New Visualization of Surfaces in Parallel Coordinates – Eliminating Ambiguity and Some "Over-Plotting"

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ABSTRACT

$\mathcal{A}$ point $P \in \mathbb{R}^n$ is represented in Parallel Coordinates by a polygonal line $\bar{P}$ (see [Ins99] for a recent survey). Earlier [Ins85], a surface σ was represented as the *envelope* of the polygonal lines representing it's points. This is ambiguous in the sense that *different* surfaces can provide the *same* envelopes. Here the ambiguity is eliminated by considering the surface σ as the envelope of it's *tangent planes* and in turn, representing each of these planes by n-1 points [Ins99]. This, with some future extension, can yield a new and unambiguous representation, $\bar{\sigma}$, of the surface consisting of n-1 planar regions whose properties correspond lead to the *recognition* of the surfaces' properties i.e. developable, ruled etc. [Hun92]) and *classification* criteria.

It is further shown that the image (i.e. representation) of an algebraic surface of degree 2 in $\mathbb{R}^n$ is a region whose boundary is also an algebraic curve of degree 2. This includes some *non-convex* surfaces which with the previous ambiguous representation could not be treated. An efficient construction algorithm for the representation of the quadratic surfaces (given either by *explicit* or *implicit* equation) is provided. The results obtained are suitable for applications, to be presented in a future paper, and in particular for the approximation of complex surfaces based on their *planar* images. An additional benefit is the elimination of the "over-plotting" problem i.e. the "bunching" of polygonal lines which often obscure part of the parallel-coordinate display.

KEY WORDS

Scientific Visualization and HMI , Multi-dimensional Visualization, Parallel Coordinates, Algebraic Surfaces

1 INTRODUCTION

Our purpose here is expository, sparing the reader from most of the mathematical tribulations and, focusing on the more intuitive aspects of the representational results. After a short review of the fundamentals, the essentials of the mathematical development are given together with some detailed examples to clarify the nuances and satisfy the more mathematically inclined.

In parallel coordinates (abbr. ‖-coords), a point in $\mathbb{R}^2$ is represented by a line and a line is represented by a point yielding a fundamental $point \leftrightarrow line$ duality. There follows the representation of p-flats (planes of dimension $2 \leq p \leq n-1$) in $\mathbb{R}^n$ in terms of indexed points [Ins85]. Naturally, for non-linear objects the representation is more complex, especially if they are also non-convex. The points of a curve in $\mathbb{R}^2$ can be mapped directly into a family of lines whose *envelope* defines a curve ("line-curve"). Actually this is awkward and also clutters the display. Instead we map the tangents of the original curve into points to obtain the "point-curve", sometimes called "dual-curve", image directly as shown in Fig. 1. In short, this approach provides a convenient *point-to-point* mapping [Ins99].

Applying these considerations it was proved that the image of an algebraic curve of degree n is also al-
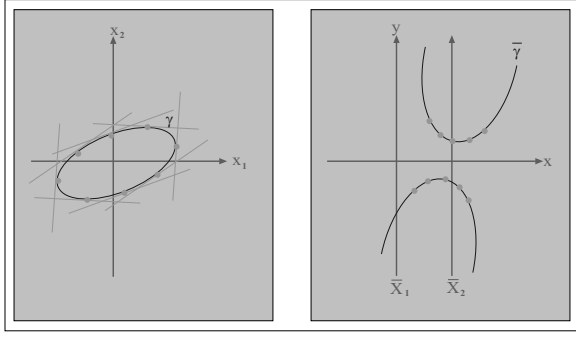
Figure 1. Point-curve mapped into point-curve, result of considering the point-curve as the envelope of it's tangents.

gebraic of degree at most $n(n-1)$ in the absence of singular points [Izh01]. This theorem is a generalization of the known result that conics are mapped into conics [Dim84] in six different ways.

Perhaps we are "pushing our luck", our intent is to apply next the point-to-point mapping in the representation of surfaces considered as the envelope of their tangent planes; with the resulting image being constructed from the representation of tangent planes [Ins99]. As has already been pointed out, planes can be represented in ∥-coords by indexed points. The collection of these planar points, grouped for each index, is the representation of the surface.

In the past [Ins85], surfaces were represented in ∥-coords by the envelope of the polygonal lines representing the surfaces' points. By itself this is ambiguous. For example the image of a sphere in n-dimensions is the same as the image of the surface obtained by the intersection of n cylinders properly aligned having the same radius. In applications this was ameliorated by accessing the correct equation of the surface when needed. Not only is the ambiguity completely removed with the new representation, but also *non-convex* surfaces can be nicely treated something that was not possible previously.

