optimization

An analog result can be found with the volume/volume form factors.

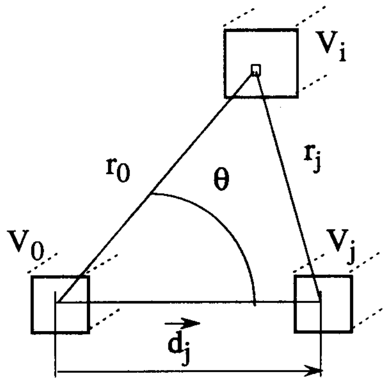


*Figure 5 : Proximity Optimization*

Using the notations of figure 5 and assuming that $\mathcal{F}_{V_i \to V_0}$ is already computed, we propose a mathematical relation between $\mathcal{F}_{V_i \to V_j}$ and $\mathcal{F}_{V_i \to V_0}$.

**Proposition 2**

If $\dfrac{d_j}{r_0} < 1$ then $\mathcal{F}_{V_i \to V_j} = \mathcal{F}_{V_i \to V_0} + \dfrac{1}{\pi} . \displaystyle\sum_{n=1}^{\infty} \int_{V_i} \int_{V_0} \frac{d_j^{\,n}}{r_0^{\,n+2}} . \frac{\sin((n+1)\theta)}{\sin\theta} dV_i dV_0$ (5)

**Demonstration**

By expressing $\dfrac{1}{r_j^2}$ in function of $\dfrac{1}{r_0^2}$, we obtain $\dfrac{1}{r_j^2} = \dfrac{1}{r_0^2} . \dfrac{1}{(1 - 2p\cos\theta + p^2)}$ where $p = \dfrac{d_j}{r_0}$.

If $\dfrac{d_j}{r_0} << 1$, we can use the classical series development into Gegenbauer polynomials $C_n^{\lambda}$ (see [GR80 p. 1029]) :

$$\frac{1}{(1-2p\cos\theta+p^2)}=\sum_{n=0}^{\infty}C_n^1(\cos\theta).p^n=\sum_{n=0}^{\infty}\frac{\sin((n+1)\theta)}{\sin\theta}.p^n$$

in order to obtain :

$$\frac{1}{r_j^2}=\frac{1}{r_0^2}\sum_{n=0}^{\infty}\left(\frac{d_j}{r_0}\right)^n\frac{\sin((n+1)\theta)}{\sin\theta} \qquad (6)$$

Substituting (6) in the definition of the form factor, we establish proposition 1 :

$$\mathcal{F}_{V_i\to V_j}=\int_{V_i}\int_{V_j}\frac{1}{r_j^2}dV_idV_j=\frac{1}{\pi}\sum_{n=0}^{\infty}\int_{V_i}\int_{V_0}\frac{d_j^n}{r_0^{n+2}}.\frac{\sin((n+1)\theta)}{\sin\theta}dV_idV_0$$

or $$\mathcal{F}_{V_i\to V_j}=\mathcal{F}_{V_i\to V_0}+\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{V_i}\int_{V_0}\frac{d_j^n}{r_0^{n+2}}.\frac{\sin((n+1)\theta)}{\sin\theta}dV_idV_0 \qquad (7)$$

### 3.3. Algorithm

The previous formula (7) can be simplified by considering a strip of voxels.

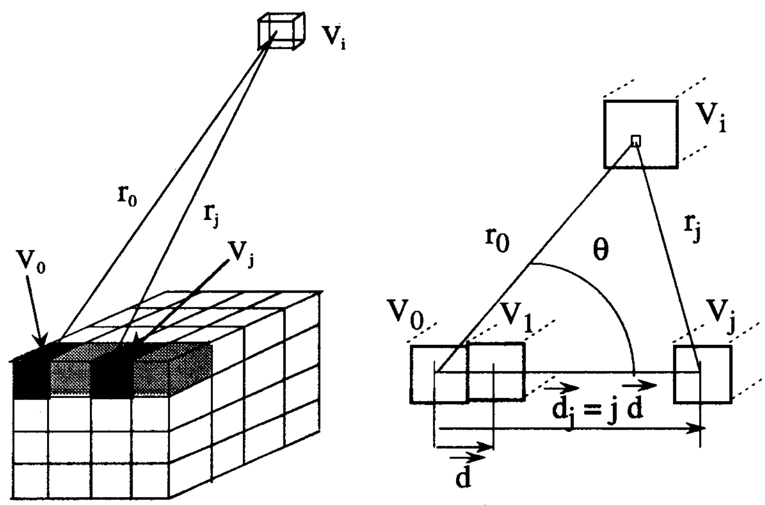



*Figure 6 : For a strip of voxels*

In this case (figure 6), we have $\vec{d_j}=j\vec{d}$ where $d$ is the length of a voxel and $j$ the index of the generic voxel in the strip. Substituting this relation in (7) gives finally :

