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Linear elastic analysis of thin laminated beams with uniform and symmetric cross-section

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Abstract

This paper deals with analyses of linear elastic thin beams which are consisted of the homogeneous orthotropic layers. The cross-sections of these beams are assumed uniform and symmetric. Governing equations of one-dimensional model are derived on the base of the Timoshenko's beam theory. An evaluation of shear correction factor consists in conservation of the shear strain energy. This factor is calculated in this paper but only in the cases of the static problem. The general static solution for the flexural and axial displacement and for the slope of the cross-section is found. Further, the possibility of calculation of the free vibrations of beams are also presented. The obtained results for the static solution are compared with the results of numerical solution based on the finite element method. The numerical model is prepared in software package MARC. As a tested example is used the uniformly loaded simply supported beam with various cross-sections.

Keywords: linear elastic beam, orthotropic layers, Timoshenko's beam theory, static solution, free vibrations

1. Introduction

At the beginning it should be pointed out that three theories are well-known in the development of the governing equations for thin beams. The first of them takes assumption that the transverse shear strains are negligible and planes of cross-section before bending remain plane and normal to the axis of the beam after bending. This classical beam theory is called Bernoulli-Euler. But the study of wave propagation in the Bernoulli-Euler beam showed that infinite phase velocities were propagated. Therefore Rayleigh applied the correction for rotary inertia. Then, the obtained results by the Rayleigh theory predicted finite propagation of velocities where their upper bounds were still greater than exact results. In 1921 Timoshenko suggested mathematical model which influence of rotary inertia and shear deformation are incorporated. The results of this theory were in good accord with the reality. More detailed description and comparison of these theories in cases of flexural waves in elastic and isotropic beams could be found in [3] and [5].

The problems of beam deformation are wide-spread and could be performed by analytical and semi-analytical approaches or by numerical methods. Analytical solutions could be either in closed form or in infinite series and could be solved by exact governing equations or could be based on variation approaches. On the other hand, the finite element method is one of the most used numerical method. The summary of computational methods including eigenvalues problem or the Fourier method which may be used to beam calculation [4] gives.

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The importance of development of methods of analyzing composite beams is connected on the one hand with the use of beams as basic elements of structures and on the other hand with the identification of mechanical properties by bending test on samples. Composite beams are very often manufactured in the form of two or more unidirectional laminae or plies stacked together at various orientations. Since beam structures are similar to that of plate composite structures, the theories for the modeling of them are the same. Two theories are commonly considered in connection with laminated materials, namely the classical laminate theory and the first-order shear deformation theory. These theories are so-called equivalent single-layer theories and are derived from the three-dimensional elasticity theory by taking assumptions about the kinematics of deformation and/or the stress distribution through the thickness of a laminate or a sandwich. With the help of these assumptions the mathematical model is reduced from a 3D-problem to a 2D-problem. Both theories (the classical laminate including the shear deformation) listed above are presented in [1] and [2]. In these references, we can find the one-dimensional solution of bending of laminate and sandwich beams with predominantly rectangular cross-sections. Also some basic free vibrations and buckling problems are shown in [1] and [2]. The higher order equivalent single layer theories by using higher order polynomials is also studied in [1].

In this paper the first order shear deformation theory is applied to calculation of the flexural and the axial displacement including the slope of the cross-section that are linked together. Furthermore, the beams are assumed thin that is why the effect of Poisson's ratio is negligible and twisting and transversally bending are not considered.

2. Governing equations of laminated thin beam

Let us consider a straight beam consisting from n layers which are perfectly bonded and are numbered from the lower to the upper face. The overall thickness of the laminated beam is h. Layers are of homogenous, orthotropic and linear elastic materials. Furthermore, the rectangular Cartesian coordinate system x_1, x_2 and x_3 is used. The orientation of coordinate axes are defined in accordance to fig. 1 where the x_1 is parallel to the longitudinal beam axis and the x_3 is directed in the direction of increasing number of the layers. Each layer k is referred to by the x_3 coordinates of its lower face h_{k-1} and upper face h_k as shown in fig. 1. Besides, orthotropic properties of layers are referred to their material axes. Angles of rotation are denoted by θ_k.

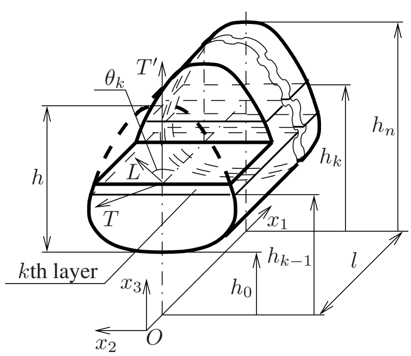


Fig. 1. A thin laminated beam with a symmetric cross-section

2.1. Strain-displacement relations

In the following subsection there have to be used another assumptions before we develop governing equations. The cross-section area of beams can have various shapes but must be uniform along the x_1-axis and symmetric to the x_3-axis. The thickness h and the width $b(x_3)$ are small relative to the beam length l. The general combination of lateral and axial loading may be applied but only bending and stretching in the $x_1 - x_3$ plane of symmetry can exist. We must, however, notice the greatest attention to this latter condition. In fact, the laminate constitutive equation shows, see [1], that Poisson's effects may cause deformations not only in the $x_1 - x_3$ plane. This effect can be neglected in cases where the length-to-width ratio is sufficiently high. Under these assumptions, the displacement field based on the first-order shear deformation theory is written in the form

$$\begin{aligned} u_1(x_1, x_2, x_3, t) &= u(x_1, t) + x_3\psi(x_1, t)\,, \\ u_2(x_1, x_2, x_3, t) &\equiv 0\,, \\ u_3(x_1, x_2, x_3, t) &= w(x_1, t)\,, \end{aligned} \tag{1}$$

