ON A DIVIDEND STRATEGY OF INSURANCE COMPANIES

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1. Introduction

Shares entitle their owners to receive the net profits of the company. After payment of debt claims is made, the remaining earnings can either be paid out in the form of dividends or used to finance the company´s investments.

To value ordinary shares, several factors (the fundamentals of the company) should be taken into account, e. g. earnings, dividends, risk, the cost of money, assets, the future growth rate together with the market expectations, management, etc. A valuation model then transforms them to the expected market value of shares. There are several models used for the valuation. In this contribution discounted cash flow models are considered as they are widely used when solving problems that we are going to discuss here.

When trying to estimate the value of a share these models are based on the assumption that it is equal to the present value of the cash flow that the shareholder expects to receive from it. If we suppose that dividend D_t is paid out at the end of the period t (usualy one year), then the present value of this stream of dividends (i. e. in time 0) is

$$\frac{D_1}{(1+i)}+\frac{D_2}{(1+i)^2}+\ldots+\frac{D_n}{(1+i)^n}+\ldots \tag{1}$$

where i means the appropriate discount rate, which is usually equal to investor´s required annual rate of return. Two models of this type became the most popular among economists. The so-called constant growth model which supposes that dividends will grow at the same rate g into the indefinite future is the simplest one. In this case (1) sums to

$$D(i-g)^{-1} \tag{2}$$

provided $i > g$ and D means the first dividend paid out. This model is advocated by the following argument: the dividend policy of a company is stable over time and, moreover, a return on new equity investments is stable. It follows that in such a case the growth in earnings equals the growth in dividends.

Instead of this simple model a natural extension is widely used- the two-period growth model based on the idea that after some years it is not reasonable (because of the lack of relevant information) to differentiate between companies but it is assumed that they all grow at the same rate. Then the following relation holds:

$$D\frac{1-\left(\frac{1+g_1}{1+i}\right)^m}{i-g_1}+D\frac{(1+g_1)^{m-1}(1+g_2)}{(i-g_2)(1+i)^m} \tag{3}$$

where the growth rate of dividends during the first m years is g_1 and after this time the company´s growth rate is g_2. (For details see, e. g. [4]).

In the following a discounted cash flow model is used in order to solve a typical problem which an insurance company, being the share company by legislation, is faced with. In the classical risk theory the surplus of an insurance company is a function of the insurer´s initial surplus (or free reserves), the premium received, the number of claims up to a certain time and the amount of claims. If in a time, say, τ , the premium collected plus the initial surplus is less than the aggregate claim amount, the so-called ruin will occur. Such a situation has been studied for many years starting with the pioneer works of Cramér and Lundberg. A handful of useful results exists which will be mentioned later. In fact, the event of ruin does almost never occur in practice. If an insurance company observes that their surplus is decreasing they will immediately increase premiums. On the other hand, if the business is successful increasing dividends can be expected. As models with a premium rate dependent on the surplus are too complicated, different strategies are preferred.

The paper is further organized as follows:

- part two discusses the so-called barrier strategy for treating the (possible) surplus of an insurance company and shows how dividends are paid out in the case the surplus is greater than a predetermined value of the barrier. We survey in short the results on the expected discounted sum of dividends and show for which value of the barrier this sum will attain the greatest value. As shown in [1] and [3] the exact solution exists provided that claims are exponentially distributed.
- part three provides the solution of the same situation, however, under completely different assumptions. It is well known that the ruin problem which is closely related to the dividend strategy problem leads to quite different results when exponentially distributed claims (the so-called small claims) are replaced by large claims. We know that insurance companies are now confronted with such claims more often than before. Let us mention catastrofic floods, earthquakes, etc. which cannot be modelled by small claim ditributions. The case of non-identically distributed claims is also treated for both small and large claims.

In the last section of the paper we discuss the results obtained and mention the possible ways how to handle some difficulties we met with.