$$\mathcal{F}_{V_i\to V_j}=\mathcal{F}_{V_i\to V_0}+\frac{1}{\pi}\sum_{n=0}^{\infty}j^n.d^n\int_{V_i}\int_{V_0}\frac{1}{r_0^{n+2}}.\frac{\sin((n+1)\theta)}{\sin\theta}dV_idV_0 \qquad (8)$$

or $$\mathcal{F}_{V_i\to V_j}=\mathcal{F}_{V_i\to V_0}+\frac{1}{\pi}\sum_{n=0}^{\infty}j^n.d^nP_n \text{ for j / } |j|<\frac{r_0}{d} \qquad (9)$$

with $$P_n=\int_{V_i}\int_{V_0}\frac{1}{r_0^{n+2}}.\frac{\sin((n+1)\theta)}{\sin\theta}dV_idV_0 \qquad (10)$$

If we replace the series development by the sum of its $N$ first terms, the error is a function of the order $N$. This approximation allows us to create the next algorithm :

```
Algorithm Formfact(Voxel Vi,Voxmap V) {
      for each strip of the voxmap V do {
            Choose one or more (if necessary) reference volumes in order to
            satisfy the constraint |j| < r0/d
            Compute the associated form factors
            Compute the set of N integrals Pn with formula (10)
            Approximate all form factors of the strip using formula (9)
      done
end
```

In practice, we can decrease time calculation by important factors. This is due to the simultaneous computations of the $N$ integrals $P_n$ where lots of terms are common or similar.

## 4. Results

### 4.1. Error estimation

Let us consider the simple example of figure 7 where we compute form factors between a voxel and a strip of 10 voxels. The first voxel $V_0$ is the reference voxel and the ultimate voxel $V_{10}$ has been chosen at the validity limit of the criteria $\frac{d_j}{r_0} < 1$.

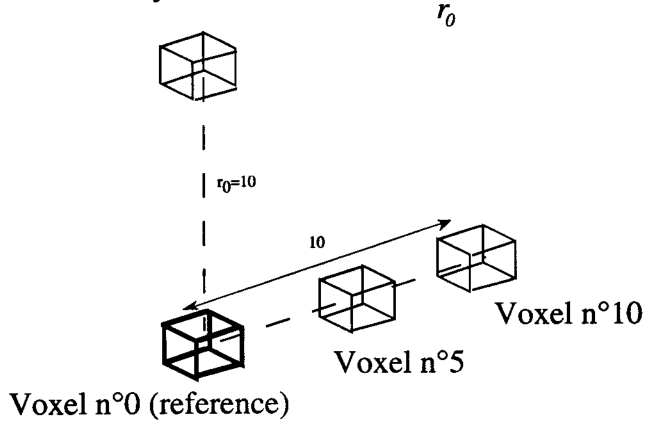



*Figure 7 : Test scene*

The following curves present the relative error between our new approximate method and an "exact" numerical integration (Gauss method).