where u and w denote reference displacements in the x_1 and x_3 directions, respectively. The symbol ψ represents rotation of the transverse normal referred to the plane $x_3 = 0$ and t is time. It is evident from (1) that the transverse normal strain $\varepsilon_3(x_1, t)$ is omitted. This may be accepted since the beam is thin. The strain-displacement equations for the first order displacement approximation give a first order strain field

$$\begin{aligned} \varepsilon_1(x_1, x_3, t) &= \frac{\partial u_1}{\partial x_1} = \frac{\partial u}{\partial x_1} + x_3\frac{\partial \psi}{\partial x_1} = \epsilon(x_1, t) + x_3\kappa(x_1, t)\,, \quad \varepsilon_2 = 0\,, \quad \varepsilon_3 = 0\,, \\ \varepsilon_5(x_1, t) &= \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = \frac{\partial w}{\partial x_1} + \psi = \gamma(x_1, t)\,, \qquad \varepsilon_4 = 0\,, \quad \varepsilon_6 = 0\,. \end{aligned} \tag{2}$$

In expressions above, the notation for the strain tensor components was reduced in the following way: $\varepsilon_{11} = \varepsilon_1$, $\varepsilon_{22} = \varepsilon_2$, $\varepsilon_{33} = \varepsilon_3$, $2\varepsilon_{23} = \varepsilon_4$, $2\varepsilon_{31} = \varepsilon_5$ and $2\varepsilon_{12} = \varepsilon_6$. We note furthermore that ϵ denotes normal strain in the reference coordinate system and κ represents curvature.

2.2. Stress-strain relations and stress resultants

While the displacements and strains in (1) and (2) are continuous and vary linearly through the total beam thickness, the stresses fulfill these conditions only in each single layer and have stress jumps at the layer interfaces. When we suppose that the stress state in the $x_1 - x_2$ plane is much larger in value than the normal out-of-plane stress we can set approximately $\sigma_3 \approx 0$. Using this, the on-principal-axis stiffness matrix (i.e in the directions L, T and T') of the kth orthotropic laminae is reduced. Constitutive equations can be rewritten by separating of transverse shear stresses and strains. The most important stress-strain relations in the kth layer with respect to (2) are expressed

$$\sigma_1^k = Q_{11}^k \varepsilon_1 \qquad \text{and} \qquad \sigma_5^k = Q_{55}^k \varepsilon_5 \tag{3}$$

by means of the reduced off-principal-axis stiffness coefficients

$$\begin{aligned} Q_{11}^k &= 4G_{LT}\cos^2\theta_k \sin^2\theta_k + \frac{E_L\cos^4\theta_k + \left(2\nu_{LT}\cos^2\theta_k + \sin^2\theta_k\right)E_T\sin^2\theta_k}{1 - \nu_{LT}\nu_{TL}}\,, \\ Q_{55}^k &= G_{LT'}\cos^2\theta_k + G_{TT'}\sin^2\theta_k\,. \end{aligned} \tag{4}$$

The elastic behavior of the laminae is described with five independent Young's moduli and one Poisson's ratio ν_{LT} since the relation $\nu_{TL}/\nu_{LT} = E_T/E_L$ is valid. If the material is transversely-isotropic in the plane $T - T'$, the following engineering constants $E_T = E_{T'}$, $G_{LT} = G_{LT'}$, $\nu_{LT} = \nu_{LT'}$ are the same and we can determine the shear modulus in the plane of isotropy as $G_{TT'} = E_{T'}/\left[2\left(1 + \nu_{TT'}\right)\right]$. Thus we get only five independent parameters in (4). The stress tensor components, $\sigma_{11} = \sigma_1$, $\sigma_{31} = \sigma_5$ etc., are denoted similarly as the strain components. It should be noted that stresses σ_2, σ_4 and σ_6 are not generally zero in the constitutive equations. Their influence on the deformation of the beam could be observed inside or outside the $x_1 - x_3$ plane.

The resultant force of a laminate by summarizing the adequate forces of all laminae is

$$N(x_1, t) = \epsilon \sum_{k=1}^{n} Q_{11}^k \int_{h_{k-1}}^{h_k} b(x_3)\, \mathrm{d}x_3 + \kappa \sum_{k=1}^{n} Q_{11}^k \int_{h_{k-1}}^{h_k} b(x_3) x_3\, \mathrm{d}x_3 = A_{11}\epsilon + B_{11}\kappa \qquad (5)$$

in the direction x_1. The normal force was derived with respect to equations (2) and (3). By analogy it follows that the resultant moment is given in the form

$$M(x_1, t) = \epsilon \sum_{k=1}^{n} Q_{11}^k \int_{h_{k-1}}^{h_k} b(x_3) x_3\, \mathrm{d}x_3 + \kappa \sum_{k=1}^{n} Q_{11}^k \int_{h_{k-1}}^{h_k} b(x_3) x_3^2\, \mathrm{d}x_3 = B_{11}\epsilon + D_{11}\kappa \qquad (6)$$

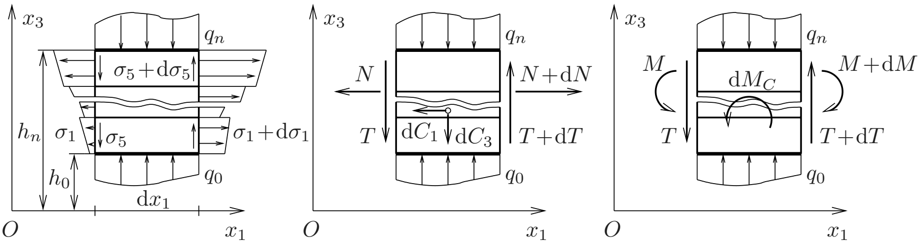
and transverse shear force is given as

$$T(x_1, t) = \gamma \sum_{k=1}^{n} Q_{55}^k \int_{h_{k-1}}^{h_k} b(x_3)\, \mathrm{d}x_3 = A_{55}\gamma = \alpha A_{55}\gamma\,. \qquad (7)$$