2. Optimal Dividend Barrier - the Case of Small Claims

We consider a homogeneous portfolio of independent, identically distributed positive claims X_k with the distribution function F and the finite expectation (mean) μ. The claims occur in random times T_n and their number in the time interval $[0, t]$ is counted by the process $N(t) = \sup\{n \geq 1, T_n \leq t\}$. If the inter-arrival times are exponentially distributed, $N(t)$ is a homogeneous Poisson process with intensity, say, λ. This is the classical Cramér-Lundberg model. (Such a model is still very popular among actuaries, though some prefer the model where the number of claims has a negatively binomial distribution.)

The corresponding process of aggregate claims is $S(t) = \sum_{i=1}^{N(t)} X_i$. Suppose that the insurer has an amount of money set aside for this portfolio at time 0. This amount of money is called the initial surplus or free reserves and is denoted by $u \geq 0$. The insurer´s surplus at any future time t is a random variable, since its value depends on the claims experience up to time t. It will be denoted by $U(t)$. So we have the model

$$U(t) = u + ct - S(t) \tag{4}$$

where c means the premium income rate in one time unit. The model is called the surplus model or risk model. It follows easily that $EU(t)/t \to c-\lambda\mu$ for $t \to \infty$. So the condition $c-\lambda\mu > 0$ is necessary for the solvency of the insurance company. However, it can happen that $U(t)$ falls below zero as a result of the last claim. In such a case we say that ruin has occured. Of course, the company wishes to keep the probability of such event as small as possible. Therefore we define the probability of ultimate ruin as

$$\Psi(u) = P\{U(t) < 0 \quad \text{for some } t \in (0, \infty)\} \tag{5}$$

and the probability of the ruin within the time interval $(0, t)$ as

$$\Psi(u, t) = P\{U(\tau) < 0 \ \text{ for some } \tau \in (0, t)\} \tag{6}$$

If we assume that a constant interest rate (or force) affects the process, these probabilities will be denoted by $\Psi_r(u)$ and $\Psi_r(u, t)$, respectively.

It is well known (see, e. g [5]) that if the solvency condittion (also known as the net profit condition) $c-\lambda\mu > 0$ holds and a constant $\nu > 0$ exists such that $\int_0^\infty e^{\nu x} \bar{F}(x)\,dx = c/\lambda$, then for all $u \geq 0$ $\Psi(u) \leq e^{-\nu u}$, where $\bar{F}(x) = 1 - F(x)$ is the tail distribution of claims. Hence the existence of the constant ν (the so-called Lundberg´s exponent) is crucial for validity of the above mentioned estimation of the ruin probability. It follows that the result may be only applied to such claims, whose moment generating function exists at least in a vicinity of zero, otherwise the constant ν cannot be found. Such claims are usually termed small claims as the probability they will exceed large values goes to zero quickly. Among them the exponential distribution plays the key role as we shall see in what follows. Exactly for this distribution the probability of ruin can be expressed by the formula

$$\Psi(u) = \frac{1}{1+\rho} \exp \frac{-\rho u}{\mu(1+\rho)} \tag{7}$$

where $\rho = c/(\lambda\mu) - 1$.

A short inspection of the formula (7) says that, even for $u=0$, this probability is always less than 1 and decreases with u increasing. It follows from this that the surplus $U(t)$ tends to infinity with probability 1. For this reason some economists suggested to reduce free reserves consistently to a predetermined value and to pay out the reduction as dividends to shareholders of the company.

Instead of the continuous process (4) we can investigate its discrete version, that is to consider the surplus only at the integral time points $t=0, 1, 2...$ Then we have the model

$$U_t = u + ct - S_t \tag{8}$$

Suppose that dividend D_t is paid out at the end of the year t. We can imagine as if the original surplus U_t has been replaced by the new process, say Q_t, for which we have

$$Q_t = U_t - \sum_{j=1}^{t-1} D_j \tag{9}$$

We suppose that $0 \le D_t \le Q_t$. If τ means the time of ruin, i. e. $Q_t < 0$ for the first time for $t = \tau$, we , of course, set $D_\tau = D_{\tau+1} = ... = 0$. Then

$$E\left[\frac{D_1}{(1+i)} + \frac{D_2}{(1+i)^2} + \dots \frac{D_{\tau-1}}{(1+i)^{\tau-1}}\right] \tag{10}$$

is the expected discounted sum of dividends paid out. Comparison of (10) with (1) shows that they are formally the same, but the operator of the expected value is applied in (10) as, this time, dividends are random variables dependent on the aggregate claims.

