UNIVERSITY OF WEST BOHEMIA

Applied and Computational Mechanics 1 (2007) 289 - 298

Applied and Computational Mechanics

Eigenfrequency sensitivity analysis of flexible rotors

J. Šašek[a,*]

[a] Faculty of Applied Sciences, UWB in Pilsen, Univerzitní 22, 306 14 Plzeň, Czech Republic

Received 10 September 2007; received in revised form 2 October 2007

Abstract

This paper deals with sensitivity analysis of eigenfrequencies from the viewpoint of design parameters. The sensitivity analysis is applied to a rotor which consists of a shaft and a disk. The design parameters of sensitivity analysis are the disk radius and the disk width. The shaft is modeled as a 1D continuum using shaft finite elements. The disks of rotating systems are commonly modeled as rigid bodies. The presented approach to the disk modeling is based on a 3D flexible continuum discretized using hexahedral finite elements. The both components of the rotor are connected together by special proposed couplings. The whole rotor is modeled in rotating coordinate system with considering rotation influences (gyroscopic and dynamics stiffness matrices).

Keywords: rotor dynamics, modal analysis, sensitivity analysis, solid elements, shaft elements

1. Introduction

The issue of the modeling of flexible rotating systems is still significant in the dynamics of engineering problems. Dynamics of turbines and dynamics of wheelsets in rail vehicles are the examples of the common applications of such systems. The presented paper is the enhancement of the previous contributions, which were made on the topic of flexible disks. It was started by [9] where derivation of equations of motion of flexible disks is described. In next work [10] the flexible connection between a disk (3D continuum) and a blade (1D continuum) was presented. The disk connection to a shaft was shown in [4]. There were described two approaches to the connection. The first is a rigid coupling which can be used for shrinkage fit modeling and the second is a flexible coupling which can be used for interlocking joint. The second approach for the disk-shaft connection is used in this article.

The main aim of this paper is the eigenfrequency sensitivity analysis of flexible rotors with respect to the chosen shape parameters of disks. Sensitivity analysis in rotor dynamics is important mainly for optimization problems in order to choose the proper design parameters. The results of sensitivity analysis can also contribute to the better understanding of complex system's behaviour. General knowledge about sensitivity analysis of the systems modeled by using finite element analysis can be found in monograph [6]. The analytical approach for the sensitivity analysis of rotating systems is developed in [3]. The example of the eigenvalue sensitivity analysis of the rotor with rigid disk is shown in [2] and of the spinning flexible disk in [1]. The sensitivity of the modal values of a flexible rotor with respect to the angular velocity is studied in [5].

The numerical approach for the eigenfrequency sensitivity analysis is used by the author of this paper. In the first part of the paper the mathematical model of the flexible disk-shaft

*Corresponding author. Tel.: +420 377 632 301, e-mail: jsasek@kme.zcu.cz.

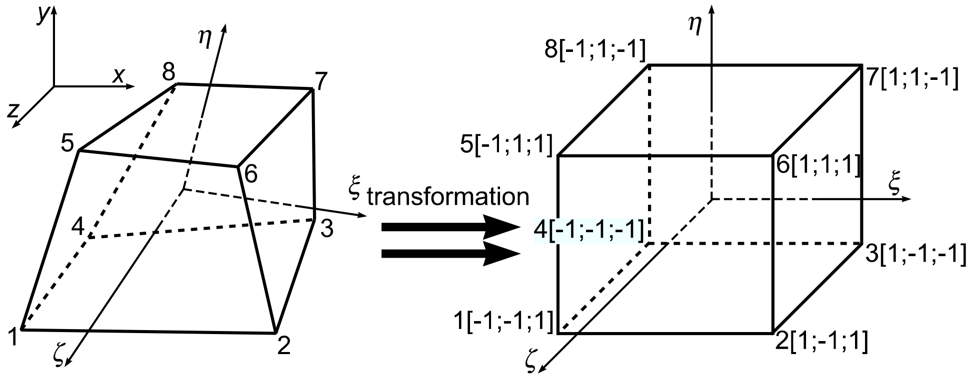

Fig. 1. Scheme of linear isoparametric hexahedral element (see [7]).

system including the mutual connection is presented. The second part of the paper deals with the example of the eigenfrequency sensitivity analysis of the simple test-rotor.

2. Mathematical model

In this section the mathematical model of rotating shaft with flexible disk will be described. The whole system consists of two subsystems – disk subsystem (subscript D) and shaft subsystem (subscript S). The disk will be modeled as three-dimensional continuum, the shaft will be modeled as one-dimensional continuum and the connection between the disk and the shaft will be provided by the flexible couplings. It is supposed that the subsystems are rotating with constatnt angular velocity ω_0 around their X-axis.