The relations of stress resultants are expressed in terms of four stiffness parameters which are well-known in the laminate theory. The first of them A_{11} is so-called the extensional stiffness, B_{11} is the coupling stiffness, D_{11} is the bending or flexural stiffness and A_{55} means transverse shear stiffness. An improvement of the last mentioned parameter is possible by its replacing by αA_{55} as applied in (7) where $\alpha = 1$. The coefficient α is so-called the shear correction factor and will be determined later.

2.3. *Equations of motion*

Now we consider a differential element of the laminated beam as isolated, see fig. 2(a). The distributed forces per length $q_0(x_1, t)$ and $q_n(x_1, t)$ act on the lower and upper face, respectively. In addition, the absence of body force is assumed. Stresses acting on the left and right section of



(a) (b) (c)

Fig. 2. Load of a differential element

the beam element may be replaced with the resultant normal (5) and shear (7) forces and variations of these quantities, as shown in fig. 2(b). Writing equations of motion in the horizontal and vertical directions for the differential element, we have

$$\mathrm{d}N(x_1,t) - \mathrm{d}C_1(x_1,t) = 0\,, \tag{8}$$

$$\mathrm{d}T(x_1,t) - \mathrm{d}C_3(x_1,t) = q(x_1,t)\,\mathrm{d}x_1\,, \tag{9}$$

after the simplification whereas the substitution $q = q_n - q_0$ was applied in (9). Inertia stresses in these equations are

$$\begin{aligned}\mathrm{d}C_1 &= \ddot{u}_1 \sum_{k=1}^{n} \rho^k \int_{h_{k-1}}^{h_k} b(x_3)\,\mathrm{d}x_3\,\mathrm{d}x_1 = \ddot{u} \sum_{k=1}^{n} \rho^k \int_{h_{k-1}}^{h_k} b(x_3)\,\mathrm{d}x_3\,\mathrm{d}x_1 + \\ &+ \ddot{\psi} \sum_{k=1}^{n} \rho^k \int_{h_{k-1}}^{h_k} b(x_3) x_3\,\mathrm{d}x_3\,\mathrm{d}x_1 = \left(\rho_{11}\ddot{u} + R_{11}\ddot{\psi}\right)\mathrm{d}x_1\end{aligned} \tag{10}$$

in the direction x_1, and

$$\mathrm{d}C_3 = \ddot{u}_3 \sum_{k=1}^{n} \rho^k \int_{h_{k-1}}^{h_k} b(x_3)\,\mathrm{d}x_3\,\mathrm{d}x_1 = \ddot{w} \sum_{k=1}^{n} \rho^k \int_{h_{k-1}}^{h_k} b(x_3)\,\mathrm{d}x_3\,\mathrm{d}x_1 = \rho_{11}\ddot{w}\,\mathrm{d}x_1 \tag{11}$$

in the direction x_3. Dots represent the differentiation with respect to time and ρ^k is the mass density of the kth material layer. New parameters are also defined in (10) and (11) such as the weight per area of the laminate ρ_{11} and the product of inertia R_{11} about the inertia axis x_2.

Summation of all moments to the origin of the global coordinate system leads to

$$\mathrm{d}M(x_1,t) - \mathrm{d}M_C(x_1,t) = T(x_1,t)\,\mathrm{d}x_1 \tag{12}$$

the third dynamic equilibrium equation. This is so since consideration of slopes and deflections of the beam are small and the higher-order contributions of the loading to the moment are neglected. In equation (12), replacing of normal stress influence to the origin of the global coordinate system by moment M is used as is shown in fig. 2(c). The influence of inertia is analogously replaced by the moment

$$\begin{aligned}\mathrm{d}M_C &= \ddot{u} \sum_{k=1}^{n} \rho^k \int_{h_{k-1}}^{h_k} b(x_3) x_3\,\mathrm{d}x_3\,\mathrm{d}x_1 + \ddot{\psi} \sum_{k=1}^{n} \rho^k \int_{h_{k-1}}^{h_k} b(x_3) x_3^2\,\mathrm{d}x_3\,\mathrm{d}x_1 = \\ &= \left(R_{11}\ddot{u} + I_{11}\ddot{\psi}\right)\,\mathrm{d}x_1\,,\end{aligned} \tag{13}$$

where I_{11} is the moment of inertia about the axis x_2 of cross-section having side of unit length.

