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Reliability analysis of beam on elastic nonlinear foundation

K. Frydrýšek [a,] *

[a] *Faculty of Mechanical Engineering, VŠB-Technical University, 17. listopadu 15, 708 33 Ostrava, Czech Republic*

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Abstract

This paper is focused on the solution of simple beam continually supported by elastic (Winkler's) foundation. The foundation contains longitudinal nonlinearity. For the calculation of displacements and bending stresses are used analytical procedures (approximate solution in the form of polynomial function) and probabilistic approaches (SBRA method, Monte Carlo Simulation Method, AntHill software). Probabilistic approach includes influences of variability of load, shape and material of the beam, and variability of modulus of the foundation. Probabilistic approach is used for the reliability expertise of the beam and calculation of safety.

Keywords: beams, elastic foundation, longitudinal nonlinearity, probability, SBRA method, reliability

1. Introduction

The analysis of bending of beams on an elastic foundation is developed on the assumption that the strains are small and the reaction forces $q_R = q_R(x)$ /Nm^{-1}/ in the foundation are proportional at every point to the deflection $v = v(x)$ /m/ of the beam at that point, etc. (first proposed by E. Winkler, Prague 1867), see fig.1.

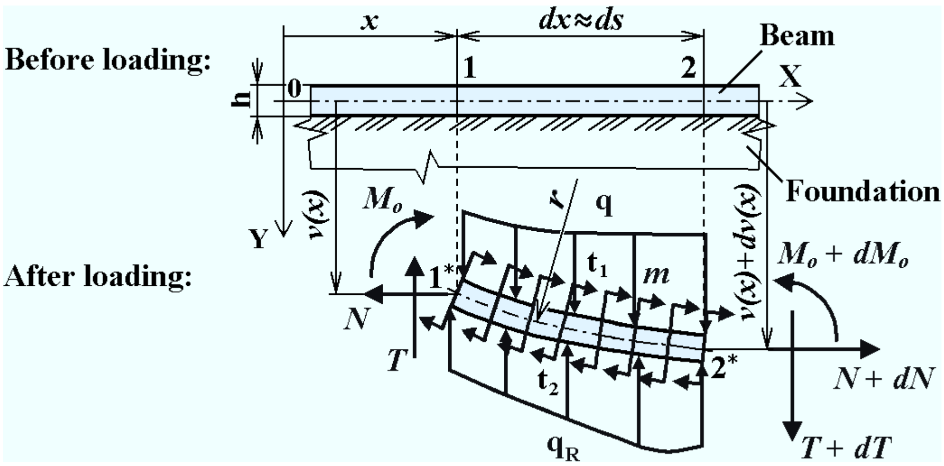


Fig. 1. Element of a Beam on Elastic Foundation.

External loads on the beam also evoke bending moment M_o /Nm/, axial (normal) force N /N/ and shearing force T /N/, see fig.1.

* Corresponding author. Tel.: +420 597324552, e-mail: karel.frydrysek@vsb.cz.

The general problem is described by ordinary differential equation:

$$\frac{d^4 v}{dx^4} - \frac{N}{EJ_{ZT}}\frac{d^2 v}{dx^2} + \frac{\beta}{GA}\frac{d^2 \mathrm{q_R}}{dx^2} + \frac{\mathrm{q_R}}{EJ_{ZT}} = \frac{1}{EJ_{ZT}}\left(\mathrm{q} - \frac{dm}{dx}\right) + \frac{\beta}{GA}\frac{d^2 \mathrm{q}}{dx^2} - \frac{\alpha_{\mathrm{t}}}{\mathrm{h}}\frac{d^2(\mathrm{t_2} - \mathrm{t_1})}{dx^2}, \qquad (1)$$

where: E /Pa/ is modulus of elasticity in tension of the beam, $J_{\mathrm{ZT}} = \int_A y^2 dA$ /m^4/ is the major principal second moment of area A /m^2/ of the beam cross-section, β /1/ is shear deflection constant of the beam, G /Pa/ is modulus of elasticity in torsion of the beam, q /Nm^{-1}/ is distributed load (intensity of force), m /N/ is distributed couple (intensity of moment), α_{t} /deg^{-1}/is coefficient of thermal expansion of the beam, h /m/ is depth of the beam and $\mathrm{t_2} - \mathrm{t_1}$ /deg/ is transversal temperature increasing in the beam. For more information about the derivation of eq. (1), see reference [1].