The disk can be discretized by isoparametric hexahedral elements (see Fig. 1). The equations of motions are derived in [9]. The mathematical model of the uncoupled disk subsystem can be written in the form

$$\boldsymbol{M}_D \ddot{\boldsymbol{q}}_D(t) + \omega_0 \boldsymbol{G}_D \dot{\boldsymbol{q}}_D(t) + (\boldsymbol{K}_{sD} - \omega_0^2 \boldsymbol{K}_{dD}) \boldsymbol{q}_D(t) = \omega_0^2 \boldsymbol{f}_D, \tag{1}$$

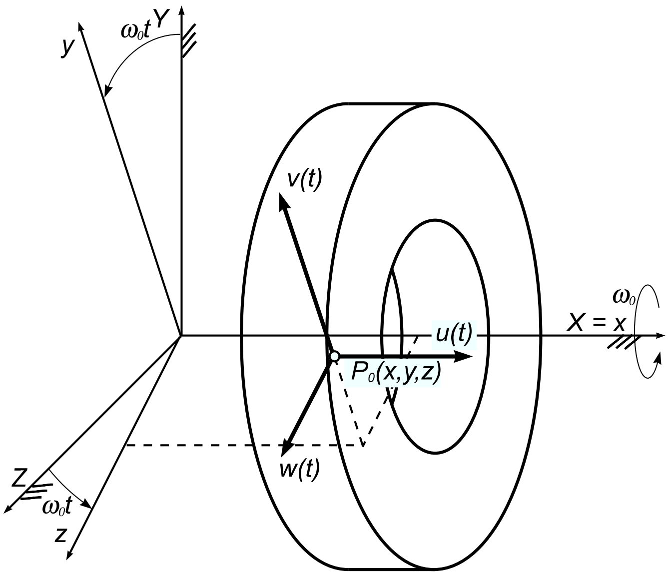

Fig. 2. Scheme of the disk coordinate systems.

where $\boldsymbol{M}_D$ is the mass matrix of the disk, the term $\omega_0 \boldsymbol{G}_D$ expresses gyroscopic effects which react on the disk, $\boldsymbol{K}_{sD}$ is the static stiffness matrix of the disk, $\boldsymbol{K}_{dD}$ is the dynamic stiffness matrix of the disk and $\omega_0^2 \boldsymbol{f}_D$ is the centrifugal load vector. The dimension of the subsystem matrices is $n_D \times n_D$. They are symmetrical ones except the gyroscopic matrix that is skew-symmetrical. The disk equations of motion are written in the configuration space defined by vector of nodal coordinates

$$\boldsymbol{q}_D = [\dots\ u_j^{(D)}\ v_j^{(D)}\ w_j^{(D)}\ \dots]^T \in \mathbb{R}^{n_D}, \tag{2}$$

where $u_j^{(D)}$ is displacement of node i in x-direction, $v_j^{(D)}$ is displacement of node i in y-direction, and $w_j^{(D)}$ is dispalacement of node i in z-direction (see Fig. 2), i.e. each node has three degrees of freedom.

The shaft is modeled as an one-dimensional continuum on assumption of the undeformable cross-section that is still perpendicular to the shaft axis. The derivation of the equations of motion is shown in [4]. This model is based on [8] where the derivation is performed in non-rotating coordinate system XYZ. But the new model is derived in rotating coordinate system xyz. The shaft is discretized using shaft finite elements (see Fig. 3) with two nodes. The dispalacement of each node i is described by six generalized coordinates – three dispalacements $u_i^{(S)}$, $v_i^{(S)}$, $w_i^{(S)}$ and three rotations $\varphi_i^{(S)}$, $\vartheta_i^{(S)}$, $\psi_i^{(S)}$. The shaft conservative mathematical model can be written in the form

$$\boldsymbol{M}_S \ddot{\boldsymbol{q}}_S(t) + \omega_0 \boldsymbol{G}_S \dot{\boldsymbol{q}}_S(t) + (\boldsymbol{K}_{sS} - \omega_0^2 \boldsymbol{K}_{dS} + \boldsymbol{K}_{BS}) \boldsymbol{q}_S(t) = \boldsymbol{0}, \tag{3}$$

where the configuration space is defined by vector

$$\boldsymbol{q}_D = [\dots\ u_j^{(S)}\ v_j^{(S)}\ w_j^{(S)}\ \varphi_i^{(S)}\ \vartheta_i^{(S)}\ \psi_i^{(S)}\ \dots]^T \in \mathbb{R}^{n_S}. \tag{4}$$

Mass matrix $\boldsymbol{M}_S$, static stiffness matrix $\boldsymbol{K}_{sS}$ and dynamic stiffness matrix $\boldsymbol{K}_{dS}$ are symmetrical, and gyroscopic matrix $\omega \boldsymbol{G}_S$ is skew-symmetrical. Rolling-element bearings are described