Now we will divide all equations of motion by $\mathrm{d}x_1$. Consequently, the derivations of (5), (6) and (7) with respect to x_1 have to be made and are substituted into the modified equilibrium equations. Thus we obtain the governing equations of motion in the terms of displacements,

$$\boldsymbol{\mathcal{M}}\ddot{\boldsymbol{q}}(x_1,t) + \boldsymbol{\mathcal{K}}\boldsymbol{q}(x_1,t) + \boldsymbol{\mathcal{F}}(x_1,t) = \mathbf{0}\,. \tag{14}$$

Operator matrices are given in the form

$$\boldsymbol{\mathcal{M}} = \begin{bmatrix} \rho_{11} & 0 & 0 \\ 0 & I_{11} & R_{11} \\ 0 & R_{11} & \rho_{11} \end{bmatrix}, \qquad \boldsymbol{\mathcal{K}} = \begin{bmatrix} -\alpha A_{55}\partial_2 & -\alpha A_{55}\partial_1 & 0 \\ \alpha A_{55}\partial_1 & \alpha A_{55} - D_{11}\partial_2 & -B_{11}\partial_2 \\ 0 & -B_{11}\partial_2 & -A_{11}\partial_2 \end{bmatrix} \tag{15}$$

with $\partial_1 = \partial/\partial x_1$ and $\partial_2 = \partial^2/\partial x_1^2$. The symbol $\boldsymbol{\mathcal{F}}$ represents the external force vector

$$\boldsymbol{\mathcal{F}} = [q,\,0,\,0]^T, \tag{16}$$

and $\boldsymbol{q}$ is the vector of unknown displacements with components $q_1 = w$, $q_2 = \psi$ and $q_3 = u$.

3. Shear correction factor

The first-order shear deformation theory described in the section above assumed the constant shear strain through the laminate thickness h, see (2). Therefore, the shear stress (3) is not generally continuous from ply to ply and is not satisfying exactly conditions at the bottom and top boundary layers. It is evident such a distribution is not realistic. A better estimate of the shear stress can be obtained by application of the equilibrium equation in the case of differential element as shown in fig. 3. It can be simplified as follows:

$$\tau_5^k(x_1, x_3, t)b(x_3)\mathrm{d}x_1 = \mathrm{d}C_\xi(x_1, x_3, t) - \mathrm{d}N_\xi(x_1, x_3, t)\,, \tag{17}$$

where τ_5^k is improved shear stress in the kth layer, $\mathrm{d}C_\xi$ and $\mathrm{d}N_\xi$ are force of inertia and normal force in the x_1-direction. Consequently, the shear correction factor α shown in (7) can be determined such that two strain energies due to the transverse shear per unit area are equal, i.e.

$$\frac{1}{2}\sum_{k=1}^{n}\int_{h_{k-1}}^{h_k}\sigma_5^k\varepsilon_5 b(x_3)\,\mathrm{d}x_3 = \frac{1}{2}\sum_{k=1}^{n}\int_{h_{k-1}}^{h_k}\tau_5^k\gamma_5^k b(x_3)\,\mathrm{d}x_3\,. \tag{18}$$

The γ_5^k is the strain tensor component and may be calculated in analogy to ε_5 in (3) but from the stress τ_5^k. Obvious result from (17) and (18) is that the shear correction factor depends on both the loading (including the force of inertia) and stacking and not on only single of them. Admittedly, higher approximation will lead to better result but also will require more expensive computational effort and the accuracy improvement will be so little in the case of the thin beam that required effort to solve more complicated equations will not be justified. Therefore, we have not already accepted inertia stresses for the α factor calculation. The equilibrium equations (12) and (8) can be put in consequence of this condition into the form

$$\begin{bmatrix} D_{11} & B_{11} \\ B_{11} & A_{11} \end{bmatrix}\begin{bmatrix} \psi'' \\ u'' \end{bmatrix} = \begin{bmatrix} T \\ 0 \end{bmatrix} \qquad \text{or, in brief,} \qquad \boldsymbol{A}\boldsymbol{u}''(x_1) = \boldsymbol{T}(x_1)\,. \tag{19}$$

The symbol $(\ldots)''$ denotes $\mathrm{d}^2/\mathrm{d}x_1^2$. Solving the linear algebraic equation system (19) gives

$$\psi''(x_1) = A_{11}T(x_1)D_T^{-1} \text{ and } u''(x_1) = -B_{11}T(x_1)D_T^{-1} \quad \text{with} \quad D_T = A_{11}D_{11} - B_{11}^2\,. \tag{20}$$

Now the shear stress τ_5^k depends only on the current width $b(x_3)$ and the normal force $\mathrm{d}N_\xi(x_1, x_3)$ which is determined in a similar way as $\mathrm{d}N$ in (5). We obtain this in brief,

$$\tau_5^k(x_1, x_3) = -\frac{1}{b(x_3)}\left\{u''(x_1)\left[Q_{11}^k I_0^k(x_3) + f_0^k\right] + \psi''(x_1)\left[Q_{11}^k I_1^k(x_3) + f_1^k\right]\right\}, \tag{21}$$

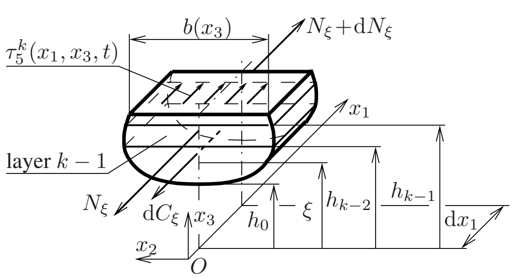


Fig. 3. The separate part of a differential element loaded in the x_1-direction

where

$$I_0^k(x_3) = \int_{h_{k-1}}^{x_3} b(\xi)\,\mathrm{d}\xi\,, \qquad f_0^k = \sum_{i=1}^{k-1} Q_{11}^i \int_{h_{i-1}}^{h_i} b(\xi)\,\mathrm{d}\xi\,,$$

$$I_1^k(x_3) = \int_{h_{k-1}}^{x_3} b(\xi)\xi\,\mathrm{d}\xi\,, \qquad f_1^k = \sum_{i=1}^{k-1} Q_{11}^i \int_{h_{i-1}}^{h_i} b(\xi)\xi\,\mathrm{d}\xi\,.$$

Substituting (20) into this relation, the shear stress in the kth layer becomes a function of the transverse shear force $T(x_1)$. The left-hand side of (18) can be also expressed with the help of the transverse shear force when we take into account (3) and (7). Inserting (21) into the right-hand side of (18) and comparing both sides of them with respect to T, the shear correction factor is given by

$$\frac{1}{\alpha} = \frac{A_{55}}{D_T^2} \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \frac{\left[f_5^k(x_3)\right]^2}{Q_{55}^k b(x_3)}\,\mathrm{d}x_3\,, \tag{22}$$

where

$$f_5^k(x_3) = B_{11}\left[Q_{11}^k I_0^k(x_3) + f_0^k\right] - A_{11}\left[Q_{11}^k I_1^k(x_3) + f_1^k\right].$$

It is seen from (22) that the factor α is invariable along the beam axis and only depending on the materials of layers and the cross-section geometry.

