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Radek Cibulka

Differential Variational Inequalities: A gentle invitation

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Differential Variational Inequalities: A gentle invitation

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Preface

These lecture notes are based on a series of lectures given at the XXIX Seminar in Differential Equations which took place in Monínec, April 14 - 18, 2014. Main goal is to provide an introduction to differential variational inequalities, which seem to be a sufficiently general framework for modeling various problems beyond equations. For example, this can be useful in contact mechanics, when one considers friction and impacts; in electronics when the diodes appear in the circuit. However, one uses many facts from other branches of mathematics such as convex analysis, variational analysis, non-smooth analysis, differential equations, differential inclusions, measure theory, numerical methods, etc. Since this work is not a book, rather than devoting a separate section to one of the previously mentioned topics, we prefer to introduce the definitions and notions precisely when they are needed for the first time. We try to explain the key ideas and illustrate them on examples instead of going into full generality. Although, we work in finite dimensions, almost all results are valid in (or can be extended in an obvious way to) Hilbert spaces or even in (reflexive) Banach spaces. Sections 1, 3, and 4 correspond to a single lecture while Section 2 was presented in two lectures. It should be noted that, usually, the order of the oral presentation was different. Section 5 contains convergence results for generalized equations obtained during last two years and has not been discussed during the seminar.

Acknowledgments

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Pilsen, March, 2016 *Radek Cibulka*

Contents

1. What, Where, Why?

This section will hopefully answer the following questions:

What is a differential variational inequality?
Where such a model arises from?
Why should one consider this model instead of other ones?

1.1. **Basic Notions.**

First, let us mention the notation used in the rest of this note. By $\mathbb{R}$ and $\mathbb{R}_+$ we denote the set of real numbers and non-negative real numbers, respectively. The space of (column) vectors $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ having n real coordinates will be denoted by $\mathbb{R}^n$. The *scalar product* in $\mathbb{R}^n$ is for each $\mathbf{x} = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, y_2, ..., y_n)^T \in \mathbb{R}^n$ defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i.$$

The symbol $\mathbf{x} \perp \mathbf{y}$ indicates that $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ and $\mathbf{x} \preceq \mathbf{y}$ means that $x_i \leq y_i$ for each $i \in \{1, 2, \ldots, n\}$. The scalar product induces the *Euclidean norm* on $\mathbb{R}^n$ which is defined by

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \Big(\sum_{i=1}^{n} x_i^2 \Big)^{\frac{1}{2}} \quad \text{for each} \quad x \in \mathbb{R}^n.$$

The closed and open ball around $\mathbf{x} \in \mathbb{R}^n$ with radius $r \geq 0$ are the sets

$$\mathbb{B}[\mathbf{x}, r] := \{\mathbf{y} \in \mathbb{R}^n : \ \|\mathbf{y} - \mathbf{x}\| \leq r\} \quad \text{and} \quad \mathbb{B}(\mathbf{x}, r) := \{\mathbf{y} \in \mathbb{R}^n : \ \|\mathbf{y} - \mathbf{x}\| < r\},$$

respectively. Given any two subsets A and B of $\mathbb{R}^n$, the *Minkowski sum* $A + B$ and the *Minkowski difference* $A - B$ *of* A *and* B are defined by

$$A + B = \{\mathbf{a} + \mathbf{b} \ : \ \mathbf{a} \in A, \ \mathbf{b} \in B\} \quad \text{and} \quad A - B = \{\mathbf{a} - \mathbf{b} \ : \ \mathbf{a} \in A, \ \mathbf{b} \in B\}.$$

If $A = \{\mathbf{a}\}$, we will write $\mathbf{a} + B$ instead of $\{\mathbf{a}\} + B$. Geometrically, this is nothing else but a shift of the set B in the direction of $\mathbf{a}$. Clearly,

$$A + B = \bigcup_{\mathbf{a} \in A} (\mathbf{a} + B) = \bigcup_{\mathbf{b} \in B} (A + \mathbf{b}).$$

For any scalar $\alpha \in \mathbb{R}$, the α*-multiple* αA *of the set* A is defined by

$$\alpha A = \{\alpha \mathbf{a} : \ \mathbf{a} \in A\}.$$

Finally, the set A is *convex* if

$$\alpha A + (1 - \alpha) A \ = \ A \quad \text{for each} \quad \alpha \in [0, 1],$$

and A is a *cone* if $\alpha A \subset A$ whenever $\alpha \geq 0$.

A *set-valued mapping (correspondence)* $\mathbf{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ associates with any $\mathbf{x} \in \mathbb{R}^m$ a subset of $\mathbb{R}^n$, denoted by $\mathbf{F}(\mathbf{x})$ and called the *value* of $\mathbf{F}$ at $\mathbf{x}$. For such a map, the set

(*i*) $\operatorname{dom} \mathbf{F} := \{\mathbf{x} \in \mathbb{R}^m : \ \mathbf{F}(\mathbf{x}) \neq \emptyset\}$ is the *domain* of $\mathbf{F}$;
(*ii*) $\operatorname{rge} \mathbf{F} := \{\mathbf{y} \in \mathbb{R}^n : \ \mathbf{y} \in \mathbf{F}(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^m\}$ is the *range* of $\mathbf{F}$;
(*iii*) $\operatorname{gph} \mathbf{F} := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m \times \mathbb{R}^n : \ \mathbf{y} \in \mathbf{F}(\mathbf{x})\}$ is the *graph* of $\mathbf{F}$.