Hung [Hun92] first applied this notion and found that regions representing *developable* surfaces consists only of the boundary curves (i.e. there are no interior points), and also that *ruled* surfaces can be recognized from characteristic properties of their corresponding regions. Encouraged by these initial results the analysis is extended to more general surfaces yielding useful criteria in the approximation of surfaces by simpler ones; but we are getting ahead of ourselves.

At first we lay the foundations, then derive the representation of quadratic algebraic surfaces and further generalize to higher dimension as well as more complex hyper-surfaces. As a result an efficient algorithm for constructing the representation of the quadratic surfaces (given either by *explicit* or *implicit* equation), and a proof that the image of an algebraic surface of degree 2 in $\mathbb{R}^n$ is also an algebraic curve and of degree 2 are obtained.

2 GENERAL REPRESENTATION OF HYPER-SURFACE

$\mathcal{I}$n general, the method employed below applies to the class of smooth hyper-surfaces in $\mathbb{R}^n$ having a unique tangent hyper-plane at each point. Equivalently, each such hyper-surface is the envelope of it's tangent hyper-planes. This is our point of departure, for it enables us to represent each tangent hyper-plane in ∥-coords by $n-1$ indexed points [Ins99]. The hyper-surface's representation consists of the $n-1$ points sets, one for each index [Hun92]. For the present we restrict our attention to algebraic hyper-surfaces and in particular those defined by quadratic polynomials. To simplify matters, most of the analysis is done in 3-dimensional space but in a way which points to the generalization for $\mathbb{R}^n$.

2.1 Hyper-Planes Representation

An n-dimensional hyper-plane π in $\mathbb{R}^n$

$$\pi : \sum_{i=1}^{n} c_i x_i = c_0, \qquad (1)$$

is represented by the $n-1$ indexed points [Ins99]. For our purposes only the first

$$\bar{\pi}_{1\ldots n} = (\sum_{i=1}^{n}(i-1)c_i, c_0, \sum_{i=1}^{n} c_i). \qquad (2)$$

needs to be studied. The remaining $n-2$ points have similar form differing only in the factor $(i-1)$ of the c_i. An important property is that the horizontal distance between the i-adjacent (in the indexing) points is the equal to the coefficient c_i; from which the sequence of indexed points can be generated from the coefficients or vice-versa.

2.2 Hyper-Surfaces Representation

Moving on to the representation of non-linear hyper-surfaces in $\mathbb{R}^n$ from their tangent hyper-planes. Let σ be a smooth n-dimensional hyper-surface generated by the differentiable function $F(x_1, \ldots, x_n) = 0$, and an arbitrary point $(x_1^0, \ldots, x_n^0) \in \sigma$. Then the hyper-surface's tangent hyper-plane at this point is given by :

$$\sum_{i=1}^{n}(x_i - x_i^0)\frac{\partial F}{\partial x_i}(x_1^0, \ldots, x_n^0) = 0.$$
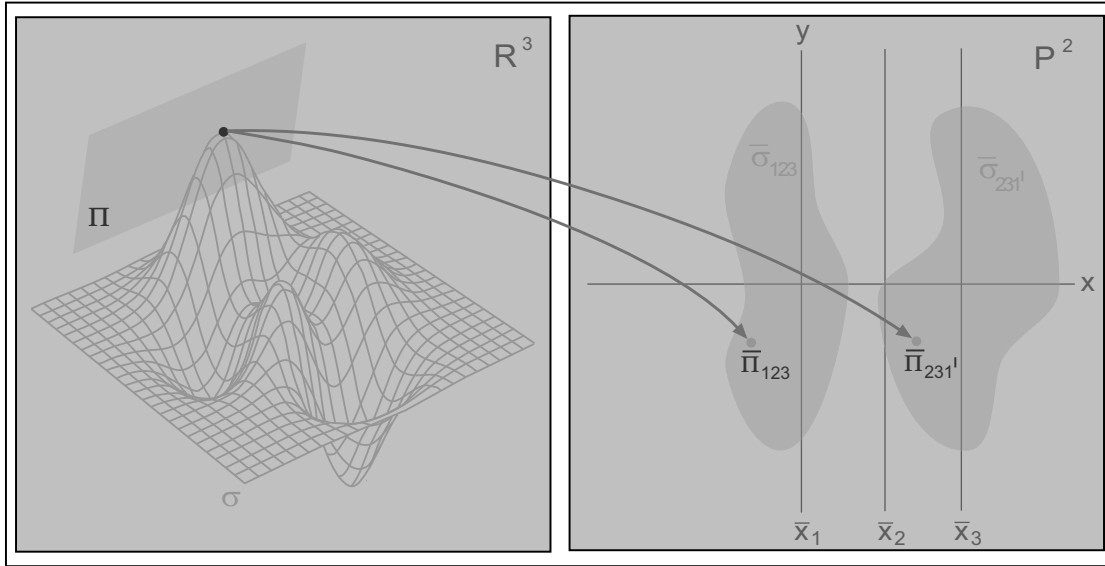
Figure 2. For a hyper-surface, it's representation's boundary curves are determined from the images of curves contained in hyper-surface.