4. Static solution

In the static problem, the general equilibrium equations (14) reduce to the form

$$\boldsymbol{\mathcal{K}}\boldsymbol{q}(x_1) + \boldsymbol{\mathcal{F}}(x_1) = \boldsymbol{0} \tag{23}$$

Thus we get the system of three ordinary differential equations of 2^{nd} order with constant coefficients. The solution can be easily obtained by utilizing the shear strain expression in (2). Integrating the first equation in (23), we obtain

$$\alpha A_{55}\left[w'(x_1) + \psi(x_1))\right] = \int q(x_1)\,\mathrm{d}x_1 + C_1\,. \tag{24}$$

Substituting this result into the second equation in (23), the general static problem could be rewritten as follows:

$$\boldsymbol{A}\boldsymbol{u}''(x_1) = \boldsymbol{F}(x_1) \qquad \text{and} \qquad w'(x_1) = \frac{1}{\alpha A_{55}} \int q(x_1)\,\mathrm{d}x_1 + \frac{C_1}{\alpha A_{55}} - \psi(x_1)\,, \tag{25}$$

where

$$\boldsymbol{F}(x_1) = \left[D^{-1}q(x_1) + C_1,\, 0\right]^T.$$

The matrix $\boldsymbol{A}$ and vector $\boldsymbol{u}(x_1)$ are defined in (19). The expression $D^{-i}q(x_1)$ which is used for $i = 1$ in vector $\boldsymbol{F}(x_1)$ means the i-fold integral (repeated integral) of function $q(x_1)$ if $D^{-1}q(0) = D^{-1}q\left(D^{-1}q(0)\right) = \ldots = 0$ is assumed. The general formulation of this is given by

$$D^{-i}q(x_1) = \underbrace{\int\cdots\int_0^{x_1}}_{i} q(x_1)\,\underbrace{\mathrm{d}x_1\cdots\mathrm{d}x_1}_{i} = \int_0^{x_1} \frac{q(\xi)\,(x_1-\xi)^{i-1}}{(i-1)!}\,\mathrm{d}\xi \quad \text{for} \quad i = 1, 2, \ldots \tag{26}$$

Now we easily determine the unknown deformations in consequence of integrating twice with respect to x_1. We get from (25)

$$\boldsymbol{u}(x_1) = \left(D^{-3}q(x_1) + 0.5\, C_1 x_1^2\right) \boldsymbol{c} + x_1 \boldsymbol{c}_2 + \boldsymbol{c}_3 \,, \tag{27}$$

where $\boldsymbol{c}_2 = [C_{2\psi},\, C_{2u}]^T$ and $\boldsymbol{c}_3 = [C_{3\psi},\, C_{3u}]^T$ are vectors of arbitrary constants and vector $\boldsymbol{c}$ is equal to $D_T^{-1}\,[A_{11},\, -B_{11}]^T$. Inserting $\psi(x_1)$ from (27) into $w'(x_1)$ in (25) and integrating once more with respect to x_1, we obtain

$$\begin{aligned} w(x_1) &= \left\{ -\frac{A_{11}}{D_T}\left[D^{-4}q(x_1) + C_1\frac{x_1^3}{6}\right] - C_{2\psi}\frac{x_1^2}{2} - C_{3\psi}x_1 + C_4 \right\} + \left\{ \frac{D^{-2}q(x_1) + C_1 x_1}{\alpha A_{55}} \right\} = \\ &= w_B(x_1) + w_S(x_1)\,. \end{aligned} \tag{28}$$

It is evident from (28) that the transverse deflection could be separated into the bending w_B and the shear w_S parts. The bending part is the same as derived in the classical theory and is usually dominant in value in the case of a thin beam. Note that C_1 and C_4 are also arbitrary constants.

However, the static problem is not yet resolved because we do not know six constants $C_1, \ldots, C_{3u}, C_4$. It means to solve the boundary conditions problem. We have to find, therefore, the solution of (23) or (25) which is satisfying in most cases

$$\boldsymbol{f}_\Gamma\left(\boldsymbol{q}'(x_a), \boldsymbol{q}'(x_b), \boldsymbol{q}(x_a), \boldsymbol{q}(x_b)\right) = \boldsymbol{0}\,. \tag{29}$$

The quantity $\boldsymbol{f}_\Gamma$ is vector function of dimension 6, and symbols x_a and x_b denote two points in the interval of definition.