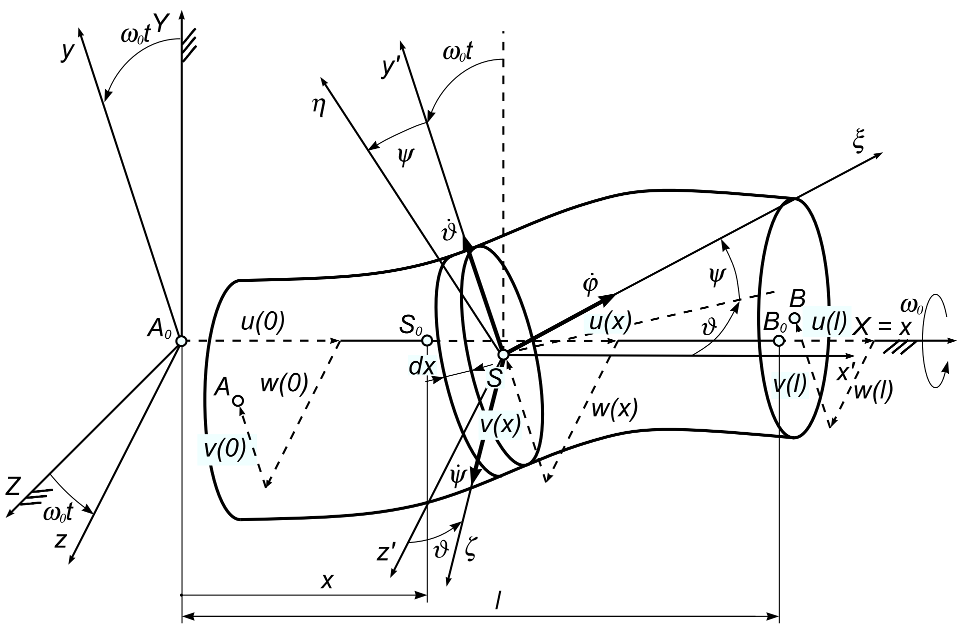

Fig. 3. Scheme of the shaft finite element.

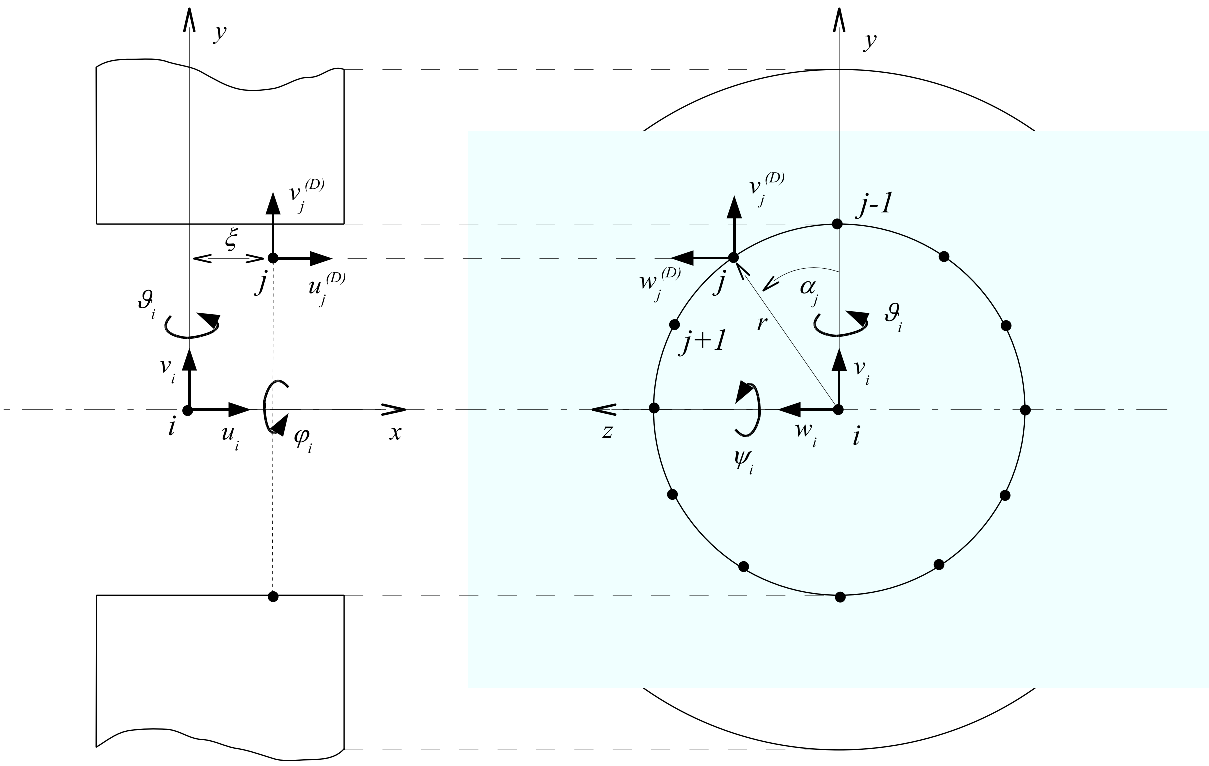

Fig. 4. Shaft and disk nodes and their displacements in the rotating coordinate system.

in this mathematical model by symmetrical stiffness matrix $\boldsymbol{K}_{BS}$. Particular forms of the bearing stiffness matrix can be found e.g. in [8] or in [11].