In the most situations, the influences of normal force, shear force and temperature can be neglected (or the beam is not exposed to them). Hence, from eq. (1) follows:

$$\frac{d^4 v}{dx^4} + \frac{\mathrm{q_R}}{EJ_{ZT}} = \frac{1}{EJ_{ZT}}\left(\mathrm{q} - \frac{dm}{dx}\right) . \qquad (2)$$

From the Winkler's theory, see [1] or [5], is evident that:

$$\mathrm{q_R} = v \times k(x) , \qquad (3)$$

$$k(x) = \mathrm{b} \times K(x) , \qquad (4)$$

where: $k(x)$ /Pa/ is stiffness of the foundation and $K(x)$ /Nm^{-3}/ is modulus of the foundation which can be expressed as functions of variable x /m/ (i.e. longitudinal nonlinearity in the foundation) and b /m/ is width of the beam. Hence, eq. (2) can be written in the form:

$$\frac{d^4 v}{dx^4} + \frac{\mathrm{b}K(x)}{EJ_{ZT}} v = \frac{1}{EJ_{ZT}}\left(\mathrm{q} - \frac{dm}{dx}\right) . \qquad (5)$$

2. Example of General Solution (Derivation)

Let us consider the short straight beam on elastic nonlinear foundation, see fig.2. The beam of length L /m/ with free ends is exposed to one vertical force F /N/, i.e. other loads

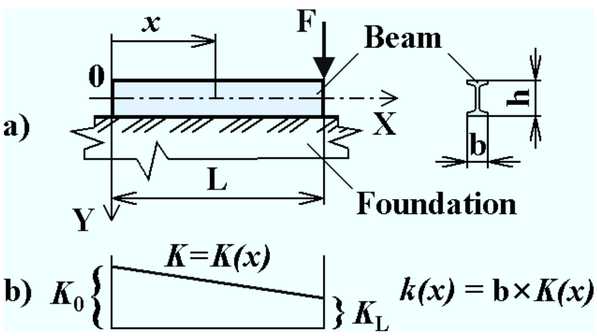


Fig. 2. Solved Example of the Beam on Elastic Nonlinear Foundation.

q and m are zero. Modulus of the foundation is given by linear function:

$$K(x) = K_0 + \frac{K_\mathrm{L} - K_0}{\mathrm{L}} x = K_0 + K_1 x \ , \tag{6}$$

Hence, in this case differential eq. (5) can be written in the form:

$$\frac{d^4 v}{dx^4} + \frac{(K_0 + K_1 x)\mathrm{b}}{EJ_{ZT}} v = 0 \ . \tag{7}$$

The approximate solution can be found in the form of polynomial function of 6$^{\mathrm{th}}$ order:

$$v = v(x) \approx b_0 + \sum_{\mathrm{i}=1}^{6} \frac{b_\mathrm{i}}{\mathrm{i}} x^\mathrm{i} \ , \tag{8}$$

where: b_0 /m/, : b_1 /1/, …, : b_6 /m^{-5}/ are unknown constants.

Equation (8) must satisfy the basic boundary conditions (at the point $x = 0$: $M_o(x = 0) = 0$, $T(x = 0) = 0$ and at the point $x = \mathrm{L}$: $M_o(x = \mathrm{L}) = 0$ and $T(x = \mathrm{L}) = \mathrm{F}$).