5. Free vibrations of beam

Formulae of free vibrations can be found easily when the vector $\boldsymbol{\mathcal{F}}$ is omitted in (14), i.e.

$$\boldsymbol{\mathcal{M}}\ddot{\boldsymbol{q}}(x_1, t) + \boldsymbol{\mathcal{K}}\boldsymbol{q}(x_1, t) = \boldsymbol{0}\,. \tag{30}$$

Let us assume the general solution of that in the following form

$$\boldsymbol{q}(x_1, t) = \boldsymbol{q}(x_1) T_t(t)\,. \tag{31}$$

If some nonzero real vector function $\boldsymbol{q}^*(x_1) \in \mathbb{R}^3$ is taken, the inner product space of $\boldsymbol{q}^*$ and (30) over the beam domain can be obtained (computed). When the solution (31) is accepted, it is readily shown that

$$-\frac{\langle \boldsymbol{q}^*(x_1), \boldsymbol{\mathcal{K}}\boldsymbol{q}(x_1)\rangle}{\langle \boldsymbol{q}^*(x_1), \boldsymbol{\mathcal{M}}\boldsymbol{q}(x_1)\rangle} = \frac{\ddot{T}_t(t)}{T_t(t)} = \lambda\,. \tag{32}$$

Hence it follows that λ must be a constant independent on variables x_1 and t. Thus ordinary differential equation of 2^{nd} order with the coefficient λ is given for $T_t(t)$. Because the solution of this equation is estimated in form $C_t \mathrm{e}^{\beta t}$, it leads to the characteristic equation for β:

$$\beta^2 - \lambda = 0\,, \quad \text{and this implies} \quad \beta = \pm\sqrt{\lambda} \quad \text{for all} \quad \lambda \in \mathbb{C} - \{0\}\,. \tag{33}$$

Hence we get

$$T_t(t) = C_{t1}\mathrm{e}^{\beta t} + C_{t2}\mathrm{e}^{-\beta t}, \tag{34}$$

where C_{t1} and C_{t2} are arbitrary constants. Substituting (31) into (30) with respect to (34), the formula of a free vibration is written as

$$\lambda \boldsymbol{\mathcal{M}} \boldsymbol{q}(x_1) + \boldsymbol{\mathcal{K}} \boldsymbol{q}(x_1) = \mathbf{0}\,, \tag{35}$$

that is independent on time. This could be below rearranged in the form

$$\lambda \boldsymbol{M} \boldsymbol{q}(x_1) - \boldsymbol{K}_2 \boldsymbol{q}''(x_1) + \boldsymbol{K}_1 \boldsymbol{q}'(x_1) + \boldsymbol{K}_0 \boldsymbol{q}(x_1) = \mathbf{0}\,, \tag{36}$$

taking into account that the notation $\boldsymbol{\mathcal{M}} \equiv \boldsymbol{M}$ is used. Remaining matrices in (36) are

$$\boldsymbol{K}_2 = \begin{bmatrix} \alpha A_{55} & 0 & 0 \\ 0 & D_{11} & B_{11} \\ 0 & B_{11} & A_{11} \end{bmatrix}, \quad \boldsymbol{K}_1 = \begin{bmatrix} 0 & -\alpha A_{55} & 0 \\ \alpha A_{55} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{K}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha A_{55} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{37}$$

Now we define new vector of variables as $\boldsymbol{q}' = \tilde{\boldsymbol{q}}$. Then this system of equations together with (36) can be reduced to the system of ordinary differential equations of first order

$$\begin{bmatrix} \boldsymbol{q}' \\ \tilde{\boldsymbol{q}}' \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{q} \\ \tilde{\boldsymbol{q}} \end{bmatrix} \qquad \text{or, in brief,} \qquad \boldsymbol{y}'(x_1, \boldsymbol{y}, \lambda) = \boldsymbol{B}(\lambda) \boldsymbol{y}(x_1, \lambda)\,, \tag{38}$$

where $\boldsymbol{I}$ is called as an identity matrix and the submatrices $\boldsymbol{B}_{21}$ and $\boldsymbol{B}_{22}$ have form

$$\boldsymbol{B}_{21}(\lambda) = \boldsymbol{K}_2^{-1} (\lambda \boldsymbol{M} + \boldsymbol{K}_0) \qquad \text{and} \qquad \boldsymbol{B}_{22} = \boldsymbol{K}_2^{-1} \boldsymbol{K}_1\,. \tag{39}$$

The solution of (38) is given by $\boldsymbol{y} = \boldsymbol{x} \mathrm{e}^{\varkappa x_1}$. Substituting this into (38), after simplification we get the standard eigenrelation for the square matrix $\boldsymbol{B}$, i.e.

$$\boldsymbol{B}\boldsymbol{x} = \varkappa \boldsymbol{x}\,, \qquad \text{which leads to} \qquad (\boldsymbol{B} - \varkappa \boldsymbol{I})\, \boldsymbol{x} = \mathbf{0}\,. \tag{40}$$

Parameter $\varkappa$ is an eigenvalue and $\boldsymbol{x}$ is an eigenvector. Since a nontrivial solution of $\boldsymbol{x}$ is expected, the next relation must be satisfied

$$\det (\boldsymbol{B} - \varkappa \boldsymbol{I}) = 0\,, \quad \text{whence} \quad \varkappa^6 - a_2(\lambda)\varkappa^4 + a_1(\lambda)\varkappa^2 - a_0(\lambda) = 0\,. \tag{41}$$

When the substitution $y = \varkappa^2$ is applied, we obtain the characteristic equation

$$y^3 - a_2(\lambda) y^2 + a_1(\lambda) y - a_0(\lambda) = 0 \tag{42}$$

from (41). The coefficients of (42) can be expressed as follows:

$$\begin{aligned} a_0 &= (c_{22}c_{33} + c_{23}c_{32})c_{11}\lambda^3 + (j_{21}c_{33} + j_{31}c_{23})c_{11}\,\lambda^2\,, \\ a_1 &= (c_{11}c_{22} + c_{22}c_{33} + c_{33}c_{11} + c_{23}c_{32})\lambda^2 + j_{21}c_{11}\lambda\,, \\ a_2 &= (c_{11} + c_{22} + c_{33})\lambda\,. \end{aligned} \tag{43}$$