Taking the x_i^0 as parameters, the coefficients of the tangent hyper-planes can written as a function of points which satisfy the hyper-surface's equation. Namely, the family of tangent hyper-planes of σ is represented in homogenous coordinates, by a collection of sets $\bar{\sigma}$ (see Fig. 2), containing the indexed points representing each member (i.e. hyper-plane) of the family. Each of the $n-1$ indexed set, $\bar{\sigma}_{j\ldots n1'\ldots(j-1)'}$, consists of the points with the same index.

In the remainder the analysis is confined to the first indexed set $\bar{\sigma}_{1\ldots n}$ using a shorter notation $\bar{\sigma}$ defined as,

$$\bar{\sigma} = \{(P(\bar{x}),\ S(\bar{x}),\ Q(\bar{x}))\ |\ \bar{x} \in \sigma)\}, \qquad (3)$$

where for $\bar{\sigma}_{1\ldots n}$ and the tuple $\bar{x} = (x_1, \ldots, x_n)$:

$$
\begin{aligned}
P(\bar{x}) &= \sum_{i=1}^{n}(i-1)\frac{\partial F}{\partial x_i} \\
S(\bar{x}) &= \sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i} \qquad (4)\\
Q(\bar{x}) &= \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}.
\end{aligned}
$$

In general, the representation in $\|$-coords is constructed via the rational transformations

$$x = \frac{P(\bar{x})}{Q(\bar{x})} \quad , \quad y = \frac{S(\bar{x})}{Q(\bar{x})}. \qquad (5)$$

The representation of all these hyper-planes transform an n-dimensional hyper-surface into subsets of $\mathbb{P}^2$; regions which are distinguished from each other by their

indices. The algorithm which constructs and describes these regions is presented next.

3 REPRESENTATION OF QUADRATIC HYPER-SURFACE

At first we treat the class of algebraic surfaces in $\mathbb{R}^3$ described by quadratic polynomials. Mercifully, the corresponding system of transformations (4) can be linearized. The next step is to determine the boundary of the regions representing the surface. Without getting into details, the existence of the boundary can be assured by selecting an appropriate spacing of axes in the system of the $\|$-coords, which eventually reflect by changing the constant multipliers of the first equation in (4).

3.1 Definition of the Regions' Boundary

Let σ be a quadratic surface whose representation is the region $\bar{\sigma}$. The boundary points are those whose every neighborhood contains both interior and exterior points. For this case both the transformation (4) and the surface, are defined by polynomials and hence are differentiable. The basic properties including continuity are therefore preserved under the transformation.