If $\mathbf{F}(\mathbf{x})$ is a singleton, we say that $\mathbf{F}$ is single-valued at $\mathbf{x}$. If $\mathbf{F}$ is single-valued at each $\mathbf{x} \in \operatorname{dom} \mathbf{F}$, then such a mapping will be denoted by $\mathbf{f} : \operatorname{dom} \mathbf{f} \to \mathbb{R}^n$ and we write $\mathbf{y} = \mathbf{f}(\mathbf{x})$ instead of $\mathbf{y} \in \mathbf{F}(\mathbf{x})$. For a subset M of $\mathbb{R}^m$, the *image of M under $\mathbf{F}$* is the set

$$\mathbf{F}(M) = \bigcup_{\mathbf{x} \in M} \mathbf{F}(\mathbf{x}).$$

Although we work in finite dimensions, from time to time, infinite dimensional spaces will appear as well. The space of all linear bounded mappings acting from a Banach space X to another Banach space Y is equipped with the standard operator norm and denoted by $\mathcal{L}(X, Y)$. We set $\mathbb{R}^{n \times m} = \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, i.e. we identify a linear bounded mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$ with its matrix representation in the standard canonical bases. Given an interval I in $\mathbb{R}$ and $K \subset \mathbb{R}^m$, by $\mathcal{C}^\infty(I, K)$ we mean functions from I with values in K possessing derivatives of arbitrary order. If $m = 1$ and $K := \mathbb{R}$, we write $\mathcal{C}^\infty[a, b]$ and $\mathcal{C}^\infty(a, b)$ for $I := [a, b]$ and $I := (a, b)$, respectively. Finally, $\mathcal{C}_0^\infty(\mathbb{R})$ denotes real valued functions of one real variable having derivatives of arbitrary order and a compact support.

1.2. **Variational Inequalities.** Given a function $\mathbf{h} : \mathbb{R}^m \to \mathbb{R}^m$ and a non-empty closed convex subset K of $\mathbb{R}^m$, the *variational inequality (VI)* is a problem to

$$\text{find} \quad \mathbf{u} \in K \quad \text{such that} \quad 0 \leq \langle \mathbf{h}(\mathbf{u}), \mathbf{v} - \mathbf{u} \rangle \quad \text{whenever} \quad \mathbf{v} \in K. \tag{1.1}$$

The set of all solutions to (1.1) will be denoted by

$$SOL(K, \mathbf{h}). \tag{1.2}$$

There are various (equivalent) ways of writing (1.1). Let us start with its geometric form.

Definition 1.1. Let K be a closed convex subset of $\mathbb{R}^m$ and $\mathbf{u} \in \mathbb{R}^m$. The *normal cone to K at $\mathbf{u}$* is the set

$$N_K(\mathbf{u}) := \begin{cases} \{\mathbf{p} \in \mathbb{R}^m : \langle \mathbf{p}, \mathbf{v} - \mathbf{u} \rangle \leq 0 \text{ for each } \mathbf{v} \in K\} & \text{if} \quad \mathbf{u} \in K, \\ \emptyset & \text{otherwise.} \end{cases}$$

In view of the above definition, solving variational inequality (1.1) means to find $\mathbf{u} \in \mathbb{R}^m$ such that

$$-\mathbf{h}(\mathbf{u}) \in N_K(\mathbf{u}) \quad \text{or equivalently} \quad \mathbf{0} \in \mathbf{h}(\mathbf{u}) + N_K(\mathbf{u}). \tag{1.3}$$

Note that any $\mathbf{u} \in \mathbb{R}^m$ satisfying (1.3) has to be an element of K (see also Figure 1). The domain of the *normal cone mapping* $\mathbf{u} \rightrightarrows N_K(\mathbf{u})$ is K.

Recall that for a subset C of $\mathbb{R}^m$ and a point $\mathbf{u} \in \mathbb{R}^m$ the *distance from $\mathbf{u}$ to C* and the *projection of $\mathbf{u}$ on C* are defined by

$$d(\mathbf{u}, C) = \inf \big\{ \|\mathbf{v} - \mathbf{u}\| : \ \mathbf{v} \in C \big\} \quad \text{and} \quad P_C(\mathbf{u}) = \big\{ \mathbf{v} \in C : \ \|\mathbf{v} - \mathbf{u}\| = d(\mathbf{u}, C) \big\},$$

respectively. Let us gather well-known properties of the above three notions (see also Figure 2 and Figure 3).

Lemma 1.2. *Let K be a non-empty closed convex subset of $\mathbb{R}^m$ and $\mathbf{u} \in \mathbb{R}^m$. Then*

(*i*) *$P_K(\mathbf{u})$ contains the only point, $\mathbf{p}_K(\mathbf{u})$ say. Moreover,*

$$\langle \mathbf{z} - \mathbf{p}_K(\mathbf{u}), \mathbf{u} - \mathbf{p}_K(\mathbf{u}) \rangle \leq 0 \quad \textit{whenever} \quad \mathbf{z} \in K;$$

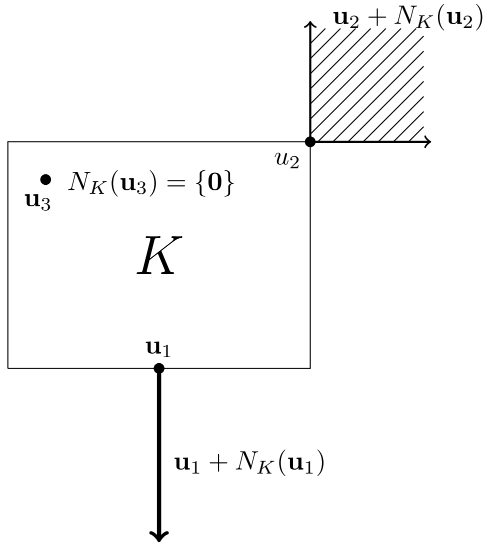

FIGURE 1. Normal cones associated with a rectangle K in $\mathbb{R}^2$.

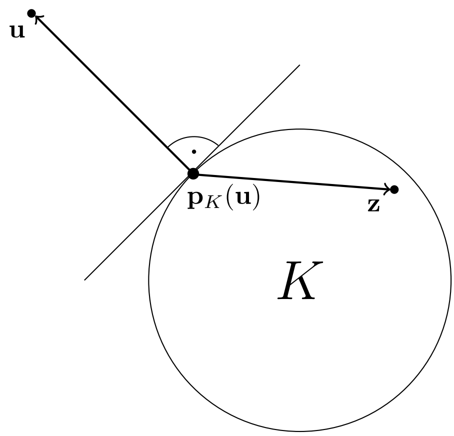

FIGURE 2. Geometric meaning of Lemma 1.2 (i).