We can find here some physical interpretation. Parameters c_{11} to c_{32} mean square of velocities while j_{21} and j_{31} represent geometric and material properties of beam. They are defined with the help of (39) and (20) as

$$\begin{aligned} j_{21} &= \alpha A_{11} A_{55} / D_T\,, & j_{31} &= \alpha B_{11} A_{55} / (h D_T)\,, \\ c_{22} &= (A_{11} I_{11} - B_{11} R_{11}) / D_T\,, & c_{32} &= (B_{11} I_{11} - D_{11} R_{11}) / (h D_T)\,, \\ c_{23} &= h(A_{11} R_{11} - B_{11} \rho_{11}) / D_T\,, & c_{33} &= (D_{11} \rho_{11} - B_{11} R_{11}) / D_T\,, \\ c_{11} &= \rho_{11} / (\alpha A_{55})\,. \end{aligned} \tag{44}$$

The cubic equation (42) has three different roots, when the discriminant of (42) is nonzero. Therefore, the solution of the characteristic equation (41) may be expected with respect of the substitution $y = \varkappa^2$ in the form: $\varkappa_i = \sqrt{y_i}$ and $\varkappa_{i+3} = -\varkappa_i$ for $i = 1, 2, 3$. Let us rewrite the eigenvector $\boldsymbol{x}$ by using of two subvectors $[\boldsymbol{x_q}, \, \tilde{\boldsymbol{x}}_{\boldsymbol{q}}]^T$. Then inserting into (40) and with regard to the form of matrix $\boldsymbol{B}$, see (38) and (39), only these eigenrelation are solved

$$[\boldsymbol{B}_{21} + \varkappa_i (\boldsymbol{B}_{22} - \varkappa_i \boldsymbol{I})] \, \boldsymbol{x}_{\boldsymbol{q}}^i = \boldsymbol{0} \qquad \text{and} \qquad [\boldsymbol{B}_{21} - \varkappa_i (\boldsymbol{B}_{22} + \varkappa_i \boldsymbol{I})] \, \boldsymbol{x}_{\boldsymbol{q}}^{i+3} = \boldsymbol{0} \, . \tag{45}$$

Hence the eigenvector

$$\boldsymbol{x}_{\boldsymbol{q}}^i = \left[c_1^i, \, 1, \, c_3^i\right]^T \qquad \text{and} \qquad \boldsymbol{x}_{\boldsymbol{q}}^{i+3} = \left[-c_1^i, \, 1, \, c_3^i\right]^T, \tag{46}$$

where

$$c_1^i = \frac{\varkappa_i}{\lambda c_{11} - \varkappa_i^2} \qquad \text{and} \qquad c_3^i = -\frac{[\varkappa_i^4 - \lambda (c_{11} + c_{22}) \, \varkappa_i^2 + (\lambda^2 c_{22} + \lambda j_{21}) \, c_{11}] \, h}{(\lambda c_{11} - \varkappa_i^2) \, \lambda c_{23}} \, ,$$

are uniquely determined except for arbitrary multiples. Taking above mentioned into consideration, the vector of deformation which is defined in (31) is

$$\boldsymbol{q}(x_1, \lambda) = \sum_{i=1}^{3} \left(C_i \boldsymbol{x}_{\boldsymbol{q}}^i \mathrm{e}^{\varkappa_i x_1} + C_{i+3} \boldsymbol{x}_{\boldsymbol{q}}^{i+3} \mathrm{e}^{-\varkappa_i x_1} \right) \tag{47}$$

where C_i, C_{i+3} for $i = 1, 2, 3$ are arbitrary constants. They are calculated from boundary conditions which may be written in analogy with (29). Still unknown parameter λ or more precisely λ_ν for $\nu = 1, \ldots, \infty$ has to be determined for nontrivial solution of $\boldsymbol{q}(x_1, \lambda)$. When we formally rewrite $\boldsymbol{q}(x_1, \lambda)$ with $\boldsymbol{q}_\nu(x_1, \lambda_\nu)$ and likewise $T_t(t)$ with $T_{t\nu}(t, \lambda_\nu)$ in (31) for concrete λ_ν, free vibrations of beams have form

$$\boldsymbol{q}(x_1, t) = \sum_{\nu=1}^{\infty} \boldsymbol{q}_\nu(x_1, \lambda_\nu) T_{t\nu}(t, \lambda_\nu) \, . \tag{48}$$

Note that the constants C_{t1}^ν and C_{t2}^ν must be determined from the initial conditions of problem.

6. Numerical examples

We perform the comparison of static deformations for some easy examples which were calculated analytically from derived relations and numerically with the help of the software MARC. It was applied to a simple supported laminate beam uniformly loaded by $q = 1$ [kNm]. As shown in fig. 4, three types of cross-section were considered that were made of six layers of

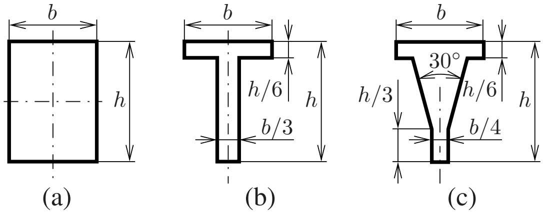


Fig. 4. Cross-sections (a) shape-A, (b) shape-B and (c) shape-C of beam

Table 1. Computed values of the shear correction factor

Orientation of layers		$[0_6]$			$[0_2/90_4]$		
Cross-sections of beams		shape-A	shape-B	shape-C	shape-A	shape-B	shape-C
Shear correction factor α	$[-]$	0.833 3	0.670 7	0.721 4	0.825 5	0.664 9	0.667 3

the same thickness. Two sequences of stacking $[0_6]$ and $[0_2/90_4]$ were mainly explored. For each transversely-isotropic layer the following characteristics of the unidirectional carbon fiber composite AS4/3501-6 were used: $E_L = 142$ [GPa], $E_T = 10.3$ [GPa], $G_{LT} = 7.2$ [GPa], $\nu_{LT} = 0.27$ [–] and $\nu_{TT'} = 0.4\,[-]$. The basic dimensions $h = 18$ [mm] and $b = 9$ [mm] of cross-section, see fig. 4, were chosen. The length of all beams was the same $l = 450$ [mm].