Geometrically we rely on the differentiability in finding those points $\bar{p} \in \bar{\sigma}$ so that we can "move" from $\bar{p}$ in any direction and still remain in region; these are interior points of $\bar{\sigma}$. Clearly the boundary points are easily found as the complement of the interior of $\bar{\sigma}$. The condition for determining whether a point is in-
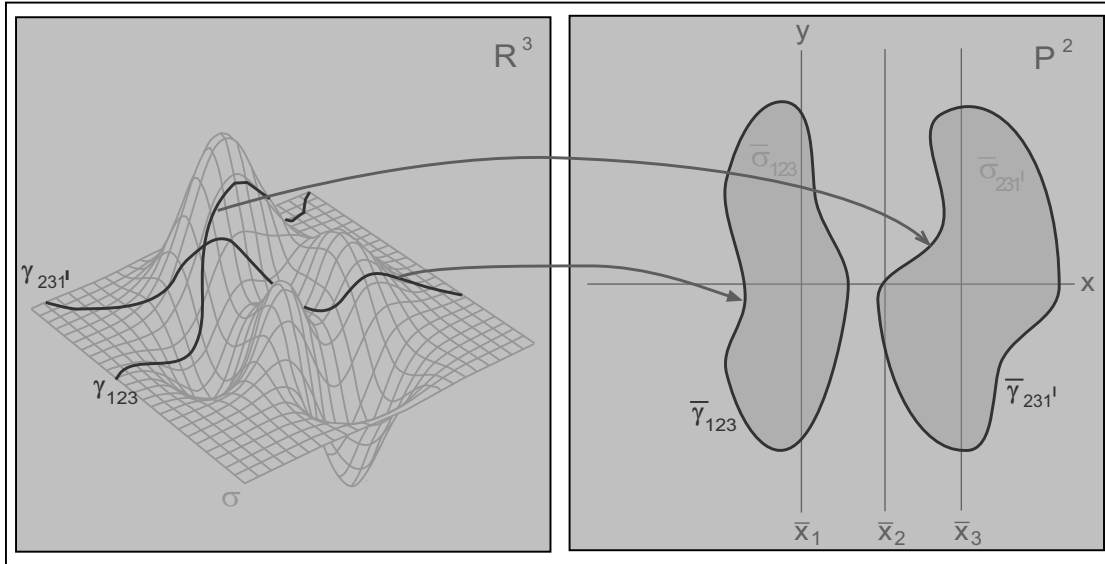
Figure 3. The boundary of each region is an image of a curve embedded in surface.

terior or not is given by *theorem of implicit function* [Mar85]. Equivalently, a point $\bar{a} \in \bar{\sigma}$ is interior point if and only if the Jacobian, $J(F)|_{\bar{a}}$, at this point is different from zero. Conversely, a point $\bar{b} \in \bar{\sigma}$ for which this is not true is necessarily a boundary point; namely, a point $\bar{b} \in \bar{\sigma}$ s.t. $J(F)|_{\bar{b}} = 0$. In essence the theorem tells us that $\bar{\sigma}$ is closed set and the complement of its interior is the sought after boundary.

Generalizing the above for $\mathbb{R}^n$ we get the mapping $(x_1, \dots, x_n) \rightarrow (\eta, \xi, \psi)$ into the projective space, were $x = \frac{\eta}{\psi}$ and $y = \frac{\xi}{\psi}$. Restating the condition in terms of differential products using homogeneous coordinates with the variables η, ξ and ψ yields,

$$(\eta d\xi d\psi - \xi d\eta d\psi + \psi d\eta d\xi)dF = 0. \qquad (6)$$

This form is more convenient for handling hypersurfaces embedded in $\mathbb{R}^n$ where $n > 3$, for $n = 3$ eq. (6) can be written equivalently in term of Jacobian as,

$$J(F)\psi^3 = Det \begin{bmatrix} \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} & \frac{\partial F}{\partial x_3} \\ \frac{\partial(\eta/\psi)}{\partial x_1} & \frac{\partial(\eta/\psi)}{\partial x_2} & \frac{\partial(\eta/\psi)}{\partial x_3} \\ \frac{\partial(\xi/\psi)}{\partial x_1} & \frac{\partial(\xi/\psi)}{\partial x_2} & \frac{\partial(\xi/\psi)}{\partial x_3} \end{bmatrix} \psi^3. \qquad (7)$$

Substituting η, ξ and ψ in terms of the x_i's yields an equation which defines an algebraic surface, σ', in $\mathbb{R}^3$. Geometrically, the boundary $\bar{\gamma}$ consist of points which represent tangent planes touching at points of σ on the intersection $\sigma' \cap \sigma$. Hence, $\bar{\gamma}$ is the image of the algebraic curve $\sigma' \cap \sigma = \gamma$ (see Fig. 3).

Combining the criterion, eq. (7), for the boundary with the equation of the surface (embedded in 3-dimensional space) and the transformation equations

(in homogenous coordinates) yields :

$$
\begin{aligned}
F(x_1, x_2, x_3) &= 0, \\
\eta Q(x_1, x_2, x_3) - \psi P(x_1, x_2, x_3) &= 0, \\
\xi Q(x_1, x_2, x_3) - \psi S(x_1, x_2, x_3) &= 0, \\
J(F(x_1, x_2, x_3))\psi^3 &= 0.
\end{aligned} \qquad (8)
$$

Solving for η, ξ and ψ yields the equation of the boundary. Note that if F is a polynomial of degree 2, then the degree of $J(F)\psi^3$ is ≤ 2 in terms of all variables, while it is linear in terms of x_1, x_2 and x_3.