(*ii*) *$N_K(\mathbf{u})$ is a non-empty closed convex cone. If, in addition, $\mathbf{u}$ is an interior point of K, then $N_K(\mathbf{u}) = \{\mathbf{0}\}$;*

(*iii*) *$\mathbf{p} \in N_K(\mathbf{u})$ if and only if $\mathbf{p}_K(\mathbf{u} + \mathbf{p}) = \mathbf{u}$.*

Proof. Clearly, (*iii*) is trivial once (*i*) is proved.

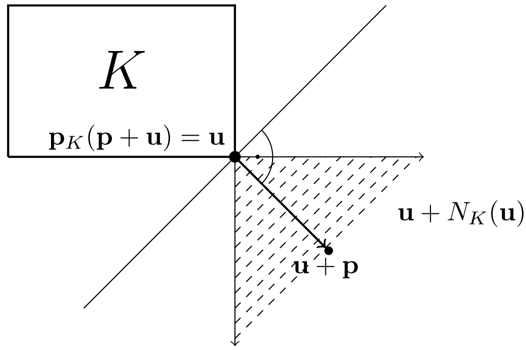

FIGURE 3. Geometric meaning of Lemma 1.2 (iii).

(i) If $\mathbf{u}\in K$, then $P_K(\mathbf{u})=\{\mathbf{u}\}$ and we are done. From now on, assume that $\mathbf{u}\notin K$. Let $(\mathbf{v}_n)_{n\in\mathbb{N}}$ be a sequence in K such that

$$d(\mathbf{u},K)\le\|\mathbf{v}_n-\mathbf{u}\|<d(\mathbf{u},K)+\frac{1}{n}\quad\text{for each}\quad n\in\mathbb{N}.$$

As $(\mathbf{v}_n)_{n\in\mathbb{N}}$ is bounded, it has a cluster point, $\mathbf{v}\in\mathbb{R}^m$ say. The above inequalities give that $d(\mathbf{u},K)=\|\mathbf{v}-\mathbf{u}\|=:r$. Clearly, the closed set K must contain $\mathbf{v}$. Hence $\mathbf{v}\in P_K(\mathbf{u})$, thus $P_K(\mathbf{u})$ is non-empty.
We `claim` that $\langle\mathbf{z}-\mathbf{v},\mathbf{u}-\mathbf{v}\rangle\le 0$ whenever $\mathbf{v}\in P_K(\mathbf{u})$ and $\mathbf{z}\in K$. Indeed, given $t\in[0,1]$, the point $\mathbf{z}_t:=(1-t)\mathbf{v}+t\mathbf{z}$ is in K thanks to the convexity. Then, for any $t\in(0,1)$, we have

$$\begin{aligned} r^2=d^2(\mathbf{u},K) &\le \|\mathbf{u}-\mathbf{z}_t\|^2=\|\mathbf{u}-(1-t)\mathbf{v}-t\mathbf{z}\|^2=\|(\mathbf{u}-\mathbf{v})-t(\mathbf{z}-\mathbf{v})\|^2\\ &= r^2-2t\langle\mathbf{z}-\mathbf{v},\mathbf{u}-\mathbf{v}\rangle+t^2\|\mathbf{z}-\mathbf{v}\|^2,\end{aligned}$$

whence $\langle\mathbf{z}-\mathbf{v},\mathbf{u}-\mathbf{v}\rangle\le(t/2)\|\mathbf{z}-\mathbf{v}\|^2$. Taking the limit as $t\downarrow 0$, we get the desired claim. Fix any $\bar{\mathbf{v}},\tilde{\mathbf{v}}\in P_K(\mathbf{u})$. Using the claim twice with $(\mathbf{z},\mathbf{v}):=(\bar{\mathbf{v}},\tilde{\mathbf{v}})$ and $(\mathbf{z},\mathbf{v}):=(\tilde{\mathbf{v}},\bar{\mathbf{v}})$, respectively, we infer that

$$\|\bar{\mathbf{v}}-\tilde{\mathbf{v}}\|^2=\langle\bar{\mathbf{v}}-\tilde{\mathbf{v}},\mathbf{u}-\tilde{\mathbf{v}}+\bar{\mathbf{v}}-\mathbf{u}\rangle=\langle\bar{\mathbf{v}}-\tilde{\mathbf{v}},\mathbf{u}-\tilde{\mathbf{v}}\rangle+\langle\tilde{\mathbf{v}}-\bar{\mathbf{v}},\mathbf{u}-\bar{\mathbf{v}}\rangle\le 0.$$

So $\bar{\mathbf{v}}=\tilde{\mathbf{v}}$, which means that $P_K(\mathbf{u})$ is singleton.
(ii) Given $\mathbf{u}\in K$, one has that

$$N_K(\mathbf{u})=\bigcap_{\mathbf{v}\in K}\{\mathbf{p}\in\mathbb{R}^m:\ \langle\mathbf{p},\mathbf{v}-\mathbf{u}\rangle\le 0\},$$

therefore $N_K(\mathbf{u})$ is a closed convex cone containing zero (at least). Suppose that $\mathbf{u}$ is an interior point of K. Let $\mathbf{p}\in N_K(\mathbf{u})$. Find $\alpha>0$ such that $\mathbf{v}:=\mathbf{u}\pm\alpha\mathbf{p}\in K$. The very definition of $N_K(\mathbf{u})$ implies that $\alpha\langle\mathbf{p},\mathbf{p}\rangle\le 0$ as well as $-\alpha\langle\mathbf{p},\mathbf{p}\rangle\le 0$. Hence $\|\mathbf{p}\|=0$. □

Let us mention some examples of the normal cones.

