Fractals and Splines

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1 Introduction

The main aim of the thesis is to bridge the gap between the spline and fractal theory. The link between fractals and splines goes through subdivision. We deal with fractals generated by IFS consisting of affine transformations and present an IFS for B-spline curves and complex Bézier curves. Resorting to complex domain shows up to be very beneficial, since we can then generate well known fractals by the de Casteljau subdivision algorithm with complex parameter. We provide rigorous justification of the main constructions and prove some of the properties of IFS for (possibly complex) subdivision curves.

We also provide a proof that the subdivision algorithm for Bézier curves leads, under suitable scaling, to the Takagi fractal curve. In this paper we focus just on the complex de Casteljau subdivision algorithm.

2 Theoretical background

We introduce an IFS for complex Bézier curves as follows. The IFS is constructed from transformations f_1 and f_2 by defining

$$f_1(X) = \mathbf{P}\mathbf{L}^\top(t)\mathbf{P}^{-1}X = \mathbf{L}X, \qquad f_2(X) = \mathbf{P}\mathbf{R}^\top(t)\mathbf{P}^{-1}X = \mathbf{R}X, \tag{1}$$

where $\mathbf{L}(t), \mathbf{R}(t)$ are the de Casteljau subdivision matrices and $\mathbf{P}$ is a square matrix which was created from the matrix of control points $(\mathbf{p}_0, \ldots, \mathbf{p}_n)$ by adding rows from identity matrix and a row of ones corresponding to homogenous component of the coordinates.

The matrices $\mathbf{R}, \mathbf{L}$ have always the form

$$\mathbf{R} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{b}_1 \\ \mathbf{0} & 1 \end{pmatrix}, \mathbf{L} = \begin{pmatrix} \mathbf{A}_2 & \mathbf{b}_2 \\ \mathbf{0} & 1 \end{pmatrix}.$$

Eigenvalues of matrices $\mathbf{A}_1, \mathbf{A}_2$ are all within unit circle [2]. Therefore, the transformation f_1, f_2 from equation (1) are eventually contractive [2]. The eventual contractivity ensures that the fixed points of transformations f_1, f_2 are unique [1].

3 Complex Bézier segment and the Takagi fractal curve

The subdivision matrices for a Bézier segment with control points 0 and 1, and subdivision parameter $t \in \mathbb{C}$ generate the transformations $f_1, f_2 : \mathbb{C} \to \mathbb{C}$ as follows:

$$f_1 \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}, \qquad f_2 \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} 1-t & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}.$$

For all $t \in \mathbb{C}$, $|t| < 1 \wedge |1-t| < 1$, the IFS consisting of f_1 and f_2 is a hyperbolic IFS and has a unique attractor A [2]. Further, for $|t| < 1 \wedge |1-t| < 1$, the attractor A of the IFS $\{\mathbb{C}; f_1, f_2\}$ is connected [2].

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The **Takagi fractal curve** is a continuous function which is nowhere differentiable, it is defined on the unit interval by

$$\mathcal{T}(x) = \sum_{n=0}^{\infty} \frac{\sigma(2^n x)}{2^n} = \sum_{n=0}^{\infty} \frac{\min_{n\in\mathbb{Z}} |2^n x - n|}{2^n}.$$

The Takagi curve can be approximated by a complex Bézier curve, which we show in the following paragraphs, the complete proof can be found in [2]. In order to formulate the statement about approximation of the Takagi curve, we define the following scaling map

$$g : \mathbb{C} \times \mathbb{R} \to \mathbb{C} \qquad g(z, y) = \mathrm{Re}(z) + y\, i\ \mathrm{Im}(z). \tag{2}$$

Theorem 3.1. *Let $A^*(y) \in \mathbb{C}$ be the attractor of the de Casteljau IFS for the Bézier segment with control points 0 and 1, and complex subdivision parameter $t = \frac{1}{2} + iy$, $|y|$ small enough. Let $T = \{x + i\mathcal{T}(x) \mid x \in [0, 1]\}$ be the graph of the Takagi function. Then the set*

$$A^* = \lim_{y\to 0} g\big(A^*(y), \frac{1}{2y}\big) \tag{3}$$

contains the set T.

The properties of the attractor $A^*(y)$ depend strongly on y. In Figure 1 we plot $g(A^*(y), \frac{1}{2y})$ for various values of y, and T for comparison. In both cases, the algorithm is iterated 15 times.

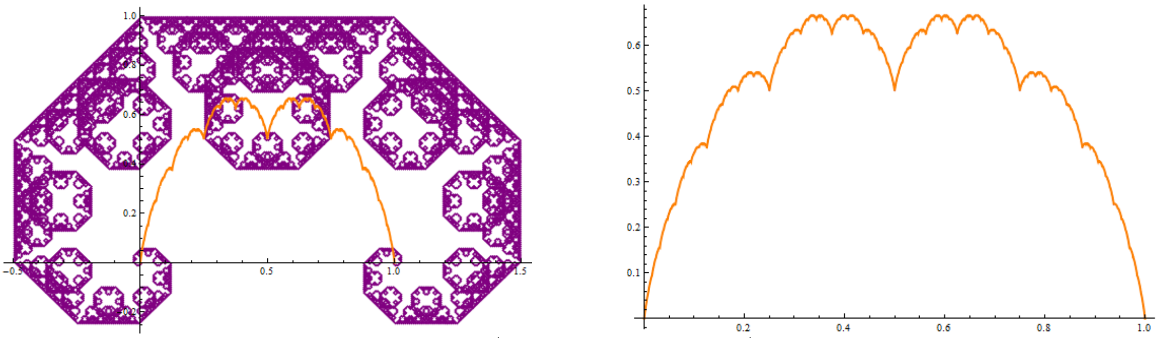

Figure 1: The purple curve is $g(A^*(y), \frac{1}{2y})$, which for $y = \frac{1}{2}$ becomes the Lévy C curve (left figure). The orange curve is the graph of Takagi curve T. On the right is $y = 2^{-10}$ and $g(A^*(y), \frac{1}{2y})$ is incident with T. That is, we can only see T.

4 Conclusion

We proved that an IFS for subdivision curves has unique fixed point. In order to do so, we used the fact that given submatrices of subdivision matrices are eventually contractive. IFS for complex Bézier curves give rise to a new way of generating fractals. We prove that the curves generated by IFS for complex Bézier curve with control points 0 and 1 are connected, and then show that a complex Bézier curve approximates the Takagi fractal curve, in a suitable limit. We conjecture that the Takagi curve is present in every Bézier curve (of higher degree as well), if the subdivision parameter has vanishing imaginary part and the real part is equal to $\frac{1}{2}$.

References

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