Thus far we have constructed a system of four equations (8) in six variables which define a mapping from the $\mathbb{R}^3$ into the projective plane $\mathbb{P}^2$. Our aim, is to determine the specific equation of the region's boundary explicitly. This involves solving this system of equations in terms of η, ξ and ψ by eliminating the variables x_1, x_2 and x_3.

The equation's structure turns out to be very advantageous. Since the last three equations are linear, the elimination can be done by isolating a variable (finding an explicit expression in term of the other variables), and substituting in the remaining linear equations. When all is said and done, each of the variables x_1, x_2 and x_3 can be expressed as a rational equation in η, ξ and ψ. Upon substitution of these expressions into F the boundary's equation in homogeneous coordinates is obtained. It follows that the boundary $\bar{\gamma}$ is a quadratic curve.

3.2 Algorithm

The algorithm's input is an equation of algebraic surface $\sigma : F(x_1, x_2, x_2) = 0$ of degree two and the output is the polynomial which describes the boundary
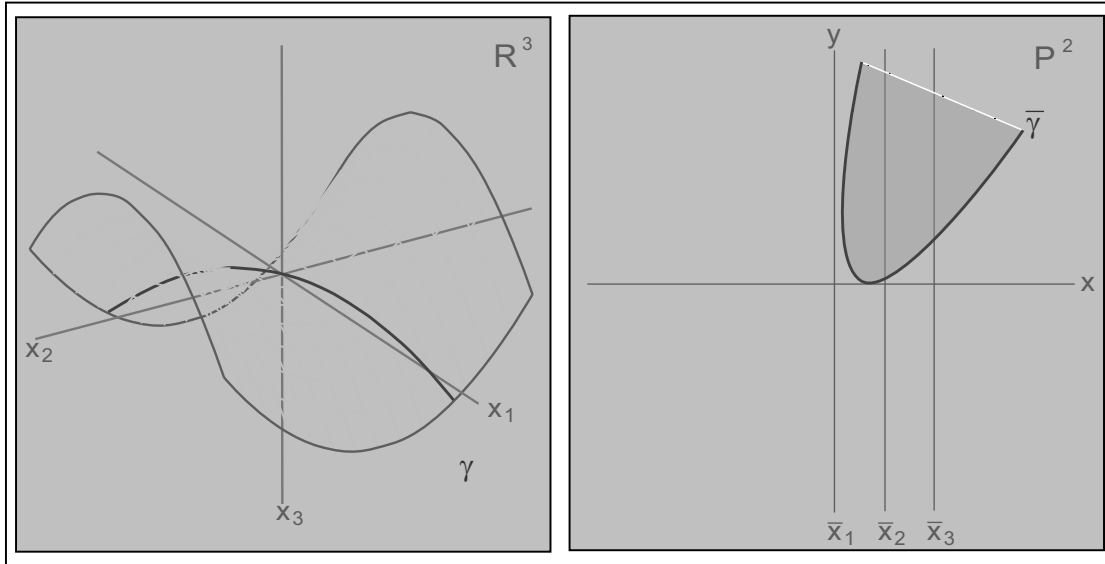
Figure 4. Saddle σ: $z = -(x/2)^2 + (y/2)^2 \rightarrow \bar{\gamma}: 16 - 16x - 4y + y^2 - 4xy + 4x^2 = 0$.

of the surface's image in $\|$-coords. It is noteworthy that the algorithm applies to implicit or explicit polynomials with or without singular points. The formal description is followed by examples which clarify the various stages and their nuances.

For a given polynomial equation $F(x_1, x_2, x_3) = 0$ of degree 2 and a spacing of axes $S_{1...n}$:

- Let :

$$\eta = \sum_{i=1}^{3}(i-1)\frac{\partial F}{\partial x_i} \ ,$$

$$\xi = \sum_{i=1}^{3} x_i\frac{\partial F}{\partial x_i} - 2F \ ,$$

$$\psi = \sum_{i=1}^{3} \frac{\partial F}{\partial x_i}.$$

- Write the three linear equations:

  a) $\quad \psi \sum_{i=1}^{3}(i-1)\frac{\partial F}{\partial x_i} - \eta \sum_{i=1}^{3} \frac{\partial F}{\partial x_i} = 0$,

  b) $\quad \psi(\sum_{i=1}^{3} x_i\frac{\partial F}{\partial x_i} - 2F) - \xi \sum_{i=1}^{3} \frac{\partial F}{\partial x_i} = 0$,

  c) $\quad J(F)\psi^3 = 0$.