The connection between the disk and the shaft is realized by the flexible coupling [4]. This methodology can be used e.g. for representing the interlocking joint, that is usual design solution in some engineering applications. The conservative mathematical model of disk and shaft subsystems mutually joined by the flexible coupling is of the form

$$\boldsymbol{M}_D\ddot{\boldsymbol{q}}_D(t) + \omega_0\boldsymbol{G}_D\dot{\boldsymbol{q}}_D(t) + (\boldsymbol{K}_{sD} - \omega_0^2\boldsymbol{K}_{dD})\boldsymbol{q}_D(t) = \omega_0^2\boldsymbol{f}_D + \boldsymbol{f}_D^C, \tag{5}$$

$$\boldsymbol{M}_S\ddot{\boldsymbol{q}}_S(t) + \omega_0\boldsymbol{G}_S\dot{\boldsymbol{q}}_S(t) + (\boldsymbol{K}_{sS} - \omega_0^2\boldsymbol{K}_{dS} + \boldsymbol{K}_{BS})\boldsymbol{q}_S(t) = \boldsymbol{f}_S^C, \tag{6}$$

where all matrices and vectors except vectors $\boldsymbol{f}_D^C$ and $\boldsymbol{f}_S^C$ are explained in the previous section of the paper. These vectors represent the coupling forces between particular subsystems. The coupling forces are acting in the chosen shaft nodes, where the disk is mounted on, and in the chosen disk nodes, that lie on the inner circumference of the disk body (see Fig. 4).

The global coupling force vector $\boldsymbol{f}_C$ in global configuration space of the disk-shaft system

$$\boldsymbol{q} = \left[\, \boldsymbol{q}_D^T \; \boldsymbol{q}_S^T \,\right]^T \tag{7}$$

can be calculated by differentiating the potential (strain) energy

$$\boldsymbol{f}_C = \left[\begin{array}{c} \boldsymbol{f}_D^C \\ \boldsymbol{f}_S^C \end{array} \right] = -\frac{\partial E_p^C}{\partial \boldsymbol{q}}. \tag{8}$$

If the disk-shaft coupling is realized using n_i shaft nodes and n_j disk nodes for each shaft node the global coupling force vector can be rewritten in the form

$$\boldsymbol{f}_C = -\boldsymbol{K}_C \boldsymbol{q} = -\sum_{i=1}^{n_i} \sum_{j=1}^{n_j} \boldsymbol{K}_{i,j}^C \boldsymbol{q}, \tag{9}$$

where stiffness matrices $\boldsymbol{K}_{i,j}$ corresponding to the particular coupling between i-th and j-th nodes are calculated as

$$\frac{\partial E_{i,j}^C}{\partial \boldsymbol{q}} = \boldsymbol{K}_{i,j}^C \boldsymbol{q}. \tag{10}$$

The coupling is characterized by three stiffnesses k_t in tangent direction to the shaft cross-section, k_r in radial direction and k_{ax} in axial direction. These stiffnesses are used for each coupling between i-th and j-th nodes. The mathematical model of the whole system is

$$\begin{aligned}&\begin{bmatrix} \boldsymbol{M}_D & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M}_S \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}}_D \\ \ddot{\boldsymbol{q}}_S \end{bmatrix} + \omega_0 \begin{bmatrix} \boldsymbol{G}_D & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G}_S \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{q}}_D \\ \dot{\boldsymbol{q}}_S \end{bmatrix} + \left(\begin{bmatrix} \boldsymbol{K}_{sD} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K}_{sS} \end{bmatrix} - \right. \\ &\left. -\omega_0^2 \begin{bmatrix} \boldsymbol{K}_{dD} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K}_{dS} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K}_{BS} \end{bmatrix} + \boldsymbol{K}_C \right) \begin{bmatrix} \boldsymbol{q}_D \\ \boldsymbol{q}_S \end{bmatrix} = \omega_0^2 \begin{bmatrix} \boldsymbol{f}_D \\ \boldsymbol{0} \end{bmatrix}.\end{aligned} \tag{11}$$

It is very useful to rewrite the motion equations of the whole disk-shaft system (9) in the following form

$$\boldsymbol{M}\ddot{\boldsymbol{q}}(t) + \omega_0 \boldsymbol{G}\dot{\boldsymbol{q}}(t) + (\boldsymbol{K}_s - \omega_0^2 \boldsymbol{K}_d)\boldsymbol{q}(t) = \omega_0^2 \boldsymbol{f}, \tag{12}$$

where the mass matrix is

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_D & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M}_S \end{bmatrix}, \tag{13}$$

the skew-symmetrical gyroscopic matrix is

$$\omega_0 \boldsymbol{G} = \omega_0 \begin{bmatrix} \boldsymbol{G}_D & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G}_S \end{bmatrix}, \tag{14}$$

the static stiffness matrix is

$$\boldsymbol{K}_s = \begin{bmatrix} \boldsymbol{K}_{sD} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K}_{sS} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K}_{BS} \end{bmatrix} + \boldsymbol{K}_C, \tag{15}$$

the dynamics stiffness matrix is

$$\boldsymbol{K}_d = \begin{bmatrix} \boldsymbol{K}_{dD} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K}_{dS} \end{bmatrix}, \tag{16}$$

and the centrifugal load vector is

$$\omega_0^2 \boldsymbol{f} = \omega_0^2 \begin{bmatrix} \boldsymbol{f}_D \\ \boldsymbol{0} \end{bmatrix}. \tag{17}$$

The dimensions of the disk-shaft system matrices is $n \times n$ where $n = n_D + n_S$. The homogenous version of this equations will be used for the eigenvalue problem in the next section.