We are looking for such a strategy which maximizes (10). In what follows, we employ the so-called barrier strategy because of its practical significance and simplicity. (See , e. g. [1] and [3] for details)

According to this strategy $D_t = \max(Q_t - B, 0)$, i. e. $D_t = Q_t - B$, provided $Q_t > B$ and $D_t = 0$ otherwise, so that no dividend is paid for $Q_t \le B$, otherwise Q_t is reduced to B, which is called the barrier. We suppose that $u < B$. It follows that the modified surplus will never excced B.

Returning to the continuous model (4) we have

$Q(t) = u + ct - S(t) - Y(t)$

where $Y(t)$ means the summed dividend, i. e. $Y(t_2) - Y(t_1)$ are dividends paid out in the interval $<t_1, t_2)$. We have $Y_t = Y_\tau$ for all $t \ge \tau$ where τ is the point of time at which the ruin occurs. Then the barrier strategy with the barrier B is as follows (It should be emphasised that in the continuous case a dividend can be paid at each point of time):

If $Q(t) < B$, no dividends are paid out. If $Q(t) = B$, the incoming premiums are paid out as dividends and $Q(t) - B$ are paid out provided $Q(t) > B$.

The aim is to maximize the continuous version of (10) with the sum replaced by integral and using the force of interest $\delta = \ln(1+i)$ as the discount factor instead of $(1+i)$. In what follows the expected discounted sum of dividends will be denoted by $V(u, B)$.

It is shown in [1] and [3] that $V(u,B)$ satisfies the integro-differential equation

$$V^{(1)}(u,B) = \frac{\lambda+\delta}{c} V(u,B) + \frac{\lambda}{c}\int_u^0 f(x) V(u-x,B)\,dx \tag{11}$$

with the boundary condition $V^{(1)}(B,B) = 1$. As expected this equation is very similar to the equation for the probability that no ruin happens.

Solving the equation (11) becomes easier if we put $V(u,B) = h(u)/h^{(1)}(B)$ for a suitable $h(u)$. Then we solve the equation

$$ch^{(1)}(x) = (\lambda+\delta)\,h(x) - \lambda\int_0^x h(x-y) f(y)\,dy \tag{12}$$

in two steps. We find a positive solution of (12) and then find such a B for which $h^{(1)}(B)$ is the minimum.

Suppose that claims have the exponential distribution with the density $f(x) = \alpha e^{-\alpha x}$ (see [1] and [3]). By differentiation of (12) we obtain

$ch^{(2)}(x) - (\lambda + \delta - c\alpha)h^{(1)}(x) - \delta\alpha h(x) = 0$

The general solution is $h(x) = C_1 \exp(r_1 x) + C_2 \exp(r_2 x)$,

where $C_1/C_2 = -(r_1 + \alpha)/(r_2 + \alpha)$ and r_1 and r_2 are the roots of the characteristic equation

$cr^2 - (\lambda + \delta - c\alpha)\, r - \delta\alpha = 0.$

We note that C_1 and C_2 have different signs and thus $h(x)$ is positive. In order to find the point B_0 at which $h^{(1)}(x)$ takes on its minimum, we solve $h^{(2)}(B_0) = 0$.

The solution is

$$B_0 = (r_1 - r_2)^{-1} \ln[-C_2 r_2^2 / C_1 r_1^2] \tag{13}$$

From this we obtain

$$V(u,B) = \frac{(\alpha + r_1)\exp(r_1 u) - (\alpha + r_2)\exp(r_2 u)}{(\alpha + r_1) r_1 \exp(r_1 B) - (\alpha + r_2) r_2 \exp(r_2 B)} \tag{14}$$

The problem becomes more complicated if the assumption that claims are identically distributed is omitted. Consider, e. g. the discrete time model (8) under the assumption that a constant interest rate affects $r > 0$ the surplus process. Then the surplus at the end of the t-th year is

$$U_t = u(1+r)^t + \sum_{k=1}^{t} \frac{c}{(1+r)^{k-t-1}} - \sum_{k=1}^{t} \frac{X_k}{(1+r)^{k-t}}$$

Put $Y_k = (X_k - c(1+r))(1+r)^{-k}$ and $Z_n = \sum_{k=1}^{n} Y_k$.

