Computational approach proposition for further processing of the fatigue curve

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Abstract

The goal of our paper is to present the numerical computational tools application for the hysteretic curve identification using Karray-Bouc and Ramberg-Osgood models. The Karray-Bouc model parameters will be determined from Ramberg-Osgood model and Manson-Coffin curve parameters. Using special Matlab\textsuperscript{\textregistered} procedures we can calculate dissipative (hysteretic) energy density per cycle and express Manson-Coffin curve in energy version.

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1. Introduction

The engineering structures components usually undergo a non-proportional and multiaxial loading. The application of the cyclic $\sigma$-$\varepsilon$ tensor response under the multiaxial loading, which depends on the loading-path, is very difficult [12,2]. Several researchers have proposed uniaxial and multiaxial fatigue criteria based on an energy evaluation [8,10,6]. Hence, the goal of the paper will be to present possibilities how to determine the energy-based fatigue curve from Manson-Coffin curve [7,10,6,15].

Considering low and high cyclic fatigue life it will be needed to express the cyclic $\sigma$-$\varepsilon$ curve mathematically. The hysteretic models can be continuous and discontinuous (e.g. linear in parts). The behaviour $\sigma$-$\varepsilon$ can have a monotonic character or a cyclic character (Fig. 1) [3,12,15].

The first models have been described by simple algebraic equations mainly for the monotonic $\sigma$-$\varepsilon$ relationship. Further, we will use one of the first models with the exponential character of hardening known as Ramberg-Osgood model (RO), i.e.

$$\varepsilon = \frac{\sigma_{RO}}{E} + \left( \frac{\sigma_{RO}}{K} \right)^{\frac{1}{n}}. \quad (1)$$

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Fig. 1. Monotonic and cyclic $\sigma$-$\varepsilon$ curve.
The parameter $E$ is Young's modulus, $K'$ is the cyclic strength coefficient, $n'$ is the cyclic strain-hardening exponent and $\sigma_{RO}$ is the yield stress. All these material constants can be obtained from experimental dates or from Manson-Coffin curve parameters; this process will be presented latter.

The Wen's differential model in one of its possible modifications will be used for the further analysis. Its fundamental mathematic form is following [14,4]

$$\dot{z}(z, x) = A \cdot x - B \cdot |x|^{n-1} \cdot |x| \cdot z - C \cdot |z|^{n} \cdot x . \tag{2}$$

The description of the material non-linearity can be formulated using so called Karray-Bouc model (KB) as a modification of Wen's model in the following form [14,4]

$$\sigma_{KB} = \alpha \cdot E \cdot \varepsilon + (1 - \alpha) \cdot z$$

$$\dot{z} = E \cdot \varepsilon - \beta \cdot \frac{E}{R^E_n} \cdot |z|^{n-1} \cdot |\varepsilon| \cdot z - \gamma \cdot \frac{E}{R^E_n} \cdot |z|^n \cdot \varepsilon , \tag{3}$$

where $\alpha = E_T / E$ and $z$ is the stress parameter. The other parameters written into the vector $p = [\alpha, \beta, \gamma, n]^T$ influence the shape of the cyclic curve.

2. Calculation of RO model from Manson-Coffin curve and the identification of the KB model

2.1. Calculation of RO model parameters from Manson-Coffin curve

Let's consider the well-known Manson-Coffin curve

$$\varepsilon = \frac{\sigma_f}{E} \cdot (2N)^b + \varepsilon_f \cdot (2N)^c \tag{4}$$

where $\sigma'_f$ is the fatigue strength coefficient, $\varepsilon'_f$ is the fatigue ductility coefficient, $b$ is the fatigue strength exponent, $c$ is the fatigue ductility exponent. As mentioned in the introduction, these parameters can be used for the calculation of the Ramberg-Osgood model coefficients [5,9]:

$$n' = \frac{b}{c} \quad \text{and} \quad K' = \frac{\sigma_f}{(\varepsilon_f')^{n'}} = \frac{\sigma_f}{(\varepsilon_f')^{\frac{b}{c}}} . \tag{5}$$

In the case of the steel STN 411373.0 the Manson-Coffin curve will be following

$$\varepsilon_{411373} = 0,003678 \cdot (2N)^{-0.078} + 0,371 \cdot (2N)^{-0.487} \tag{6}$$

and RO parameters can be calculated as follows

$$n'_{411373} = \frac{b}{c} = \frac{-0.078}{-0.487} = 0.1602 ,$$

$$K'_{411373} = \frac{\sigma_f}{(\varepsilon_f')^{n'}} = \frac{743}{0.351,0,1602} = 878,6 . \tag{7}$$

Tab.1 presents the results of our calculation for the chosen kinds of steels.
2.2. Identification of the KB model