Force equation of equilibrium can be also satisfied, i.e. equation:

$$\int_0^\mathrm{L} \mathrm{q_R}\, dx = \int_0^\mathrm{L} k(x) v\, dx = \int_0^\mathrm{L} \mathrm{b}\,(K_0 + K_1 x)\left(b_0 + \sum_{\mathrm{i}=1}^{6} \frac{b_\mathrm{i}}{\mathrm{i}} x^\mathrm{i} \right) dx = \int_0^\mathrm{L} (k_0 + k_1 x)\left(b_0 + \sum_{\mathrm{i}=1}^{6} \frac{b_\mathrm{i}}{\mathrm{i}} x^\mathrm{i} \right) dx = \mathrm{F} \ ,$$

where: $k_0 = K_0\,\mathrm{b}$ /Pa/ is the stiffness in the foundation at the point $x = 0$ and $k_1 = K_1 \mathrm{b} = \frac{K_\mathrm{L} - K_0}{\mathrm{L}} \mathrm{b}$ /Nm^{-3}/ is the slope of a given linear function $k(x)$.

From the above five conditions can be expressed constants b_0 and b_2, …, b_5 as functions of two constants b_1 and b_6.

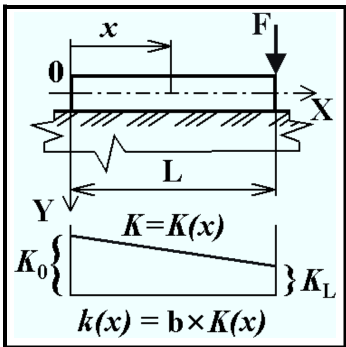
The last two constants (i.e. b_1 and b_6) can be derived via variational principles or via satisfaction of differential equation (7) at chosen points. For more details about it see [2].

The auxiliary constants $\overline{\mathrm{A}}$ /N^{2}m^{-1}/, $\overline{\mathrm{B}}$ /N^2/, $\overline{\mathrm{C}}$ /N^{-3}m^{-2}/ and remaining polynomial constants (b_1 and b_6) are derived in the Tab.1.

$k_0 = K_0 \mathrm{b}$	$k_1 = K_1 \mathrm{b} = \frac{K_\mathrm{L} - K_0}{\mathrm{L}} \mathrm{b}$	$b_6 = -2\overline{\mathrm{C}}\left[840\overline{\mathrm{B}} k_1 + (21k_0 + 11k_1\mathrm{L})\overline{\mathrm{A}}\right]$
$\overline{\mathrm{C}} = \dfrac{\mathrm{F}}{15EJ_{ZT}\mathrm{L}^2\left(560EJ_{ZT}\left[6k_0(k_0 + k_1\mathrm{L}) + k_1^2\mathrm{L}^2\right] + 3\overline{\mathrm{A}}(2k_0 + k_1\mathrm{L})\mathrm{L}\right)}$		$K(x) = K_0 + K_1 x$
$b_1 = \dfrac{\overline{\mathrm{C}}}{6}\left[604800\overline{\mathrm{B}} EJ_{ZT} - 180\left(41k_0k_1\mathrm{L} + 9k_1^2\mathrm{L}^2 + 38k_0^2\right)EJ_{ZT}\mathrm{L}^4 + (3k_0 + 2k_1\mathrm{L})\overline{\mathrm{A}}\mathrm{L}^5\right]$		
$\overline{\mathrm{B}} = EJ_{ZT}(k_1\mathrm{L} + 3k_0)$	$\overline{\mathrm{A}} = k_0(k_0 + k_1\mathrm{L})\mathrm{L}^3$	

Tab. 1. Solved Example (Auxiliary Constants and Used Polynomial Constants).

The analytical results (i.e. v, slope of the beam $\frac{dv}{dx}$ /rad/, M_o and T) are written in tab.2.