We have $U_n = (1+r)^n (u - Z_n)$ and thus $\Psi_r(u,n) = P(\max Z_k \rangle u, k = 1,...n)$. For the exponential distribution we obtain

$$\Psi_r(u,n) = P(\max Z_k \rangle u, k = 1,...n) \le K(r,n)e^{-\alpha u} \quad (15)$$

where $K(r,n) = e^{\alpha \frac{(1+r)^n - 1}{r(1+r)^{n-1}}}$. If $r = 0$, the coefficient at $e^{-\alpha u}$ is n.

3. The Case of Large Claims

Actuaries distinguish several types of distributions to describe insurance data: gamma (exponential being its special case), log-gamma, log-normal, gamma + log-gamma, gamma + log-normal and Pareto being the most important. However there are big differences among them. As mentioned earlier, gamma belongs to the so-called small claim distributions. The risk theory is well developed for them mainly because approaches based on existence of the moment generating function may be used. This is, however, not the case in the situation we are going to discuss now.

Let us begin with Pareto distribution for with $\bar{F}(x) = (1+x)^{-\alpha}, \alpha > 1$. No Lundberg´s exponent exists since $\int_0^\infty e^{vx}(1+x)^{-\alpha}\,dx = \infty$

for every $v > 0$. Such distributions are called heavy-tailed. Among them are claim size distributions with regularly varying tails such as Pareto, Burr, log-gamma. For them the existence of some $\alpha \ge 0$ with the property $\lim_{x\to\infty} \bar{F}(t.x)/\bar{F}(x) = t^{-\alpha}$ for all $t \ge 0$ is typical. However, there exist many distributions which violate the property of regularly varying tails, but they belong to the class of heavy tailed distributions. It turns out that the most interesting subclass is the class of subexponential distributions, which is characterized by the following property:

$$\lim_{x\to\infty} \frac{P\{X_1 + X_2 + \dots X_n > x\}}{P\{X_1 > x\}} = n$$

for all $n \ge 2$.

This subclass is usually denoted by **S**. Why is this subclass so important for actuaries? The reason is obvious. It can be proved that subexponential distributions have the following property:the distribution of the sum of the first, say, n claims is determined by the distribution of their maximum. In other words, they are suitable for modellig the so-called extreme events, which insurance companies encounter more often now than before. They are distributions that converge to zero slower than exponential distribution does. Hence this does not belong to **S**. On the other hand, all distributions with regularly varying tails belong to **S**. The natural question, whether there exists a distribution which does not belong to the regularly varying class but is from **S**, can be answered in the affirmative. The Weibull distribution and log-normal are typical examples.

At the beginning let us restrict our attention to the ruin problem we mentioned in the first part of the paper. As the Lundberg exponent does not exist for heavy tailed distributions a different approach has to be applied for deriving the probability of ruin. We start with Pareto distributed claims and the probability of ruin for discrete time model under the assumption that a constant interest rate $r > 0$ affects the surplus process.

Using the same notation as above we have $U_n = (1+r)^n(u - Z_n)$ and thus $\Psi_r(u,n) = P(\max Z_k \rangle u, k = 1,...n)$.

If $Y_k \in \mathbf{S}$, then $\Psi_r(u,n) = P(Z_n > u)$ because of the above mentioned property of subexponential distributions. It follows that

$$\Psi_r(u,n) = \sum_{k=1}^{n} P\{Y_k > u\} = \sum_{k=1}^{n} P\left\{\frac{R_k}{(1+r)^k} > u\right\}$$

where $R_k = X_k - c(1+r)$. Let X_k be Pareto distributed with tail $(1+x)^{-\alpha}$, $\alpha > 0$ Then

$$\Psi_r(u,n) = \frac{\left(1-(1+r)^{-n\alpha}\right)}{(1+r)^{\alpha}-1}(1+u)^{-\alpha} \quad (16)$$

For $r = 0$ the ruin probability is $n(1+u)^{-\alpha}$. It must be said that the result holds only for large u.