Example 1.3. (1) If K is a linear subspace of $\mathbb{R}^m$ then $N_K(\mathbf{u})$ is nothing else but the orthogonal complement of K;

(2) Given $\bar{\mathbf{u}} \in \mathbb{R}^m$ and $r > 0$, let $K := \mathbb{B}[\bar{\mathbf{u}}, r]$. Then

$$N_K(\mathbf{u}) := \begin{cases} \{\mathbf{0}\} & \text{if} \quad \|\mathbf{u} - \bar{\mathbf{u}}\| < r, \\ \{\lambda(\mathbf{u} - \bar{\mathbf{u}}) : \lambda \geq 0\} & \text{if} \quad \|\mathbf{u} - \bar{\mathbf{u}}\| = r, \\ \emptyset & \text{otherwise;} \end{cases}$$

(3) Given a differentiable convex function $\mathbf{h} : \mathbb{R}^m \to \mathbb{R}$, let

$$K := \{\mathbf{u} \in \mathbb{R}^m : \mathbf{h}(\mathbf{u}) \leq 0\}.$$

Then

$$N_K(\mathbf{u}) := \begin{cases} \{\mathbf{0}\} & \text{if} \quad \mathbf{h}(\mathbf{u}) < 0, \\ \{\lambda \nabla \mathbf{h}(\mathbf{u}) : \lambda \geq 0\} & \text{if} \quad \mathbf{h}(\mathbf{u}) = 0, \\ \emptyset & \text{if} \quad \mathbf{h}(\mathbf{u}) > 0. \end{cases}$$

In particular, for $K := \mathbb{R}^m$, we see that the model (1.3) (respectively (1.1)) covers equations $\mathbf{h}(\mathbf{u}) = \mathbf{0}$.

Definition 1.4. Let K be a non-empty closed convex cone in $\mathbb{R}^m$. The set

$$K^* := \{\mathbf{p} \in \mathbb{R}^m : \langle \mathbf{p}, \mathbf{v} \rangle \geq 0 \text{ for all } \mathbf{v} \in K\}$$

is called the *dual cone* to K.

Next, let us establish the relationship between the dual and the normal cone.

Proposition 1.5. *If K is a non-empty closed convex cone in $\mathbb{R}^m$, then so is K^*. Moreover, $(K^*)^* = K$ and*

$$K \ni \mathbf{u} \perp \mathbf{p} \in K^* \quad \Leftrightarrow \quad -\mathbf{p} \in N_K(\mathbf{u}) \quad \Leftrightarrow \quad -\mathbf{u} \in N_{K^*}(\mathbf{p}). \tag{1.4}$$

In particular, when $K = \mathbb{R}^m_+$, then

$$\mathbf{0} \preceq \mathbf{u} \perp \mathbf{p} \succeq \mathbf{0} \quad \Leftrightarrow \quad -\mathbf{p} \in N_{\mathbb{R}^m_+}(\mathbf{u}) \quad \Leftrightarrow \quad -\mathbf{u} \in N_{\mathbb{R}^m_+}(\mathbf{p}).$$

Proof. By the very definition,

$$K^* = \cap_{\mathbf{v} \in K} \{\mathbf{p} \in \mathbb{R}^m : \langle \mathbf{p}, \mathbf{v} \rangle \geq 0\},$$

so it is a non-empty closed convex cone as the intersection of closed half-spaces.

To prove (1.4), it suffices to prove the first equivalence. Indeed, using symmetry together with $(K^*)^* = K$, one immediately obtains the latter one. Suppose that the first assertion in (1.4) holds true. Let $\mathbf{v} \in K$ be arbitrary. Then the complementarity relation along with $\mathbf{p} \in K^*$ yields that

$$\langle \mathbf{p}, \mathbf{u} \rangle = 0 \leq \langle \mathbf{p}, \mathbf{v} \rangle.$$

Therefore the second assertion in (1.4) is proved. On the other hand, assume that the second assertion in (1.4) is valid. Since $\mathbf{u}$ lies in the cone K, so do $\mathbf{v} := \mathbf{0}$ and $\mathbf{v} := 2\mathbf{u}$. Therefore

$$0 \leq \langle \mathbf{p}, -\mathbf{u} \rangle \quad \text{and} \quad 0 \leq \langle \mathbf{p}, \mathbf{u} \rangle,$$

which means that $\mathbf{u} \perp \mathbf{p}$. Now, for a fixed $\mathbf{v} \in K$, we have that $\langle \mathbf{p}, \mathbf{v} \rangle \geq \langle \mathbf{p}, \mathbf{u} \rangle = 0$. Thus $\mathbf{p} \in K^*$.

Now, we prove that $(K^*)^* = K$. By the very definition,

$$(K^*)^* := \{\mathbf{v} \in \mathbb{R}^m : \langle \mathbf{v}, \mathbf{p} \rangle \geq 0 \text{ for all } \mathbf{p} \in K^*\}.$$

Fix any $\mathbf{v} \in K$. For any $\mathbf{p} \in K^*$ we have $\langle \mathbf{p}, \mathbf{v} \rangle \geq 0$. Hence $\mathbf{v} \in (K^*)^*$. Thus $K \subset (K^*)^*$. On the other hand, fix any $\mathbf{v} \notin K$. Set $\mathbf{u} = \mathbf{p}_K(\mathbf{v})$ and $\mathbf{p} = \mathbf{u} - \mathbf{v}$. Then $\mathbf{p}$ is non-zero and $\mathbf{p}_K(\mathbf{u} - \mathbf{p}) = \mathbf{u}$. By Lemma 1.2 (iii), we have $-\mathbf{p} \in N_K(\mathbf{u})$. The first equivalence in (1.4), gives that $\mathbf{p} \in K^*$ and $\langle \mathbf{p}, \mathbf{u} \rangle = 0$. Therefore, $\langle \mathbf{v}, \mathbf{p} \rangle = \langle \mathbf{u} - \mathbf{p}, \mathbf{p} \rangle = -\|\mathbf{p}\|^2 < 0$. This reveals that $\mathbf{v} \notin (K^*)^*$.

To conclude the proof, observe that $\left(\mathbb{R}^m_+\right)^* = \mathbb{R}^m_+$. □

Example 1.6. Given $\mathbf{v} \in \mathbb{R}^m$, suppose that one wants to find $\mathbf{u} \in \mathbb{R}^m$ such that

$$\mathbf{0} \preceq \mathbf{u} \perp \mathbf{v} + \mathbf{u} \succeq \mathbf{0}.$$

Fix any $i \in \{1, 2, \ldots, m\}$. Then $v_i + u_i \geq 0$ and $u_i \geq 0$. The complementarity relation implies that $u_i(v_i + u_i) = 0$. If $v_i = 0$, then $u_i = 0$. If $v_i < 0$, then $u_i \geq -v_i > 0$, which means that $u_i = -v_i$. Finally, when $v_i > 0$, then $v_i + u_i > 0$, and thus $u_i = 0$. To sum up, $u_i = \max\{-v_i, 0\} =: (v_i)^-$. Therefore

$$\mathbf{u} = \mathbf{v}^- := (\max\{-v_1, 0\}, \max\{-v_2, 0\}, \ldots, \max\{-v_m, 0\})^T.$$

Also

$$\mathbf{v} + \mathbf{u} = \mathbf{v}^+ := (\max\{v_1, 0\}, \max\{v_2, 0\}, \ldots, \max\{v_m, 0\})^T.$$

One can derive various calculus rules concerning the normal cones. Let us mention the obvious one here.