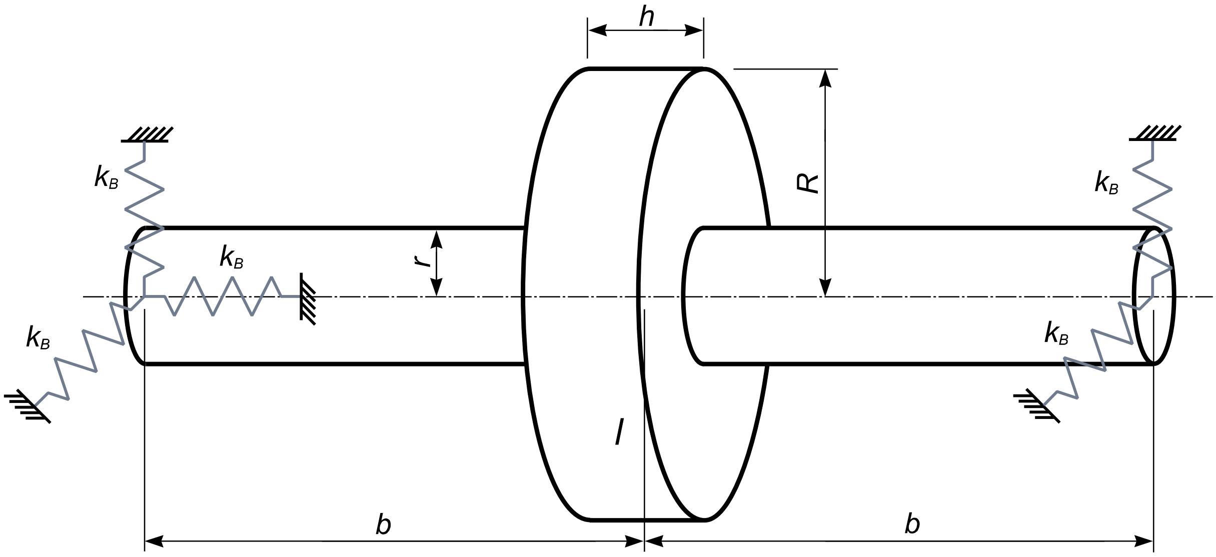

Fig. 5. Scheme of the test rotor.

3. Eigenfrequency sensitivity analysis of rotors

The eigenfrequency sensitivity analysis of the disk-shaft system (10) will be presented. The identity $\boldsymbol{M}\dot{\boldsymbol{q}} - \boldsymbol{M}\dot{\boldsymbol{q}} = \boldsymbol{0}$ should be added to the homogenous form of the equations (10), when we want to consider the gyroscopic effects. These equations can be written in matrix form

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{M} \\ \boldsymbol{M} & \omega\boldsymbol{G} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}} \\ \dot{\boldsymbol{q}} \end{bmatrix} + \begin{bmatrix} -\boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K}_s - \omega^2\boldsymbol{K}_d \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{q}} \\ \boldsymbol{q} \end{bmatrix} = \boldsymbol{0}. \tag{18}$$

The eigenvalue problem of the system (16) is then given by

$$\boldsymbol{A} - \lambda\boldsymbol{E} = \boldsymbol{0}, \tag{19}$$

where

$$\boldsymbol{A} = \begin{bmatrix} -\omega\boldsymbol{M}^{-1}\boldsymbol{G} & -\boldsymbol{M}^{-1}\left(\boldsymbol{K}_s - \omega^2\boldsymbol{K}_d\right) \\ \boldsymbol{E} & \boldsymbol{0} \end{bmatrix}, \tag{20}$$

λ is eigennumber, and $\boldsymbol{E}$ is the unit matrix. The matrices of the system (18) are square of the $2n$-th order.

The sensitivity analysis will be applied on a test rotor (disk-shaft-bearing system) shown in Fig. 5. The reference dimensions are following. The shaft radius is $r = 25\ mm$, the disk radius $R = 80\ mm$, the disk width $h = 40\ mm$, and the shaft lengths $a = b = 140\ mm$. The shaft is discretized using 16 one-dimensional shaft elements and the disk is discretized using 576 solid hexahedral elements. The support is provided by the isotropic bearings with stiffness $k_B = 10^9 N/m$ in the outside nodes of the shaft (left bearing – radial and axial direction, right bearing – radial direction).

The coefficients of the flexible couplings are chosen with respect to the ratio of the global coupling stiffnesses and the shaft stiffnesses that can be analytically expressed (see [8]). The whole system has 2622 degreeses of freedom. Standard steel material properties are considered. The original in-house software is created in MATLAB system based on the developed modeling methodology. The first fifteen eigenfrequencies of non-rotating rotor ($\omega_0 = 0$) are shown in Tab. 1 with brief characterization.

i	f_i [Hz]	Note	i	f_i [Hz]	Note
1	0	free rotation around x-axis	9, 10	5939	1 nod. diam. of the disk
2, 3	708	shaft bending	11, 12	6291	shaft ends oscillation
4	1402	axial bearings oscillation	13	7065	2 nod. diam. of the disk
5, 6	2226	shaft bending	14	7072	2 nod. diam. of the disk
7	4564	torsional oscillation	15	7432	1 nod. cir. of the disk
8	5878	torsional oscillation	16	10254	axial shaft oscillation

Tab. 1. The list of eigenfrequencies with brief specification.