At first, we evaluate computed values of the shear correction factor. If we first look at $\alpha = 0.833\,3$ in tab. 1, we observe that is identical in value of correction factor well-known in the case of the isotropic beam with a rectangular cross-section. The same result is true for all beams with the rectangular cross-section and the identical orientation of all layers when we use (22). It is also evident from tab. 1 that α is or not sensitive to shape of cross-sections and orientation of layers, see shape-B and shape-C.

When we want to find analytical relations for $w_A(x_1)$, $\psi_A(x_1)$ and $u_A(x_1)$ of the simple supported beam fixed at the lower ends $x_1 = 0$ and $x_1 = l$, the boundary conditions are

$$w_A(0) = 0\,, \quad u_A(0) = 0\,, \quad M(0) = 0\,, \quad w_A(l) = 0\,, \quad N(l) = 0\,, \quad M(l) = 0\,, \tag{49}$$

where N and M are resultant force and moment, respectively. Numerical values of deformation w_N and u_N were calculated thank to development of finite element models. The mesh of all models have 100 elements of the same length in axial direction. Furthermore, each layer consists of two elements through the thickness. These elements are three-dimensional and isoparametric and use triquadratic interpolation functions to represent the coordinates and displacements. There are type 21 in the software MARC. Boundary conditions of numerical models are applied in line with (49) at the lower ends of beams, i.e. $w_N(0) = 0$, $u_N(0) = 0$ and $w_N(l) = 0$.

The static deformations of beams obtained from analytical and numerical calculation are mostly compared in tab. 2. We can see a good correspondence between maximum beam deflection w_N and w_A. Since the influence of boundary conditions of finite element models on the

Table 2. Comparison in percents of numerically and analytically computated values of displacements

Orientation of layers		$[0_6]$			$[0_2/90_4]$		
Cross-sections of beams		shape-A	shape-B	shape-C	shape-A	shape-B	shape-C
$(w_N - w_A)/w_A$	$x_3 = 0$	1.49	2.34	2.46	0.56	0.78	0.87
for $x_1 = l/2$	$x_3 = 0^*$	0.59	0.90	0.77	0.39	0.48	0.47
$\|w_N - w_A\| \,/\, \|w_A\|$	$x_3 = 0$	1.87	2.97	3.19	0.61	0.89	1.02
for $w_N, w_A \in L_2(0,l)$	$x_3 = 0^*$	0.64	0.98	0.85	0.39	0.49	0.48
	$x_3 = 0$	4.57	6.13	5.93	1.90	2.91	3.09
$\|u_N - u_A\| \,/\, \|u_A\|$	$x_3 = h$	1.75	1.48	0.77	0.98	1.49	1.19
for $u_N, u_A \in L_2(0,l)$	$x_3 = 0^*$	2.69	3.61	3.11	1.18	1.80	1.75
	$x_3 = h^*$	0.58	0.63	0.63	0.39	0.42	0.58

* Computated with a correction

accuracy of calculated deformations is observed, the correction (difference $w_N - w_A$ in the first element is eliminated in the rest of values w_N) is employed. Then more better agreement was found as shown tab. 2. The comparison of results from analytical and numerical solutions were moreover accomplished with respect to the $L2$-norm. It is seen to be better way to confront w_N and w_A or u_N and u_A because these values are monitored not only at a single point. The errors in percents were comparable in size with errors of maximum beam deflection.

In addition to the deformations of beam with rectangular cross-section and with orientation of layers $[45_6]$ and $[0/30/45_2/60/90]$ were analyzed. Large differences were reached in beam deflections. The error was about $150\,\%$ in the case $[45_6]$ and about $50\,\%$ in the second one.

7. Conclusion

In this paper, the derivation of equations for static and free vibration problem in-plane of laminated beams with symmetric cross-sections is given. In our one-dimensional model coupling among beam‘s transversal deflection, rotation of cross-section and axial displacement is considered. Deformations are determined only in-plane of loading, i.e. in plane of symmetry. Moreover the relation for calculation of the shear correction factor is found.

It is following from numerical examples that derived formulas work only for the specific beam configurations. An excellent agreement between analytical and numerical results is discovered in cases of orthotropic and cross-ply laminates. But obtained results of beam deflection give fatal errors for symmetric laminate $[45_6]$ and laminate with sequences of stacking $[0/30/45_2/60/90]$. It is the reason why we cannot recommend to use beam deflection calculation in [1], chapter 7, because it is applied for symmetric laminated beams in spite of cross-section characteristics are determined by similar way as in this work.

However, we can expect that likewise our solution may be used for angle-ply laminates when the thickness of layers is small against their width. The Poisson’s effect is then negligible. The advantage of this solution is also that sandwich beams with full core and not only rectangular cross-section can be calculated. It suffices to define appropriate material properties in inner layers.

Finally, we point out that developing and analyzing this model of beam will be connected with identification of material properties on samples in future. Of particular interest is the possibility of experimental identification of these properties by measurement of natural frequencies.

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