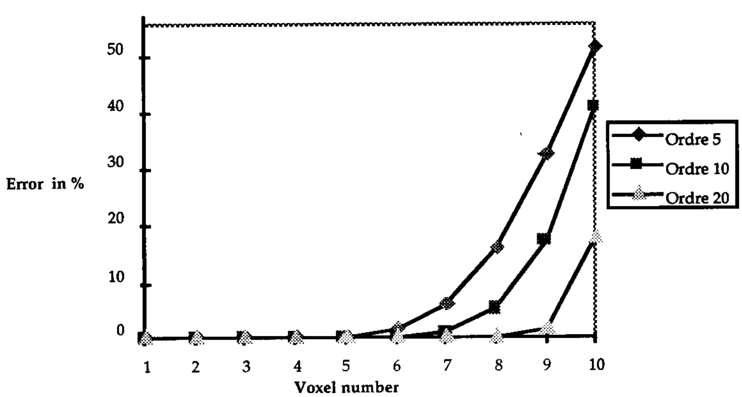



*Figure 8 : Relative errors*

Each curve corresponds to a different development order $N$ and presents the error in function of the position of the voxel (from 1 to 10). Curves have the same general characteristics : for a given strip, errors are stable and small (<1%) up to a given threshold and then increase quickly when the ratio $\frac{d_j}{r_0}$ tends to 1. In practice, by choosing a high value of $N$ (20 or 30) and a small ratio $\frac{d_j}{r_0}$ (for example 0.5), we can always stay in the minimal error zone of the curve.

### 4.2. Computation time

Next computational times are obtained on a classical IBM RS6000 workstation. For a simple configuration (a single voxel looking at a set of voxels in figure 9), we compare times obtained with a classical numerical integration (with Gauss technique), with those obtained with our previous approximation (formula 8).

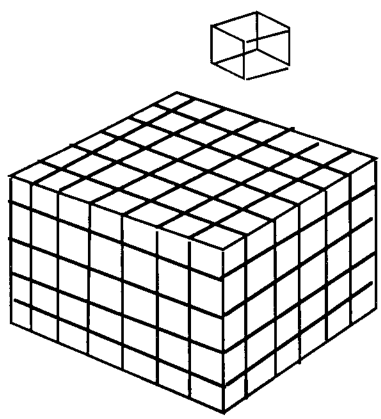


*Figure 9 : A simple test*

| | 10*10*10 voxels | 20*20*20 voxels |
|---|---|---|
| Exact volume/volume method (Gauss integration) | 91 s. | 680 s. |
| Approximation with a Gegenbauer polynomial series | 20 s. | 100 s. |

In practice, our results are from 3 to 20 time faster, depending on the number of voxels in a strip and on the relative position of the objects.

### 4.3. Pictures

Next pictures present two different participating media. In one hand, stained-glass windows are defined by a participating medium, strongly absorbing and slightly diffusive. On the other hand, the medium representing the air (smoke, dust ...) is slightly absorbing and strongly diffusive.