<table>
<tr><td rowspan="4">
</td><td colspan="3">$M_o = -EJ_{ZT}\frac{d^2v}{dx^2} \approx \frac{Fx^2(x-L)}{L^2} - 5EJ_{ZT}x^2(x-L)^2 b_6$</td></tr>
<tr><td colspan="3">$T = -EJ_{ZT}\frac{d^3v}{dx^3} \approx 10EJ_{ZT}x\left(3xL-2x^2-L^2\right)b_6 + \frac{Fx(3x-2L)}{L^2}$</td></tr>
<tr><td colspan="3">$\frac{dv}{dx} \approx b_1 + \frac{x^3\left(10L^2-15xL+6x^2\right)}{6}b_6 + \frac{Fx^3(4L-3x)}{12EJ_{ZT}L^2}$</td></tr>
<tr><td>$K(x) = K_0 + K_1 x$</td><td>$k_0 = K_0 b$</td><td>$x \in (0;L)$</td></tr>
<tr><td colspan="4">$v \approx b_1 x + \frac{k_0x^4\left(5L^2-6xL+2x^2\right)-120EJ_{ZT}L^2}{12k_0}b_6 + \frac{F\left[k_0x^4(5L-3x)-120EJ_{ZT}L\right]}{60k_0EJ_{ZT}L^2}$</td></tr>
<tr><td colspan="3">Constants b_1 and b_6 are derived in Tab.1.</td><td>$k_1 = K_1 b = \frac{K_L - K_0}{L} b$</td></tr>
</table>

Tab. 2. Solved Example (Analytical Results of the Beam on Elastic Nonlinear Foundation).

The accuracy of the derived results (tab. 2) was also checked by ANSYS software. However, the derived results fits very well for short beams (i.e. for the situations when the length of the beam $L \le 2$ m). For longer beams must be used higher approximation, i.e. function: $v = v(x) \approx b_0 + \sum_{i=1}^{n}\frac{b_i}{i}x^i$, where $n \ge 7$.

3. Probability Analysis of the Beam

Deterministic approach (i.e. all inputs are constant) is the elder but simple way how to get the solution of mechanical systems. However, the deterministic approach cannot trully includes variability of all inputs.

But this example is solved via probabilistic approach (i.e. all inputs are given by bounded (truncated) histograms) which is the modern and new trend of the solution of mechanical systems.

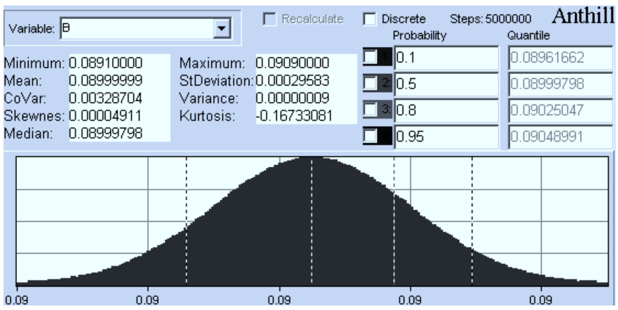


Fig. 3. Histogram of Input Parameter b /m/.

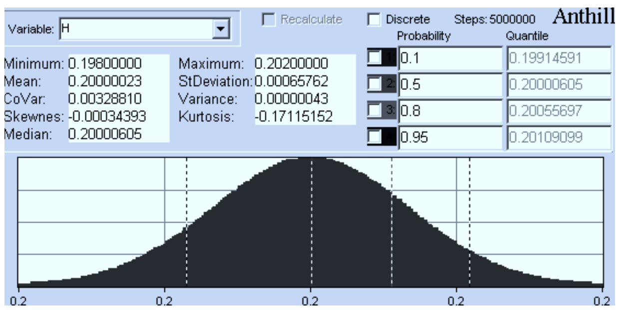


Fig. 4. Histogram of Input Parameter h /m/ .

Probability analysis (see [7] and [8]) of the presented beam (see fig.2) includes influences of variability of “I“ shape ($b = 0.09 \pm 9 \times 10^{-4}$ /m/, $h = 0.2 \pm 2 \times 10^{-3}$ /m/, $J_{ZT} = 2.16 \times 10^{-5} \pm 6.5 \times 10^{-7}$ /m⁴/), material: ($E = 1.8 \times 10^{11} \pm 9 \times 10^{9}$ /Pa/, yield stress $R_e = 162.361\times10^{11}\,{}^{+77.587}_{-43.345}$ /MPa/), load ($F = 157324.2\,{}^{+168773.1}_{-75524.2}$ /N/) and modulus of the foundation: ($K_0 = 1.125 \times 10^{11} \pm 3.375 \times 10^{10}$ /Nm^{-3}/, $K_L = 1.125 \times 10^{11} \pm 3.375 \times 10^{10}$ /Nm^{-3}/), see fig.4 to 10 (i.e. inputs for AntHill software, Simulation-Based Reliability Assessment (SBRA) Method).