Proposition 1.7. *Consider two non-empty closed convex sets $K_1 \subset \mathbb{R}^l$ and $K_2 \subset \mathbb{R}^d$. Then*

$$N_{K_1 \times K_2}(\mathbf{u}) = N_{K_1}(\mathbf{u}_1) \times N_{K_2}(\mathbf{u}_2) \quad \textit{for each} \quad \mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in K_1 \times K_2.$$

Proof. A vector $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2)$ belongs to $N_{K_1 \times K_2}(\mathbf{u})$ if and only if, for every $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in K_1 \times K_2$ we have

$$0 \geq \langle \mathbf{p}, \mathbf{v} - \mathbf{u} \rangle = \langle \mathbf{p}_1, \mathbf{v}_1 - \mathbf{u}_1 \rangle + \langle \mathbf{p}_2, \mathbf{v}_2 - \mathbf{u}_2 \rangle.$$

In particular, letting $\mathbf{v}_1 := \mathbf{u}_1$, we get $\mathbf{p}_2 \in N_{K_2}(\mathbf{u}_2)$. Similarly, the choice $\mathbf{v}_2 := \mathbf{u}_2$ yields that $\mathbf{p}_1 \in N_{K_1}(\mathbf{u}_1)$. The reverse implication is trivial. □

Example 1.8. Let $\mathbf{u} = (u_1, \ldots, u_n)^T \in \mathbb{R}^m_+$. Then

$$\mathbf{p} = (p_1, \ldots, p_n)^T \in N_{\mathbb{R}^m_+}(\mathbf{u}) \iff \begin{cases} p_j \leq 0 & \text{for } j \text{ with } u_j = 0, \\ p_j = 0 & \text{for } j \text{ with } u_j > 0. \end{cases}$$

1.3. **Problem Formulation.** Suppose that functions $\mathbf{f} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\mathbf{g} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ are continuous, that K is a closed convex subset of $\mathbb{R}^m$ and that $b > a$. *Differential variational inequality (DVI)* is a problem to find an absolutely continuous function $\mathbf{x} : [a, b] \to \mathbb{R}^n$ and an integrable function $\mathbf{u} : [a, b] \to \mathbb{R}^m$ such that for almost all $t \in [a, b]$ one has:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)), \tag{1.5}$$

$$0 \leq \langle \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)), \mathbf{v} - \mathbf{u}(t) \rangle \quad \text{whenever} \quad \mathbf{v} \in K, \tag{1.6}$$

$$\mathbf{u}(t) \in K, \tag{1.7}$$

where $\dot{\mathbf{x}}(t)$ is the derivative of $\mathbf{x}(\cdot)$ at t. Of course, one has to prescribe additional initial (or boundary) conditions, but we will come to this issue later. The above model was formally introduced and studied by Jong-Shi Pang and David E. Stewart in [24]. The name (DVI) is based on the fact that an ordinary differential equation (1.5) is linked together with an algebraic constraint represented by the inequality (1.6). Note that the derivative of $\mathbf{u}(\cdot)$ does not appear in (1.5), therefore $\mathbf{u}$ is called an *algebraic variable*. On the other hand, $\mathbf{x}$ is called a *differential variable*. The requested "quality" of $\mathbf{x}(\cdot)$ and $\mathbf{u}(\cdot)$, we are searching for, depends on a particular application. Very often, our setting is too strong especially when impacts come into play (see Example 2.7). First, we will focus on the algebraic constraints.

If (1.6) and (1.7) hold for a fixed $t \in [a,b]$, then $\mathbf{u}(t)$ solves variational inequality (1.1) with $\mathbf{h} := \mathbf{g}(t, \mathbf{x}(t), \cdot)$, that is,

$$\mathbf{u}(t) \in SOL(K, \mathbf{g}(t, \mathbf{x}(t), \cdot)).$$

Therefore the properties of the *solution mapping* $(t, \mathbf{x}) \rightrightarrows SOL(K, \mathbf{g}(t, \mathbf{x}, \cdot))$ will play a key role in our consideration.

For $K := \mathbb{R}^m$, Lemma 1.2 (ii) implies that the problem (1.5) - (1.7) boils down to the so-called *differential algebraic equation (DAE)*, which is a problem to find functions $\mathbf{x} : [a,b] \to \mathbb{R}^n$ and $\mathbf{u} : [a,b] \to \mathbb{R}^m$ such that

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)) \text{ and } \mathbf{0} = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)) \quad \text{for almost all } t \in [a,b]. \tag{1.8}$$

Therefore differential algebraic equations are special cases of differential variational inequalities. In view of Lemma 1.2 (iii), one can always convert a DVI to a particular DAE. Indeed, (1.6) − (1.7) are equivalent to

$$P_K\big(\mathbf{u}(t) - \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t))\big) - \mathbf{u}(t) = \mathbf{0} \quad \text{for almost all} \quad t \in [a,b].$$

However, since the projection mapping is non-smooth, one looses nice properties (such as smoothness) of the function $\mathbf{g}$.