Now we come to the probability of an ultimate ruin. As shown in [5] for subexponential distributions the following holds:

$$\Psi(u) \sim 1/\rho\mu \int_u^\infty \bar{F}(y)\,dy \quad (17)$$

for $u \to \infty$ provided $\rho > 0$

It is worth mentioning that the formula (17) means that $\lim_{u\to\infty} \rho\mu\Psi(u)/\int_u^\infty \bar{F}(y)dy = 1$. Unlike small claim distributions where the approximation of by a function of $\Psi(u)$ Lundberg´s exponent holds for moderate values of u , the situation for heavy tailed distributions is quite different. In order to get the reliable values of the ruin probability the right hand side of (17) may be used only for large values of u. There are several papers discussing this problem, which say that the convergence is very slow(see, e. g. [11]). We recall that the formula for $\Psi_r(u)$ is given in [6].

Another important question in this context is the choice of a proper claim distibution. If we consider mistakenly a wrong distribution for individual claims, especially large ones, it may lead to large mistakes for the aggregate claim distribution as shown in [11]. Therefore testing the quality of fit is recommended. (See, e. g. [10]).

It can be shown that testing the hypothesis on the parameter of Pareto distribution leads to a statistics depending on gamma distribution.

Now we come to the question of the barrier dividend strategy for portfolios of large claims. It can be shown (see, e. g. [1]) that the integro-differential equation (11) holds also in this case. However, it turns out, that a satisfactory solution does not exist for Pareto distributed claims and an alternative approach should be used. The same is true for Weibull and log-normal distributions. The problem will be dealt with in a separate paper.

4. Discussion

The problem of finding the optimal dividend strategy is very important for insurance companies. We showed that the problem is closely related to the problem of ruin, which plays the key role in decision taking for insurers. It follows from the results presented in the paper that following the barrier strategy would inevitably lead to ruin. Then the natural question arises, namely, what should be done after such an event happens. As pointed out by many actuaries, instead of maximizing the expected discounted dividends, the difference between them and the deficit at ruin should be maximized. There are several proposals how to solve this problem discussed in [2].

At the end of the paper we would like to pay attention to some problems, which may be met when one tries to solve the above mentioned questions. Besides the choice of a proper claim distribution parameter estimation is also important especially for large claim distributions. Testing hypotheses on parameters is in general a difficult problem which remains open for many situations encountered in practice. (See, e. g. ([10] and [11]). We recommend the papers [7] and [8] as a survey of other possibilities in this area. We remind that the surplus model may be used not only for the situation described here but also for problems related to pension funds. (See, e. g. [11] and [12]).

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ABSTRACT

ON A DIVIDEND STRATEGY OF INSURANCE COMPANIES

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The problem of finding the optimal dividend strategy is very important for insurance companies. In order to solve this problem a discounted cash flow model has been used which is a special case of valuation models. We analyse the surplus of an insurance company with attention focused on probability of ruin. The usual strategy of insurers to allow the surplus go to infinity in order to avoid ruin has been crticised by several economists (De Finetti and others) as too conservative. In the paper we discuss the so-called barrier strategy for treating the surplus of an insurance company and show how dividends are paid out in the case the surplus is greater than a predetermined value of the barrier. Using the results in (1(and [3] we present the expected discounted sum of dividends and show for which value of the barrier this sum will attain the greatest value.

In the first part of the paper the problem is analysed in details provided that claims are exponentially distributed. The purpose of the paper is to solve the same situation, however, for the so-called large claims. It is well known that the ruin problem which is closely related to the dividend strategy problem leads to different results for such claims. On the other hand insurance companies are now confronted with such claims more often than before. Let us mention catastrofic floods, earthquakes, etc. which cannot be modelled by exponential and similar distributions. The problem of non-identically distributed claims is also treated.

The last section of the paper deals with the question how the dividend strategy should be changed after ruin happens. A solution is mentioned suggested by several actuaries that instead of maximizing the expected discounted sum of dividends the difference between them and the deficit at ruin should be maximized. At the end our attention is paid to the questions of parameter estimation and the choice of a proper claim distribution.

Key Words: *dividend strategy, probability of ruin, small and large claims*

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