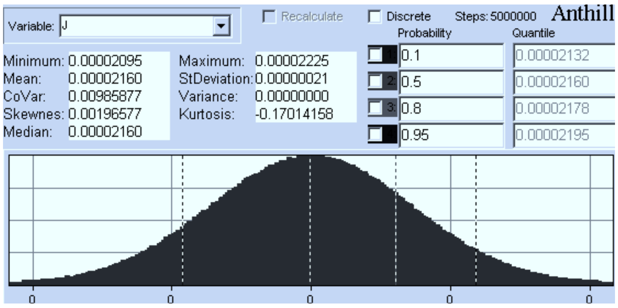


Fig. 5. Histogram of Input Parameter J_{ZT} /m^4/ .

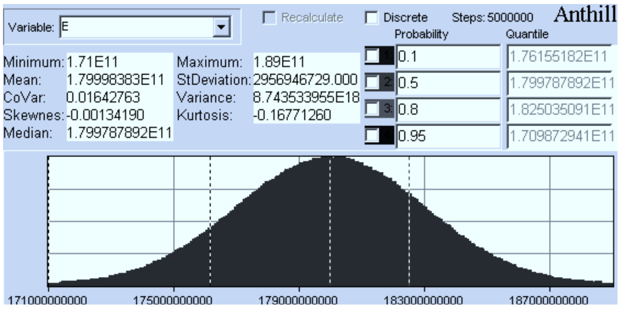


Fig. 6. Histogram of Input Parameter E /Pa/.

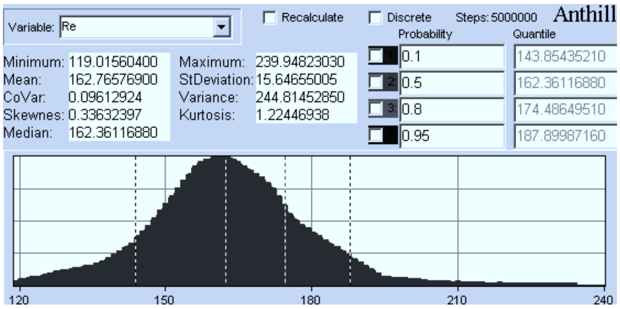


Fig. 7. Histogram of Input Parameter R_e /MPa/.

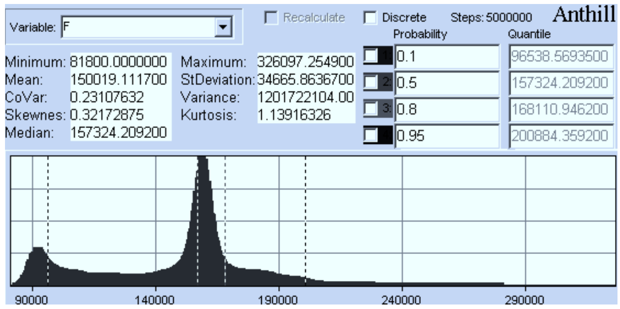


Fig. 8. Histogram of Input Parameter F /N/.

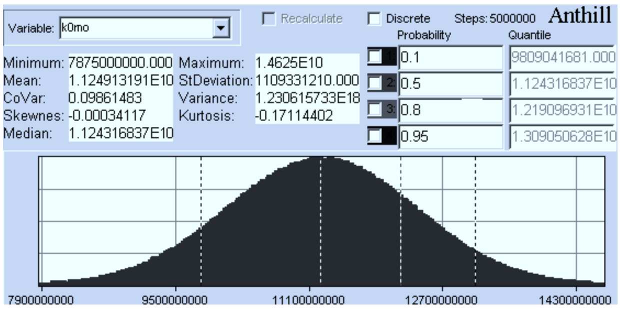


Fig. 9. Histogram of Input Parameter K_0 /Nm^{-3}/ .