Assuming that the behaviour of monotonic curves \( \sigma - \varepsilon \) will be equivalent for both models RO and KB, the vector \( p \) can be obtained by a numerical approach using an optimising method implemented in Matlab. The objective function can be formulated [9] as follows

\[
F(p) = \min_0 \int \left[ \sigma_{KB}(p) - \sigma_{RO}(K', n') \right] \, d\varepsilon \quad \text{or} \quad F(p) = \left\| \sigma_{KB}(p) - \sigma_{RO}(K', n') \right\| \rightarrow \min. \quad (8)
\]

where \( p = [\alpha, \beta, \gamma, n]^T \) will be the vector of the optimised variables. The relationship \( \sigma_{KB}(p) - \varepsilon \) can be expressed as follows

\[
\sigma_{KB}(p) = \alpha \cdot 2,02 \cdot 10^{11} \cdot \varepsilon + (1 - \alpha) \cdot z(p) \quad \text{where}
\]

\[
\dot{z}(p) = 2,02 \cdot 10^{11} \cdot \dot{\varepsilon} - \beta \cdot \frac{2,02 \cdot 10^{11}}{\left(299 \cdot 10^6\right)^n} \cdot |\varepsilon|^{(n-1)} \cdot |\dot{\varepsilon}| \cdot z - \gamma \cdot \frac{2,02 \cdot 10^{11}}{\left(299 \cdot 10^6\right)^n} \cdot |\varepsilon|^n \cdot \dot{\varepsilon} \quad (9)
\]

and \( \sigma_{RO}(K', n') - \varepsilon \) can be formulated as

\[
\varepsilon = \frac{\sigma_{RO}}{2,02 \cdot 10^{11}} \quad \text{if} \quad \sigma_{RO} \leq 299 \cdot 10^6
\]

\[
\varepsilon = \frac{\sigma_{RO}}{2,02 \cdot 10^{11} + \left( \frac{\sigma_{RO}}{878.6 \cdot 10^6} \right)^{0.1602}} \quad \text{if} \quad \sigma_{RO} > 299 \cdot 10^6. \quad (10)
\]

Minimizing (8) the mathematical expression of the KB model describing material STN 411 353.0 for strain amplitude \( \varepsilon \in (0; 0.0282) \) the following expressions have been obtained

\[
\sigma_{KB-411353} = 0.0246 \cdot 2,02 \cdot 10^{11} \cdot \varepsilon + (1 - 0.0246) \cdot z \quad \text{and}
\]

\[
\dot{z} = 2.02 \cdot 10^{11} \cdot \dot{\varepsilon} - 0.5710 \cdot \frac{2.02 \cdot 10^{11}}{\left(299 \cdot 10^6\right)^{0.4057}} \cdot |\varepsilon|^{0.5945} \cdot |\dot{\varepsilon}| \cdot z - 0.3488 \cdot \frac{2.02 \cdot 10^{11}}{\left(299 \cdot 10^6\right)^{0.4057}} \cdot |\varepsilon|^{0.4057} \cdot \dot{\varepsilon} \quad (11)
\]

Fig. 2 shows the graphic representation of the identification process using Nelder-Mead optimising method for steel STN 411 373.0.
Results of the identification process for other materials are presented in Tab. 2. Using these coefficients we can propose a material computational model (KB, RO) for FEM or other analyses.

Using this numerical approach the KB computational model has been prepared for the hysteretic loop analysis (Fig. 3) and for the numerical calculation of the total strain-energy density per a cycle or the hysteretic energy per a cycle for different strain amplitudes.

Tab. 2. Table of RO and KB material parameters.

<table>
<thead>
<tr>
<th>Steel</th>
<th>K</th>
<th>n</th>
<th>α</th>
<th>β</th>
<th>γ</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>11 373.0</td>
<td>878.6</td>
<td>0.1602</td>
<td>0.0139</td>
<td>0.3182</td>
<td>0.5935</td>
<td>0.3053</td>
</tr>
<tr>
<td>11 523.1</td>
<td>1163.5</td>
<td>0.1986</td>
<td>0.0111</td>
<td>0.4123</td>
<td>0.5143</td>
<td>0.1915</td>
</tr>
<tr>
<td>12 010.1</td>
<td>873.8</td>
<td>0.1992</td>
<td>0.0150</td>
<td>0.7490</td>
<td>0.2034</td>
<td>0.1972</td>
</tr>
<tr>
<td>15 320.6</td>
<td>1045.0</td>
<td>0.1104</td>
<td>0.0083</td>
<td>0.5505</td>
<td>0.5336</td>
<td>1.2892</td>
</tr>
</tbody>
</table>

Fig. 2. Monotonic part \( \sigma-\varepsilon \) curve for material STN 411353.0 before and after the identification process.

Fig. 3. \( \sigma-\varepsilon \) hysteretic loop (STN 411353) for amplitudes \( \varepsilon_{\text{max}} = 0.0282 \), \( \varepsilon_{\text{max}} = 0.01065 \) and \( \varepsilon_{\text{max}} = 0.00435 \).