By Proposition 1.5, if K is a non-empty closed convex cone, then (1.5) - (1.7) reduce to the *differential generalized complementarity problem (DGCP)*, i.e. one wants to find functions $\mathbf{x} : [a,b] \to \mathbb{R}^n$ and $\mathbf{u} : [a,b] \to \mathbb{R}^m$ such that

$$\begin{aligned} \dot{\mathbf{x}}(t) \quad &= \quad \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)), \\ K \quad &\ni \quad \mathbf{u}(t) \perp \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)) \in K^* \quad \text{for almost all} \quad t \in [a,b]. \end{aligned} \tag{1.9}$$

In particular, when $K = \mathbb{R}^m_+$ and both $\mathbf{f}$ and $\mathbf{g}$ are affine, we arrive at the *differential linear complementarity problem (DLCP)*, which is also called the *linear complementarity system (LCS)* in the literature. More precisely, this model reads as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{p}, \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{q}, \\ \mathbf{0} \preceq \mathbf{u}(t) &\perp \mathbf{y}(t) \succeq \mathbf{0} \qquad \text{for almost all} \quad t \in [a,b], \end{aligned} \tag{1.10}$$

for given matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{m\times n}$, $\mathbf{D} \in \mathbb{R}^{m\times m}$, and vectors $\mathbf{p} \in \mathbb{R}^n$, $\mathbf{q} \in \mathbb{R}^m$.

On the other hand, DVI means that

$$\dot{\mathbf{x}}(t) \in \mathbf{f}\big(t, \mathbf{x}(t), SOL(K, \mathbf{g}(t, \mathbf{x}(t), \cdot))\big) \quad \text{for almost all} \quad t \in [a,b].$$

Let us define $\mathbf{F} : \mathbb{R} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ for each $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n$ by

$$\mathbf{F}(t, \mathbf{x}) = \mathbf{f}\big(t, \mathbf{x}, SOL(K, \mathbf{g}(t, \mathbf{x}, \cdot))\big).$$

We arrive at

$$\dot{\mathbf{x}}(t) \in \mathbf{F}(t, \mathbf{x}(t)) \quad \text{for almost all} \quad t \in [a, b].$$

The above problem is known as *differential inclusion (DI)* in the literature. When $\mathbf{F}$ has closed graph, non-empty compact convex values and satisfies certain growth properties then the theory of differential inclusions can be applied (see Section 3.1). However, this can be a non-trivial task. The importance of understanding the behavior of the solution mapping emerges again.

In summary, the differential variational inequalities occupy a niche between differential algebraic equations and differential inclusions and one can profit from the special structure of the problem (1.5)–(1.7). However, one can find a different model called "differential variational inequality" in [1]. Namely, given $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^n$ and a closed convex subset C of $\mathbb{R}^n$, the problem to find an absolutely continuous function $\mathbf{x} : [a, b] \to \mathbb{R}^n$ such that:

$$\dot{\mathbf{x}}(t) \in -\mathbf{h}(\mathbf{x}(t)) - N_C(\mathbf{x}(t)) \quad \text{for almost all} \quad t \in [a, b], \tag{1.11}$$

$$\mathbf{x}(t) \in C \quad \text{for all} \quad t \in [a, b]. \tag{1.12}$$

Such a model is called *variational inequality of evolution (VIE)* in [24]. When C is a cone, then Proposition 1.5 implies that

$$C \ni \mathbf{z} \perp \mathbf{u} \in C^* \quad \Leftrightarrow \quad -\mathbf{u} \in N_C(\mathbf{z}).$$

Therefore, (1.11)–(1.12) can be equivalently rewritten as

$$\dot{\mathbf{x}}(t) = -\mathbf{h}(\mathbf{x}(t)) + \mathbf{u}(t) \quad \text{and} \quad C \ni \mathbf{x}(t) \perp \mathbf{u}(t) \in C^*,$$

which is (1.9) with $K := C^*$, $\mathbf{f}(t, \mathbf{x}, \mathbf{u}) := -\mathbf{h}(\mathbf{x}) + \mathbf{u}$, and $\mathbf{g}(t, \mathbf{x}, \mathbf{u}) = \mathbf{x}$.

For a general closed convex set C, introducing an additional variable, (1.11)–(1.12) can be reformulated as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= -\mathbf{h}(\mathbf{x}(t)) + \mathbf{w}(t), \\ \mathbf{0} &= \mathbf{x}(t) - \mathbf{y}(t), \\ 0 &\leq \langle \mathbf{w}(t), \mathbf{v} - \mathbf{y}(t) \rangle \quad \text{for each} \quad \mathbf{v} \in C, \\ \mathbf{y}(t) &\in C. \end{aligned}$$

This is DVI with $K := \mathbb{R}^n \times C$, $\mathbf{u} := (\mathbf{w}, \mathbf{y})$, $\mathbf{f}(t, \mathbf{x}, \mathbf{u}) := -\mathbf{h}(\mathbf{x}) + \mathbf{w}$, and $\mathbf{g}(t, \mathbf{x}, \mathbf{u}) := (\mathbf{x} - \mathbf{y}, \mathbf{w})$. Hence, in general, DVIs cover a broader class of problems. Nevertheless, one can study both DVIs and VIEs in the following unified framework

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(t, \mathbf{x}(t), \mathbf{y}(t), \mathbf{w}(t)), \\ \mathbf{0} &= \mathbf{g}(t, \mathbf{x}(t), \mathbf{y}(t), \mathbf{w}(t)), \\ 0 &\leq \langle \mathbf{w}(t), \mathbf{v} - \mathbf{y}(t) \rangle \quad \text{for each} \quad \mathbf{v} \in \widetilde{K}, \\ \mathbf{y}(t) &\in \widetilde{K}, \end{aligned}$$

with given $\widetilde{K} \subset \mathbb{R}^m$, $\mathbf{f} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ and $\mathbf{g} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$.

1.4. **Application in Electronics.** In this section, we discuss some examples appearing in the theory of electrical circuits. The electrical circuit consists of wires connecting the other elements such as voltage sources, resistors, capacitors and inductors. There is a current (usually denoted by i) flowing through each branch that is measured by a real number. A *node* is a junction (connection) within a circuit were two or more circuit elements are connected or joined together giving a connection point between two or more branches. A node is indicated by a dot. A *loop* is a simple closed path in a circuit in which no circuit element or node is encountered more than once. The state of the circuit is characterized by the currents in each branch together with the voltage or, more precisely, the voltage drop across each branch (usually denoted by v). It is a convention that the voltage in the branch (element) is oriented in the opposite direction than the corresponding current flowing through it, i.e. voltage decreases in the direction of positive current flow.