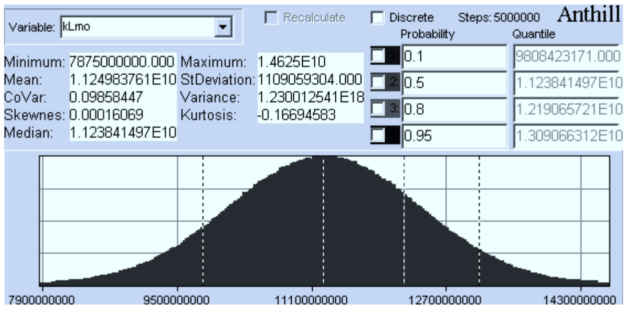


Fig. 10. Histogram of Input Parameter K_L /Nm^{-3}/ .

The values of results parameters (i.e. stiffness of the foundation $k(x)$, displacement $v(x)$, maximal displacement $v_{MAX} = v(x = L)$, bending stress $\sigma(x)$ and maximal bending stress $\sigma_{MAX} = \sigma(x \approx 0.63) = \frac{|M_{o\,MAX}|}{W_o} = \frac{|M_{o\,MAX}|h}{2J_{ZT}}$) were calculated for 5×10^6 simulations by Monte Carlo Method. Results are plotted by histograms in the following Figures 11 to 14 and Tab.3.

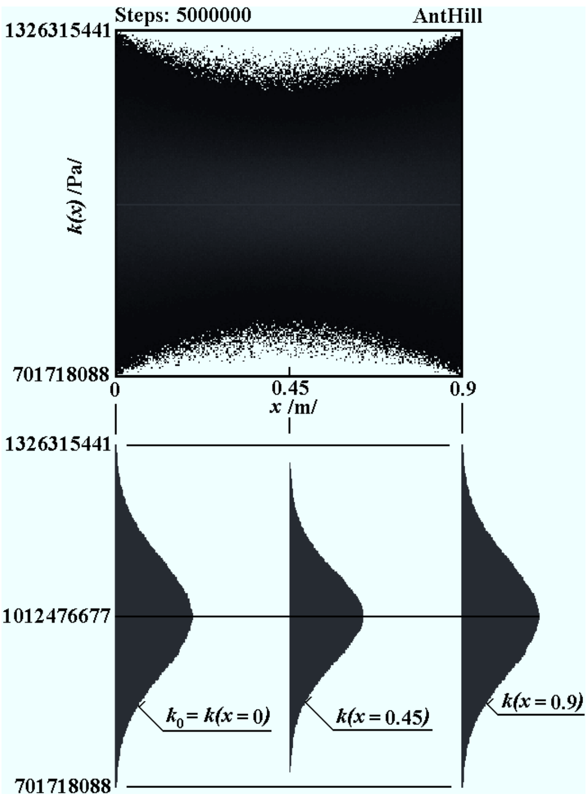


Fig. 11. 2D Histogram and its Sections for Output Parameter $k = k(x)$.