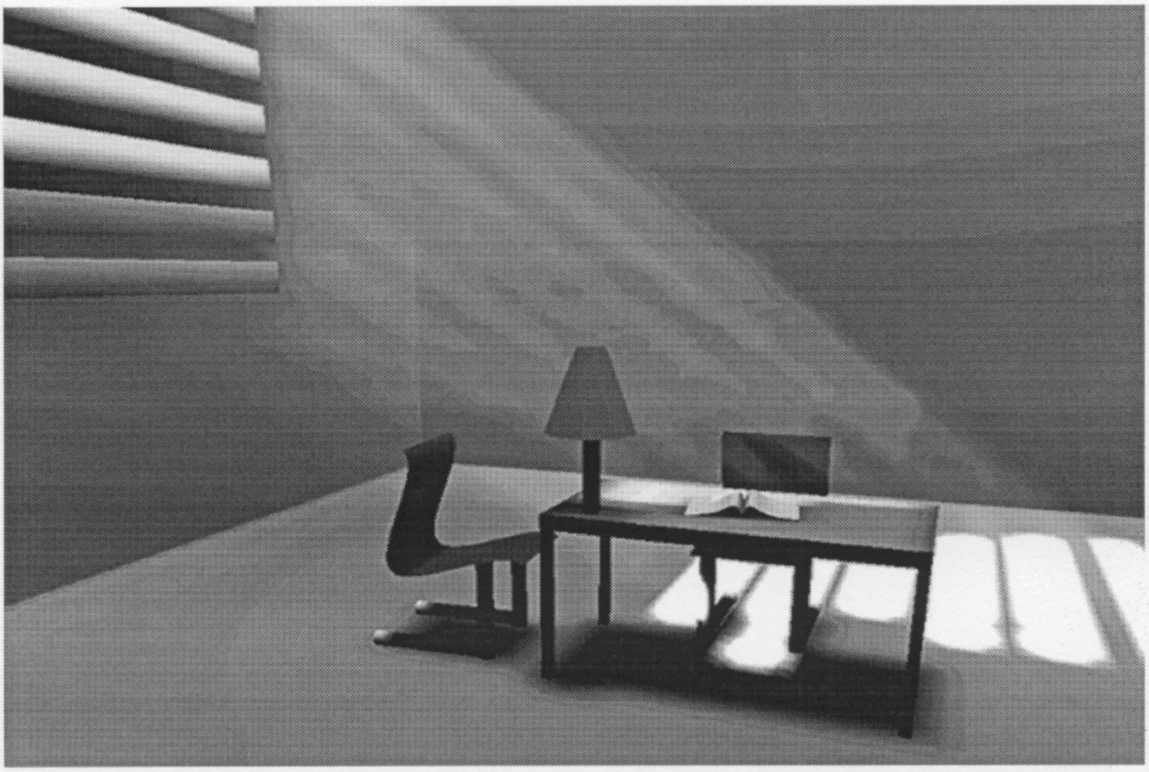


*Figure 10 : A scene including participating medium*

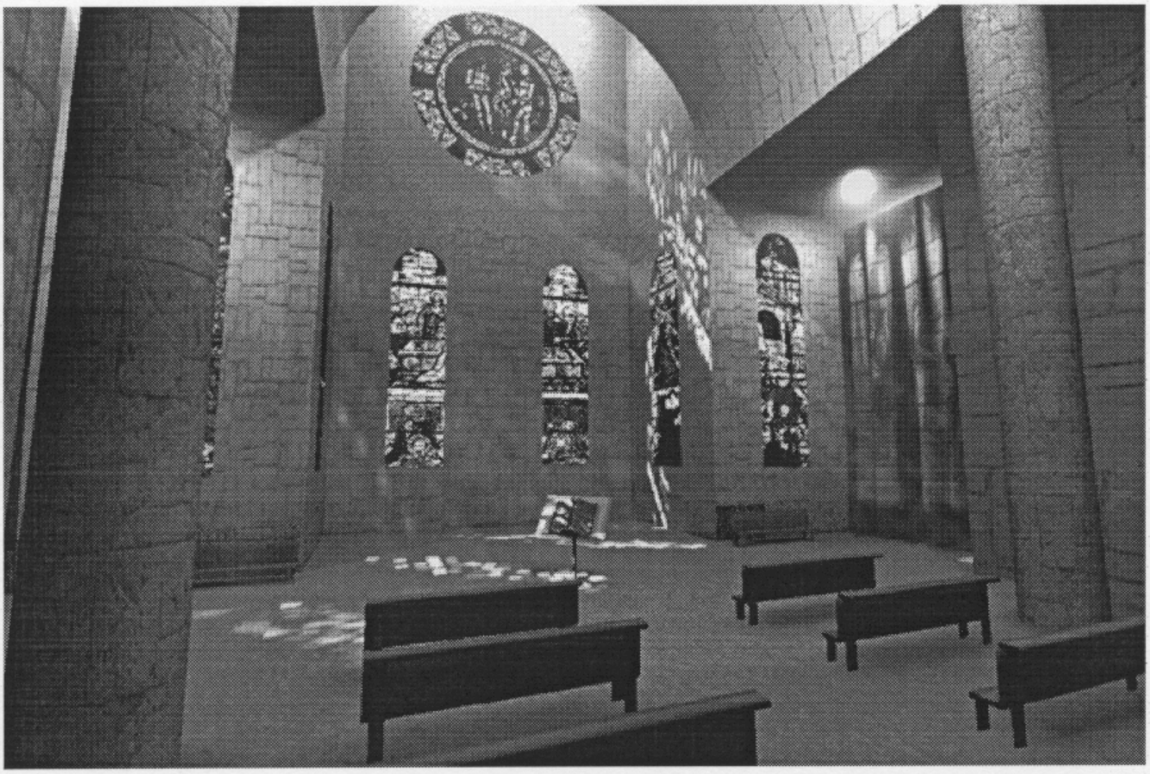


*Figure 11 : A church with texture and stained-glass windows*
*About 120000 patches and 1,5 million of voxels*

*Figure 12 : Another point of view in the church*

## 5. Conclusion

In this paper, we have presented a new technique, that allows us to considerably decrease, with an accurate error control, computation times due to the zonal method . The computation of the form factor values is accelerated by using a series development of orthogonal functions (Gegenbauer polynomials).

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