- Using substitution write

$$x_i = f_i(\eta, \xi, \psi) \ , \quad \text{for } i = 1, 2, 3.$$

- Substitute

$$F(f_1(\eta, \xi, \psi), f_2(\eta, \xi, \psi), f_3(\eta, \xi, \psi)) = 0.$$

- Retain the equation's numerator.

- The output is obtained by substitution:

$$\eta \leftarrow x \ , \quad \xi \leftarrow y \ , \quad \psi \leftarrow 1.$$

All this falls into place with the following examples.

4 EXAMPLE OF QUADRATIC SURFACE AND THEIR TRANSFORMS

$\mathcal{I}$n the first example is quite detailed to accommodate the readers wishing to follow the application of the algorithm in depth. Let σ be 3-dimensional saddle (see Fig 4) generated by the polynomial equation,

$$F(\bar{x}) = \left(\frac{x_1}{2}\right)^2 - \left(\frac{x_2}{2}\right)^2 + x_3 = x_1^2 - x_2^2 + 4x_3 = 0,$$

where $\bar{x} = (x_1, x_2, x_3)$, and $S_{1...n}$ the standard spacing of axes.

step 1 Let :

$$
\begin{array}{llll}
P(\bar{x}) = & \sum_{i=1}^{3}(i-1)\frac{\partial F}{\partial x_i} & = & -2x_2 + 8, \\
S(\bar{x}) = & \sum_{i=1}^{3} x_i\frac{\partial F}{\partial x_i} - 2F & = & -4x_3, \\
Q(\bar{x}) = & \sum_{i=1}^{3} \frac{\partial F}{\partial x_i} & = & 2x_1 - 2x_2 + 4.
\end{array}
$$

step 2 Write[1] the three linear equations:

a) $\eta Q(\bar{x}) - \psi P(\bar{x}) =$
$$2(\eta x_1 + (\psi - \eta)x_2 + 2\eta - 4\psi) = 0,$$

b) $\xi Q(\bar{x}) - \psi S(\bar{x}) =$
$$2(\xi x_1 - \xi x_2 + 2\psi x_3 + 2\xi) = 0,$$

c) $J(F(\bar{x}))\psi^3 = (x_1(\psi - \eta) + x_2\eta - \xi)$.

Notice: substitution of η, ξ and ψ in terms of x_1, x_2 and x_3 yields a surface in $\mathbb{R}^3$,