There are two basic laws from physics. The first is *Kirchhoff's current law* which says that *the total current flowing into a node is equal to the total current flowing out of that node.* This means that current is conserved. The other is *Kirchhoff's voltage law* which says that *the sum of the voltages in any closed loop is zero.* This idea is known as the conservation of energy. The direction of a current and the polarity of a voltage source can be assumed arbitrarily. To determine the actual direction and polarity, the sign of the values also should be considered. For example, a current labeled in left-to-right direction with a negative value is actually flowing right-to-left.

Let us mention several common circuit elements (see Figure 4). A basic element

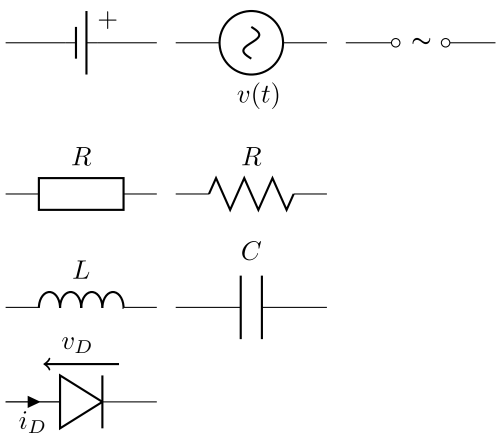

FIGURE 4. Schematic symbols of circuit elements (voltage sources, resistors, an inductor, a capacitor, and a diode).

is a *(linear) resistor*. It has two terminals across which electricity must pass, and it is designed to drop the voltage of the current as it flows from one terminal to the other. Resistors are primarily used to create and maintain known safe currents within electrical components. In this case, *Ohm's law* says that

$$v_R(t) = Ri_R(t),$$

where $R > 0$ is a given resistance. In general, we may have $v_R = \varphi(i_R)$ with a given (non-linear) function $\varphi : \mathbb{R} \to \mathbb{R}$. The graph of φ is called the (Ampere-Volt) *characteristic* of the resistor. Another important element is an *inductor* with the relationship

$$v_L(t) = L\frac{\mathrm{d}i_L}{\mathrm{d}t}(t),$$

where $L > 0$ is a given inductance. An inductor is typically made of a wire wound into a coil. When the current flowing through an inductor changes, a time-varying magnetic field is created inside the coil, and a voltage is induced. The third basic element is a *capacitor* described by

$$v_C(t) = \frac{1}{C}\int_0^t i_C(\tau)\mathrm{d}\tau,$$

where $C > 0$ is a given capacitance. The capacitors contain at least two electrical conductors separated by an insulator. A *voltage source* is a circuit element where the voltage across it is independent of the current flowing through it. Finally, one can come across various types of *diodes* which have Ampere-Volt characteristic described by a non-smooth function or even set-valued function, e.g. *ideal diode* can be described by

$$v_D \in N_{\mathbb{R}_+}(i_D) \quad \Leftrightarrow \quad 0 \leq -v_D \perp i_D \geq 0 \quad \Leftrightarrow \quad -i_D \in N_{\mathbb{R}_+}(-v_D).$$

The above non-smooth law describes the fact, that the current can flow in one direction only, i.e. the diode is blocking in the opposite direction. In practice, the above model is not appropriate because the diode blocks unless the voltage exceeds some value called *breakdown voltage* $V_b > 0$. This value depends on the diode (e.g., it may be 100 V). More appropriate model could be

$$v_D \in F(i_D), \quad \text{where} \quad F(y) := \begin{cases} -V_b, & y < 0, \\ [-V_b, 0], & y = 0, \\ 0, & y > 0. \end{cases}$$

Example 1.9. Let us consider the circuit involving a series connection of a load resistance $R > 0$, an input-signal source generating the voltage $v(t)$ at time $t > 0$, an inductor with inductance $L > 0$, a capacitor with capacitance $C > 0$, and an ideal diode (see Figure 5).

Then the current, denoted by i, is the same for all the elements. Using the Kirchhoff's voltage law, we have

$$v(t) = v_R(t) + v_L(t) + v_C(t) + v_D(t) = Ri(t) + L\frac{\mathrm{d}i}{\mathrm{d}t}(t) + \frac{1}{C}\int_0^t i(\tau)\mathrm{d}\tau + v_D(t),$$

with $v_D(t) \in N_{\mathbb{R}_+}(i(t))$. Setting

$$u(t) = -v_D(t), \quad x_1(t) := \int_0^t i(\tau)\mathrm{d}\tau \quad \text{and} \quad x_2(t) := \dot{x}_1(t) = i(t),$$

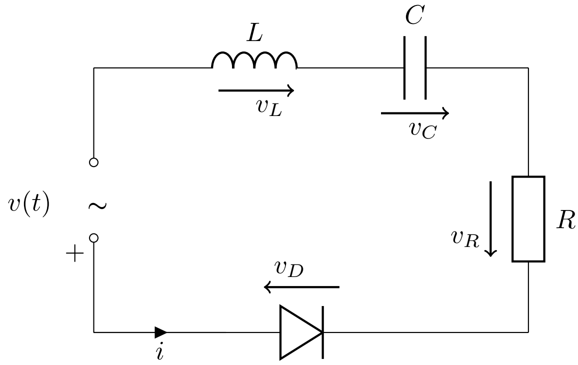

FIGURE 5. The circuit from Example 1.9.

we have

$$L\dot{x}_2(t) = -\frac{1}{C}x_1(t) - Rx_2(t) + v(t) + u(t).$$

Hence, dividing by L, we arrive at the dynamic system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix} v(t) + \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix} u(t)$$

with

$$-u(t) \in N_{\mathbb{R}_+}\left((0\ 1) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \right).$$

Set $\mathbf{x} = (x_1, x_2)^T$,

$$\mathbf{A} := \begin{pmatrix} 0 & 1 \\ -1/(LC) & -R/L \end{pmatrix}, \quad \mathbf{b} := \begin{pmatrix} 0 \\ 1/L \end{pmatrix}, \quad \text{and} \quad \mathbf{c} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

As $-u \in N_{\mathbb{R}_+}(\langle \mathbf{c}, \mathbf{x} \rangle)$ if and only if $-\langle \mathbf{c}, \mathbf{x} \rangle \in N_{\mathbb{R}_+}(u)$, one arrives at the differential variational inequality with

$$\mathbf{f}(t, \mathbf{x}, u) := \mathbf{b}v(t) + \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad \mathbf{g}(t, \mathbf{x}, u) = \langle \mathbf{c}, \mathbf{x} \rangle, \quad \text{and} \quad K := \mathbb{R}_+.$$