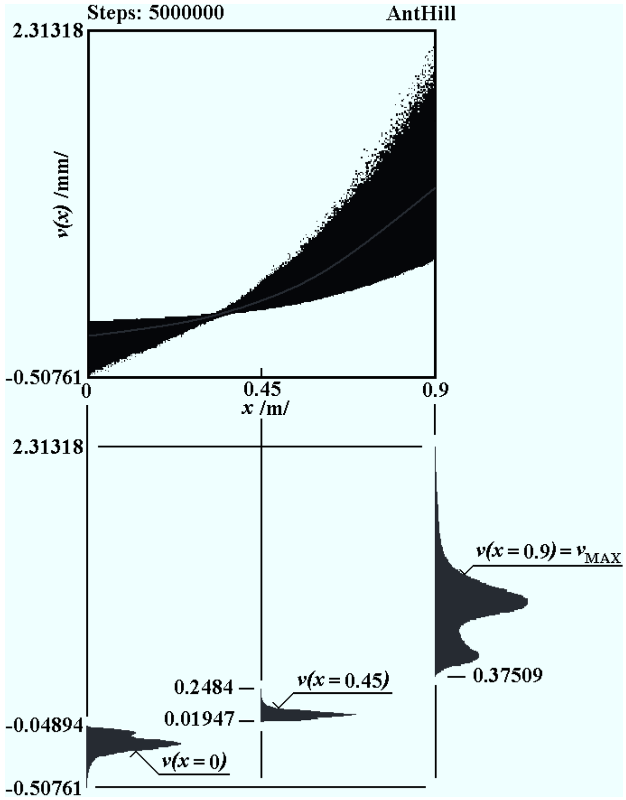


Fig. 12. 2D Histogram and its Sections for Output Parameter $v = v(x)$.

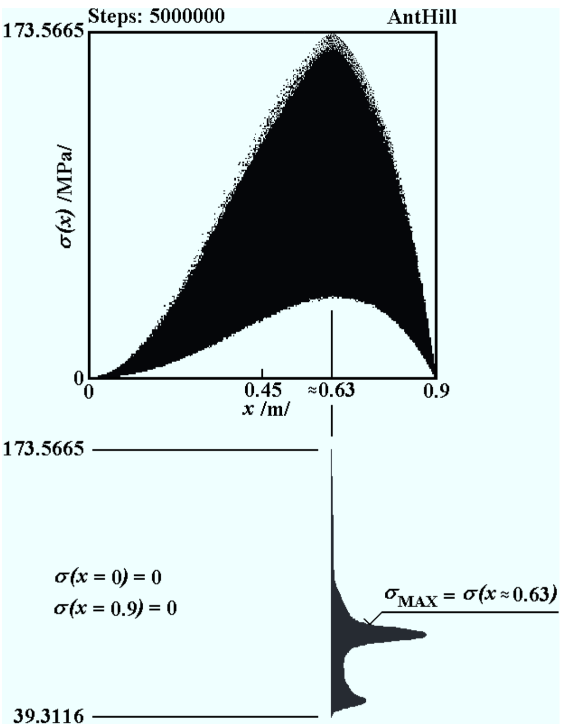


Fig. 13. 2D Histogram and its Section for Output Parameter $\sigma = \sigma(x)$.

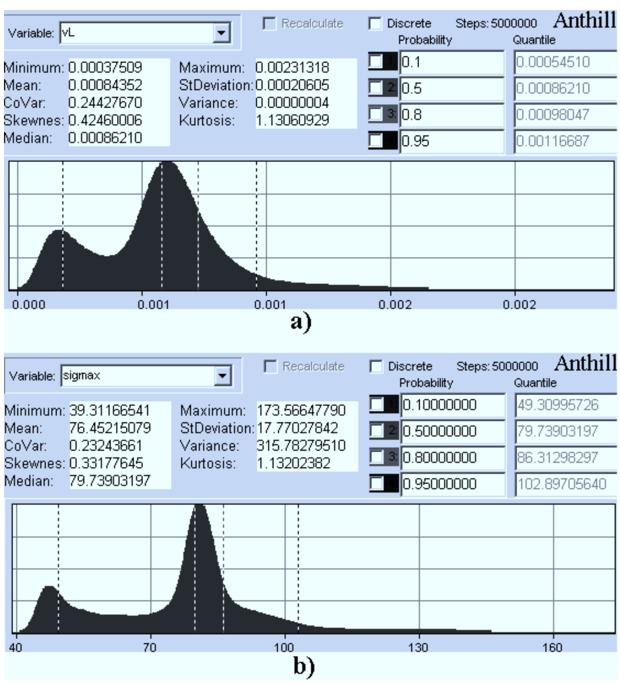


Fig. 14. Histograms of Output Parameters:
a) $v_{MAX} = v(x = L = 0.9\,\mathrm{m})$,
b) $\sigma_{MAX} = \sigma(x \approx 0.63\,\mathrm{m})$.