$$\sigma': 32(-x_2^2 + 4x_2 + x_1^2 + 2x_3 - 2x_1) = 0.$$

Hence $\gamma = \sigma' \cap \sigma$,

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[1]The authors acknowledge and are grateful for the use of the symbolic manipulation program *Singular* developed by the Algebraic Geometry Group, Department of Mathematics, University of Kaiserslautern, Germnay.
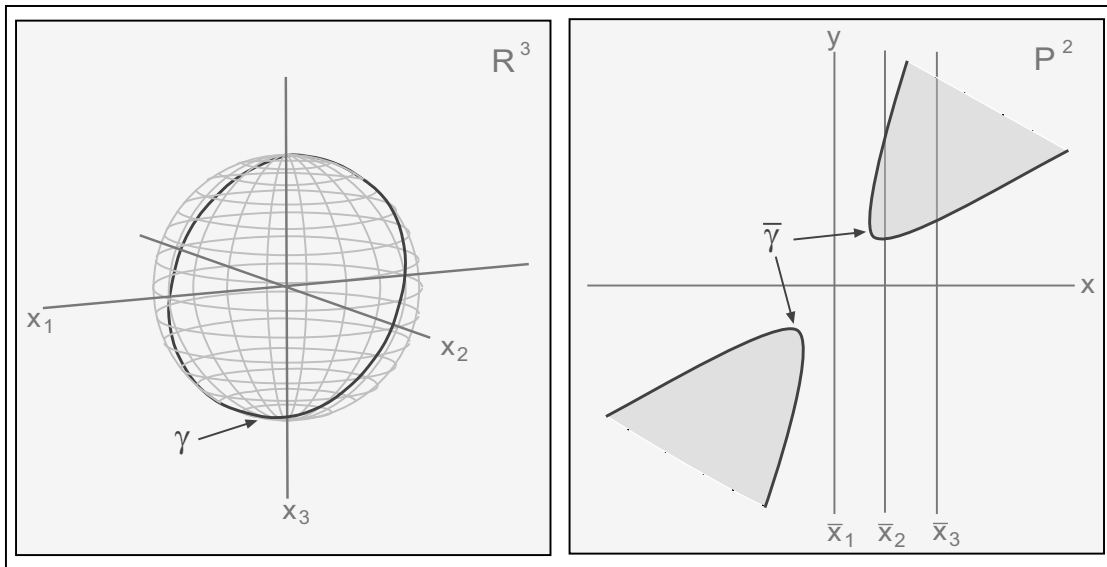
Figure 5. Sphere σ: $x^2 + y^2 + z^2 = 2 \;\;\rightarrow\;\; \bar{\gamma} : x^2 - 4xy + y^2 + 1 = 0$.
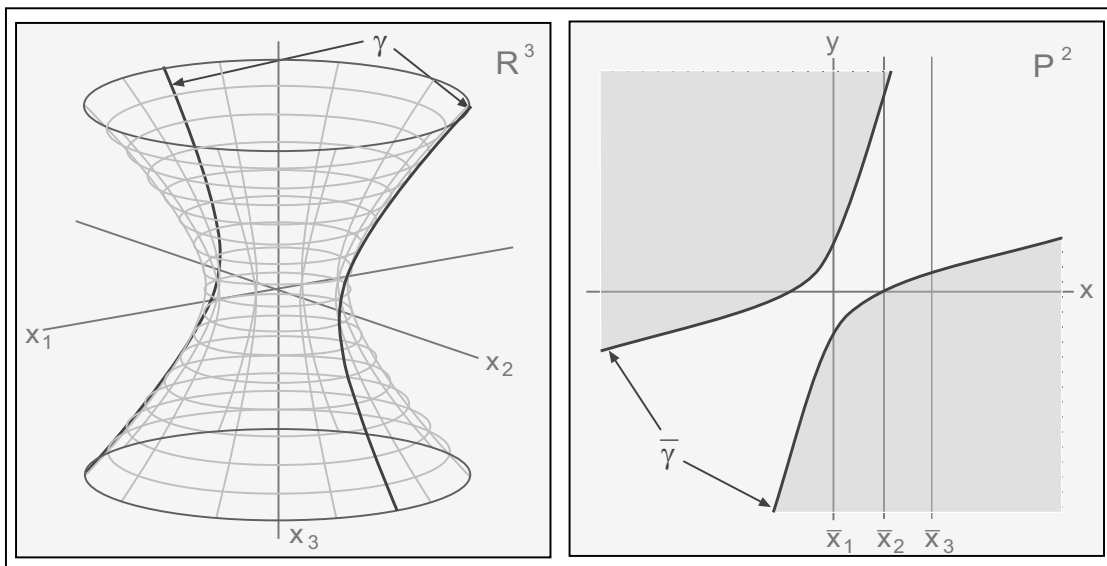

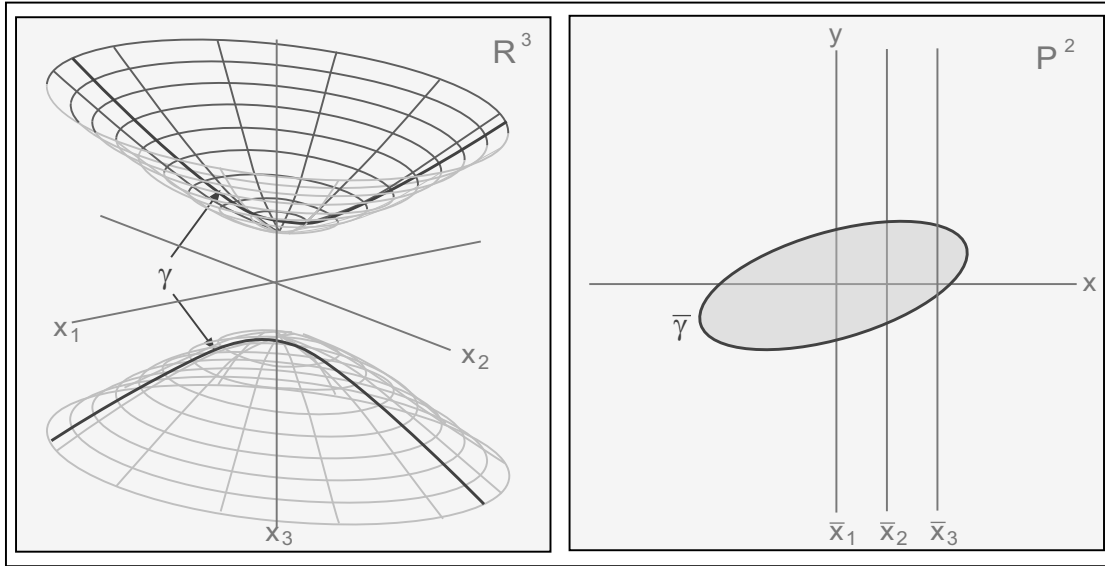Figure 6. Hyperboloid of one sheet $\sigma : x^2 + y^2 - z^2 = 1 \;\;\rightarrow\;\; \bar{\gamma} : x^2 - 4xy + y^2 - 1 = 0$.