The first eigenfrequency is zero because the rotor can freely rotate around its axis of rotation. The bending eigenmodes are characterized by pairing eigenmodes which have nearly equal eigenfrequency. Torsional and axial eigenmodes are separated. Chosen eigenmodes are shown in following figures. The 2^{nd} eigenmode is characterized by shaft bending, whereas the 14^{th} eigenmode is characterized by disk bending (see Fig. 6). In the case of the 4^{th} eigenmode the rotor oscillates in bearings in axial direction (see Fig. 7 – left). The 7^{th} eigenmode is torsional (see Fig. 7 – right).

The Campbell diagram figures the first six eigenfrequencies dependence on angular velocity from $\omega_0 = 0\ rpm$ to $\omega_0 = 9000\ rpm$ (see Fig. 8). The Campbell diagram shows the independence of the separated eigenfrequencies (4^{th}, 7^{th}) on angular velocity. The pairing eigenmodes (2^{nd}, 14^{th}) are characterized by two eigenfrequency roots, where the first root decreases, and the second root increases with the grow of angular velocity.

Two parameters are chosen for the eigenfrequency sensitivity analysis. The first parameter is the disk radius $R \in \langle 80,\ 96 \rangle\ mm$ and the second parameter is the disk width $h \in \langle 40,\ 48 \rangle\ mm$.

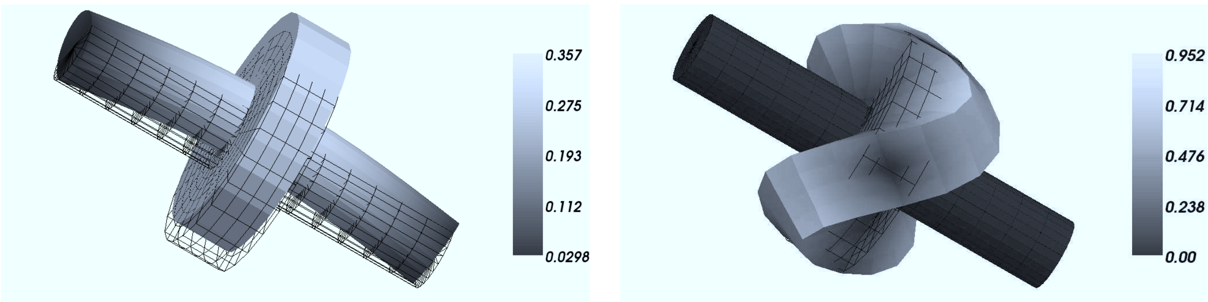

Fig. 6. The 2^{nd} and 14^{th} bending eigenmodes ($f_2 = 708\ Hz$, $f_{14} = 7072\ Hz$).

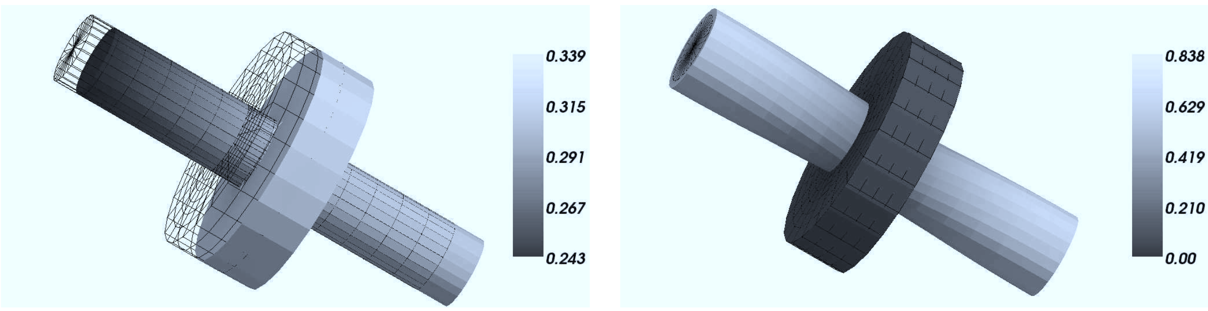

Fig. 7. The 4^{th} axial and 7^{th} torsional eigenmodes ($f_4 = 1402\ Hz$, $f_7 = 4564\ Hz$).