3. Strain-energy density and hysteretic energy calculation

Considering the harmonic character of the strain amplitude \( \varepsilon = \varepsilon_{\text{max}} \cdot \sin(2\pi \cdot t) \), the eq. (11) will be solved for \( t \in <0,10 \text{ sec.}> \), \( \varepsilon_{\text{max}} \in <0,00001; 0,028> \). By the obtained results it is possible to describe \( \sigma-\varepsilon \) hysteretic loop for the analysed material and a value of the stress amplifi-
tude $\sigma_{\text{max}}$. Using the expression $\sigma - \varepsilon$ in a discrete form the dissipative (hysteretic) energy density can be calculated [1,11] as follows

$$W_D = \int_{t_1}^{t_2} \sigma \cdot \dot{\varepsilon} \cdot dt,$$

(12)

where $t_1$ is the time of the first stress maximum achievement and $t_2$ is the time of the second stress maximum achievement. In the case of the total strain energy density it is necessary to add the elastic part, i.e.

$$W_E = \frac{\sigma_{\text{max}}^2}{2 \cdot E}.$$

(13)

A calculation of the total strain energy density per half cycle can be realised using a well-known Feltner’s term [13]

$$W_F = \sigma_{\text{max}} \cdot \varepsilon_{\text{max},p} \cdot \frac{2}{1 + n'} + \frac{\sigma_{\text{max}}^2}{2 \cdot E},$$

(14)

and for the hysteretic energy density per cycle the Morrow’s relationship is often applied [13],

$$W_M = 4 \cdot \sigma_{\text{max}} \cdot \varepsilon_{\text{max},p} \cdot \frac{1 - n'}{1 + n'}.$$

(15)

It is important to note that these relationships are approximate, very simple, what is their facility. Our proposed numerical approach is more complicated but much more accurate. Of course, the relation $\sigma - \varepsilon$ cannot be solved explicitly.

4. Energy-based fatigue curve

Applying the KB computational model, the energy-based fatigue curve will be obtained from Manson-Coffin curve (4). Using (14) or (15) it is possible to express the analysed curve in an explicit form, mainly from the comparison point of view. The Feltner’s explicit expression is following [13]

$$W_F = \sigma_f' \cdot (2N_f)^b \cdot \left[ \frac{2}{1 + n'} \cdot \varepsilon_f' \cdot (2N_f)^c + \frac{\sigma_f'}{E} \cdot (2N_f)^b \right].$$

(16)

This expression can be applied for both the low and the high cycle fatigue analysis. Morrow’s energy fatigue curve is applicable for the low cycle first of all. Its mathematical notation is known in the following form [13]

$$W_M = 4 \cdot \frac{1 - n'}{1 + n'} \cdot \sigma_f' \cdot \varepsilon_f' \cdot (2N_f)^{b+c}.$$
Considering the strain energy density as the fatigue parameter the proposed solution can be very problematic, because it is not possible to determine the loading type from history of the energy \( W(t) \). This problem is typical for the case of the rainflow analysis. In \([7,8]\) the authors defined \( W(t) \) taking into account the signs of stresses and strains in order to distinguish the energy under the tension (+) and under the compression (-) as follows

\[
W'_i(t) = \frac{1}{2} \sigma(t) \cdot \varepsilon(t) \cdot \frac{\text{sign}(\sigma) + \text{sign}(\varepsilon)}{2}.
\]  

(18)
For distinguishing between the positive and negative parameter of a strain energy density, the function \( \text{sign} \) has been introduced into the elastic energy density expression \([7,8]\). Another computational approach considering the positive and negative character of the strain energy density may be defined (from author’s experience) as follows

\[
W_2(t) = \frac{1}{2} \cdot \sigma(t) \cdot \varepsilon(t) \cdot \text{sign}(\varepsilon). 
\]  

Character of the expressions (18), (19) and their comparison with strain behaviour is shown on Fig. (8). Energy-based fatigue life estimation can be used for multiaxial loading settings mainly in the cases of the non-proportional conditions \([1]\).

Proposed computational techniques can be compared in following simple example. Let’s consider a random behaviour of \( \varepsilon(t) \). By (18), (19) we can calculate the positive or negative energy density \( W_1(t) \) or \( W_2(t) \) and also the rainflow analysis apply. A cumulative damage \( D \) is assumed as a comparative parameter. Finally, it is compared the classic Manson-Coffin damage calculation with energy damage calculation using \( W_1(t) \) and \( W_2(t) \). Results of this short study are presented in Tab.3.

<table>
<thead>
<tr>
<th>Manson-Coffin cumulative damage</th>
<th>( W_1 ) calculation technique</th>
<th>( W_2 ) calculation technique</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Feltner</td>
<td>Morrow</td>
</tr>
<tr>
<td>0.093</td>
<td>0.86</td>
<td>0.089</td>
</tr>
</tbody>
</table>

Tab. 3. Table of the cumulative damage \( D \).

5. Conclusion

The goal of this paper was to present the chosen computational tools for an energy-based fatigue life estimation. A numeric approach for an identification of the hysteretic Karray-Bouc material model from Ramberg-Osgood model parameters, which have been calculated from Manson-Coffin curve, has been proposed. Using Karray-Bouc model the energy fatigue curve has been obtained and consequently compared with Feltner’s and Morrow’s curves. Results shown in figures 6 and 7 indicate that the energy fatigue curves behaviours are a bit different. It leads to a claim that the energy fatigue life prediction using Feltner’s or Morrow’s methods can give a slightly garbled information. However, the numerical tests using rainflow decomposition indicate feasible values of the damage.
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References