Figure 7. Hyperboloid of two sheets σ: $x^2 - 4y^2 + 2z^2 = -2 \quad \rightarrow \quad \bar{\gamma}: x^2 - 2xy + 4y^2 - 1 = 0$.

$$\gamma: \quad \begin{cases} x_1^2 - x_2^2 + 4x_3 & = 0 \\ -x_2^2 + 4x_2 + x_1^2 + 2x_3 - 2x_1 & = 0 \end{cases}.$$

step 3 Using simple substitution write

$$b) \Rightarrow x_3 = -\frac{\xi(x_1 - x_2 + 2)}{2\psi},$$

$$c) \Rightarrow x_2 = -\frac{(\psi - \eta)x_1 - \xi}{\eta}.$$

Then using equation $a)$ we get,

$$x_1 = -\frac{-2\eta^2 + \eta\xi - \psi\xi + 4\psi\eta}{\psi(\psi - 2\eta)},$$

$$x_2 = \frac{2\eta^2 - \eta\xi - 6\psi\eta + 4\psi^2}{\psi(\psi - 2\eta)},$$

$$x_3 = \frac{\xi(-\xi + 2\eta + 2\psi)}{2\psi(\psi - 2\eta)}.$$

step 4 Substitute in F,

$$x_1^2 - x_2^2 + 4x_3 =$$

$$-\frac{16\psi^2 - 16\psi\eta - 4\psi\xi + \xi^2 - 4\eta\xi + 4\eta^2}{\psi(\psi - 2\eta)} = 0.$$

step 5 Retain the equation's numerator,

$$16\psi^2 - 16\psi\eta - 4\psi\xi + \xi^2 - 4\eta\xi + 4\eta^2 = 0.$$

step 6 Finally, the output is obtained by substitution,

$$\eta \leftarrow x \ , \ \xi \leftarrow y \ , \ \psi \leftarrow 1:$$

$$\bar{\gamma}: \quad 16 - 16x - 4y + y^2 - 4xy + 4x^2 = 0.$$

The surface and its image including the boundary curve are shown in Fig 4. The representation of other quadratic surfaces is illustrated in figures 5, 6 and 7.

5 GENERALIZATION FOR MULTIDIMENSIONAL QUADRATICS HYPER-SURFACES

$\mathcal{S}$ummarizing, the representation of 3-dimensional quadratic surfaces is obtained from :

1. the criterion for the boundary points, and

2. solving the system of equations.

For dimension $n = 3$ there are only two indexed regions. The generalization to arbitrary dimension n, to be covered in a future paper, develops along similar lines.

The boundary criterion, (6), in homogenous coordinates is defined for any dimension. The following part involves the elimination requiring that the number of equations be greater then the number of variables. Hence the system needs to be extended to at least $n+1$ equations. Without entering into the details it is possible to obtain the additional equations to complete the construction algorithm. Then the hyper-surface's representation consists of $n - 1$ regions stemming from the $n - 1$ indexed points representing each of the tangent hyper-planes.

6 CONCLUSION

$\mathcal{T}$he new representation

- is constructive,

- enables the representation of non-convex objects,

- maps algebraic surfaces to regions having algebraic curves as boundaries,

- has properties which enable the easy recognition of the some surface properties, and

- provides a new basis for the approximation of complex surfaces, from their corresponding *planar* regions.

An important "fringe benefit" is the avoidance of the "over-plotting" problem in ∥-coords where polygonal lines obscure portions of the display.

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