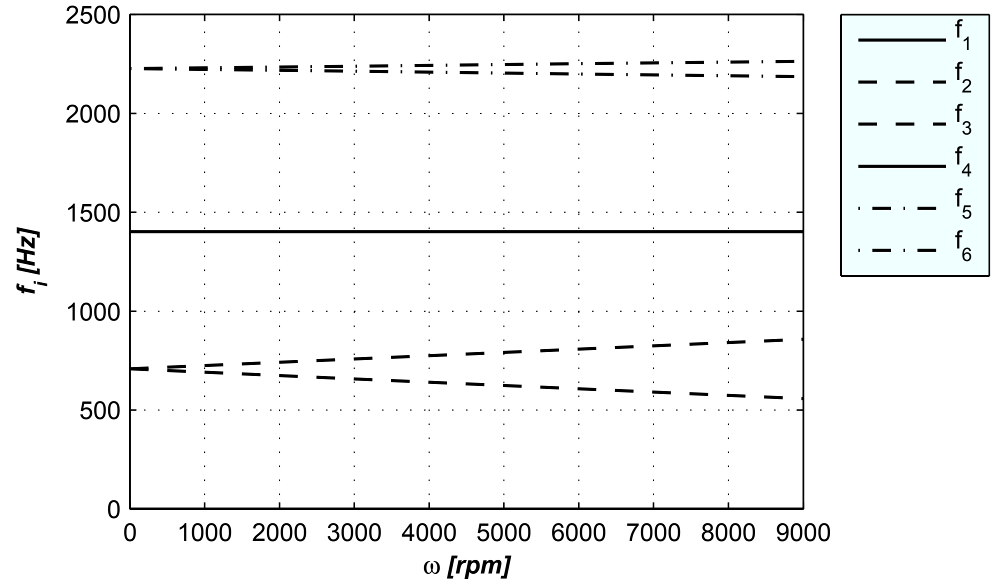

Fig. 8. The Campbell diagram for the angular velocity from $\omega_0 = 0\ rpm$ to $\omega_0 = 9000\ rpm$.

The both parameters are changed in 20%. The frequency dependence on design parameters of non-rotating rotor is shown in Fig. 9.

In the event of the first six eigenmodes the eigenfrequency is declining with growing width h. Whereas, the other eigenfrequencies rise because of the disk is stiffened (the disk oscillates more). In the case of disk radius (parameter R) the eigenfrequencies go down in all eigenmode cases. The system is generally more sensitive to disk radius changes. In the event of dominant

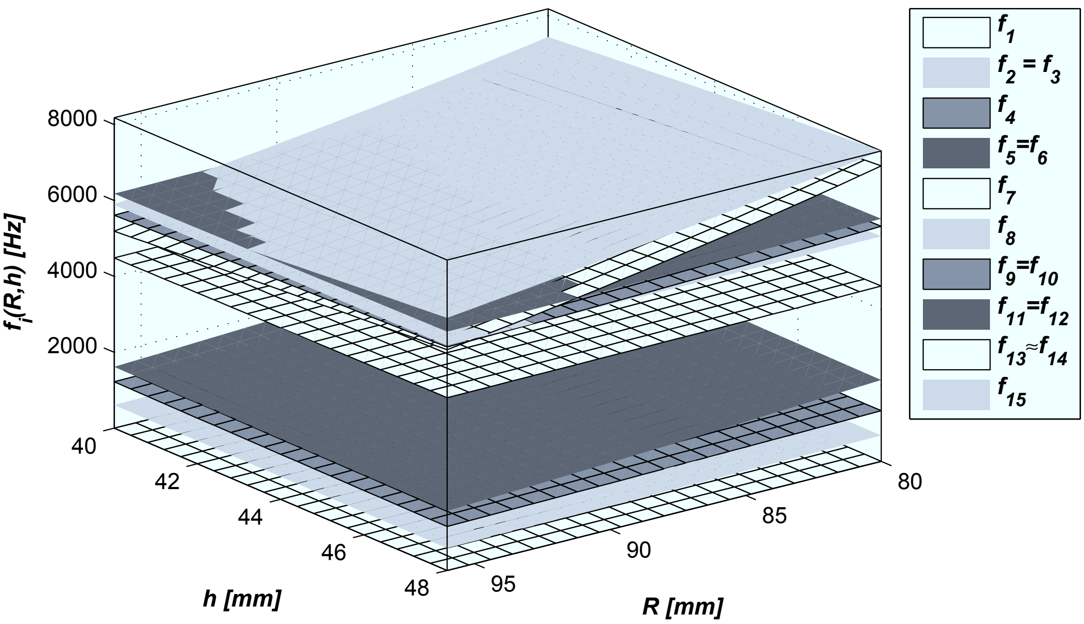

Fig. 9. The frequency dependence on design parameters of non-rotating rotor.

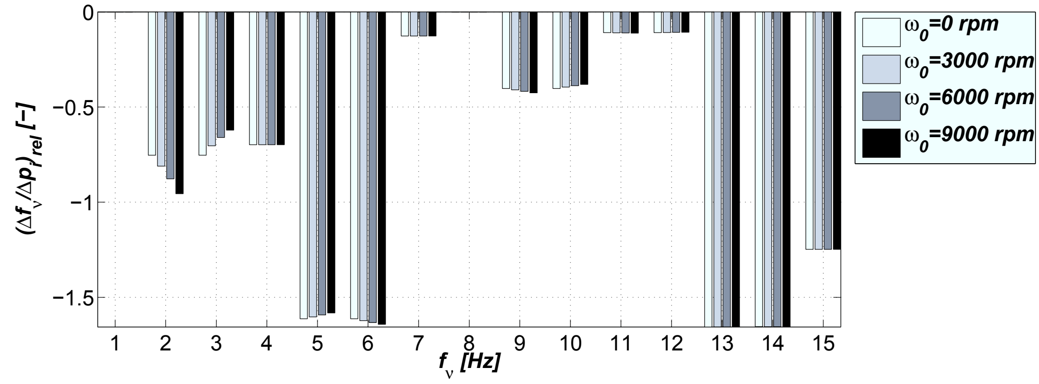

Fig. 10. The relative sensitivity to the disk radius ($p_i = R$).