Output Variables:	Minimum:	Median:	Maximum:	See Figures:
$k(x)$ /Pa/	701718088	1012476677	1326315441	11
v_{MAX} /mm/	3.75×10^{-1}	8.62×10^{-1}	2.31	12 and 14a)
σ_{MAX} /MPa/	39.31	79.74	173.57	13 and 14b)

Tab. 3. Solved Example (Results of AntHill Software).

Hence, from the presented results is evident that maximal displacement is at the right end of the beam (i.e. at the point x = L = 0.9 m) and maximal stress is at the point $x \approx 0.63$ m.

Probability analysis can be also used for reliability expertise of the beam (AntHill software, SBRA Method). Hence, the function of safety F_S (reliability factor) is defined by:

$$F_S = R_e - \sigma_{\mathrm{MAX}} \ , \tag{9}$$

see also fig.15 and 16. Hence, it is evident that the safe situation occurs when $F_S > 0$ (i.e. yield stress R_e is greater than maximal bending stress σ_{MAX}).

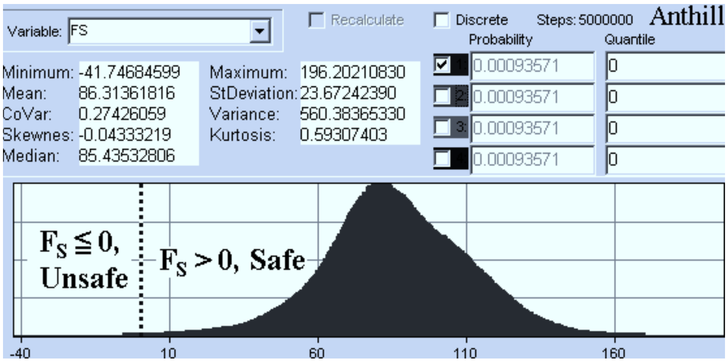


Fig. 15. Histogram of Output Parameters F_S /MPa/ .

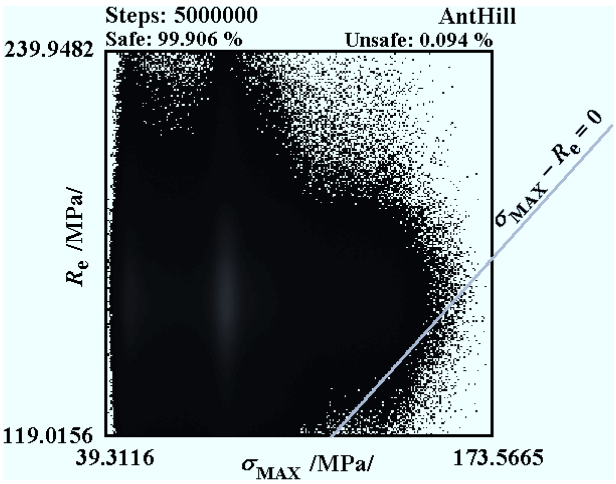


Fig. 16. 2D Histogram of Output Parameters For Calculation of F_S .

The above function of safety F_S was analyzed by Anthill software. Hence, the probability that $F_S \leq 0$ is 9.3571×10^{-4} (i.e. the yield stress and plastic deformations will be reached with a probability of 9.3571×10^{-4}). In other words, $9.3571 \times 10^{-4} \approx 0.094\%$ of all states will result in yielding.

4. Conclusion

General solution for the chosen beam on nonlinear elastic foundation was derived in the form of polynomial function of 6^{th} order. Derived results were used for probabilistic analysis (SBRA method, Monte Carlo Simulation Method, Anthill software).

Finally, the probability that the plastic deformations occurs in the beam is 0.094%. Figure 16 shows distribution of yield stress versus maximal bending stresses and calculation of safety, which is 99.906%.

Another examples of the applications of SBRA method are shown in references [2] and [3], [4], [6], [7] and [8].

Another examples of the solutions of the beams on nonlinear elastic foundation are shown in references [2] and [5].

Acknowledgements

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