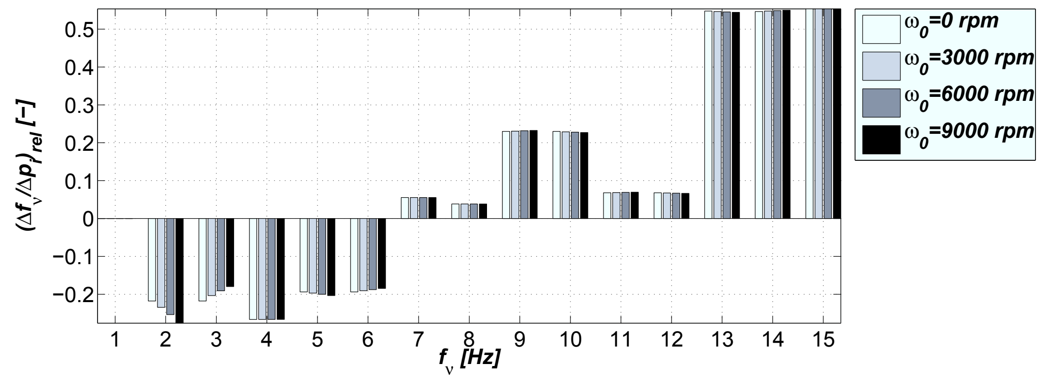

Fig. 11. The relative sensitivity to the disk width ($p_i = h$).

disk oscillation and the eleventh and twelfth eigenmodes the eigenfrequency is declining with growing radius (less stiff), the eigenfrequency rises with the growing width (more stiff). The eighth eigenfrequency is independent on disk radius changes.

The relative (effective) sensitivity (see [3]) is more useful for the comparison of sensitivity to parameter difference than eigenfrequencies dependence. It can be expressed in numerical form

$$\left(\frac{\Delta f_\nu}{\Delta p_i}\right)_{rel} = \frac{f_\nu - f_{\nu,0}}{p_i - p_{i,0}} \cdot \frac{p_{i,0}}{f_{\nu,0}}, \tag{21}$$

where f_ν is the ν-th eigenfrequency, $f_{\nu,0}$ is the ν-th eigenfrequency of reference system, p_i is the value of i-th parameter, and $p_{i,0}$ is the value of i-th reference parameter. The reference values of parameters are $p_{1,0} = R = 80\ mm$, $p_{2,0} = h = 40\ mm$. The relative sensitivities to parameter R are shown in Fig. 10, and the relative sensitivities to parameter h are shown in Fig. 11. The dependence on the angular velocity ω_0 can be recognized here. The described phenomenon (see above) are confirmed in these diagrams.

4. Conclusion

The several methodology for the flexible rotors modeling was presented in this paper. The eigenfrequency sensitivity analysis of rotors with flexible disks was then performed. It is useful to model rotors with flexible disks in case of e.g. high frequency excitation. The modal sensitivity analysis allows to recognize the eigenfrequency response to design parameters changes. It can also help for the better identification of important dynamic properties.

Acknowledgements

The work has been supported by the Fund for University Development FRVŠ F23/2007/G1.

References

[1] J. Chen, Vibration and sensitivity analysis of a spinning disk under tangential edge loads, Journal of Sound and Vibration 215 (1998) 1-15.

[2] N. Guedria, H. Smaoui, M.Chouchane,A direct algebraic method for eigensolution sensitivity computation of damped asymmetric systems, Interbational Journal for Numerical Methods in Engineering 68 (2006) 674-689.

[3] M. Hajžman, Modelování, dynamická analýza a optimalizace rozsáhlých rotujících systémů, Ph.D. thesis, UWB in Pilsen, Pilsen, 2006.

[4] M. Hajžman, J. Šašek, V. Zeman, Vibrations of rotors with flexible disks, Proceedings of conference Engineering Mechanics 2007, Svratka, Institute of Thermomechanics, Academy of Sciences of the Czech Republic, 2007, pp.73-74 (full text on cd-rom).

[5] M.A. Hili, T. Fakhfakh, M. Haddar, Vibration analysis of a rotating flexible shaft-disk system, Journal of Engineering Mathematics 57 (2007) 351-363.

[6] M. Kleiber, H. Antúnez, T.D. Hien, P. Kowalczyk, Parameter Sensitivity in Nonlinear Mechanics, John Wiley & Sons Ltd., Chichester, 1997.

[7] S.S. Rao, The finite element method in engineering, Pergamon Press, Oxford, 1989.

[8] J. Slavík, V. Stejskal, V. Zeman, Základy dynamiky strojů, Vydavateství ČVÚT, Praha, 1997.

[9] J. Šašek, V. Zeman, M. Hajžman, Modal properties of rotating disks, Proceedings of 22nd conference Computational Mechanics 2006, Hrad Nečtiny, University of West Bohemia in Pilsen, 2006, pp. 593-600.

[10] V. Zeman, J. Kellner, J. Šašek, Contribution to modelling of bladed disk vibration, Proceedings of colloquium Dynamics of Machines 2007, Prague, Institute of Thermomechanics AS ČR, 2007, pp. 213-220.

[11] V. Zeman, M. Hajžman, Modelling of shaft system vibration with gears and rolling-element bearings, Proceedings of colloquium Dynamics of Machines 2005, Prague, Institute of Thermomechanics AS ČR, 2005, pp. 163-170.