Consider a practical diode instead of the ideal one, with $V_b = 100$ V say. Put $u = v_D$ and $K = [-100, 0]$. Then

$$u \in F(\langle \mathbf{c}, \mathbf{x} \rangle) \quad \text{with} \quad F(y) := \begin{cases} -100, & y < 0, \\ [-100, 0], & y = 0, \\ 0, & y > 0. \end{cases}$$

So $\langle \mathbf{c}, \mathbf{x} \rangle \in F^{-1}(u) = N_K(u)$. One obtains a differential variational inequality with

$$\mathbf{f}(t, \mathbf{x}, u) := \mathbf{b}v(t) + \mathbf{A}\mathbf{x} - \mathbf{b}u \quad \text{and} \quad \mathbf{g}(t, \mathbf{x}, u) = -\langle \mathbf{c}, \mathbf{x} \rangle.$$

Example 1.10. Let us consider the four diodes bridge full-wave rectifier involving four diodes (supposed to be ideal), a resistor with the resistance $R > 0$, a capacitor with the capacitance $C > 0$ and an inductor with the inductance $L > 0$ (see Figure 6).

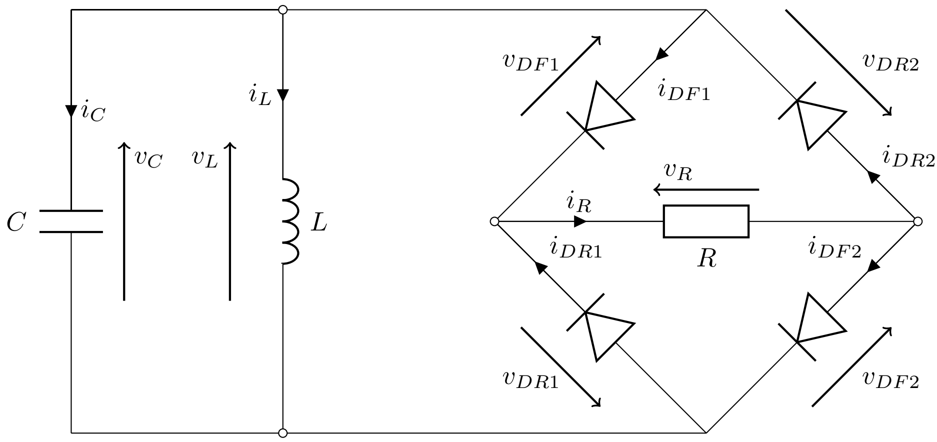

FIGURE 6. The circuit from Example 1.10.

This circuit allows unidirectional current through the load during the entire input cycle; the positive signal goes through unchanged whereas the negative signal is converted into a positive one.

The Kirchhoff's laws can be written as:

$$\begin{cases} v_L = v_C, \\ v_L = v_{DF1} - v_{DR1}, \\ v_{DF2} + v_R + v_{DR1} = 0, \\ i_C + i_L + i_{DF1} - i_{DR2} = 0, \\ i_{DF1} + i_{DR1} = i_R, \\ i_{DF2} + i_{DR2} = i_R. \end{cases}$$

Setting $x = \begin{pmatrix} v_C \\ i_L \end{pmatrix}$, this can be rewritten as a differential linear complementarity problem (1.10) with

$$\mathbf{A} = \begin{pmatrix} 0 & -1/C \\ 1/L & 0 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 0 & 0 & -1/C & 1/C \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} -v_{DR1} \\ -v_{DF2} \\ i_{DF1} \\ i_{DR2} \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 1 & 0 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} 1/R & 1/R & -1 & 0 \\ 1/R & 1/R & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{p} = \mathbf{0}, \quad \mathbf{q} = \mathbf{0}.$$

2. Reduction to an ODE

In this section, we start to investigate the existence of a solution to a differential variational inequality. First, we focus on the possibility to apply standard results on ordinary differential equations.

2.1. **Index for DAEs.** To motivate our consideration, suppose that we want to reduce a differential algebraic equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)) \quad \text{and} \quad \mathbf{0} = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)) \quad \text{for all} \quad t \in [a, b], \tag{2.1}$$

to an (equivalent) ordinary differential equation. The *index* of (2.1) measures its singularity when compared to the ODE. This key concept has evolved over several decades, and today a number of definitions with different emphasis exist. The minimum number of differentiation steps required to transform a DAE into an ODE is known as the *differential/differentiation index* of (2.1) [31]. Index 0 corresponds either to the case of an ODE without any algebraic constraint or to the case when one can neglect this constraint without differentiation.

Example 2.1. Suppose that $\mathbf{g} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable, and also that both $\dot{\mathbf{x}}(t)$ and $\dot{\mathbf{u}}(t)$ exist for all $t \in [a, b]$. Differentiating the second equation in (2.1) one gets, for each $t \in [a, b]$, that

$$\nabla_{\mathbf{x}} \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t))\, \dot{\mathbf{x}}(t) + \nabla_{\mathbf{u}} \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t))\, \dot{\mathbf{u}}(t) + \nabla_t \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)) = \mathbf{0},$$

where $\nabla_{\mathbf{x}} \mathbf{g}(\bar{t}, \bar{\mathbf{x}}, \bar{\mathbf{u}}) \in \mathbb{R}^{m \times n}$ is the *partial Jacobian* of $\mathbf{g}$ at $(\bar{t}, \bar{\mathbf{x}}, \bar{\mathbf{u}}) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ with respect to $\mathbf{x}$ (similarly for $\nabla_{\mathbf{u}} \mathbf{g}(\bar{t}, \bar{\mathbf{x}}, \bar{\mathbf{u}}) \in \mathbb{R}^{m \times m}$ and $\nabla_t \mathbf{g}(\bar{t}, \bar{\mathbf{x}}, \bar{\mathbf{u}}) \in \mathbb{R}^m$). If $\nabla_{\mathbf{u}} \mathbf{g}(\bar{t}, \bar{\mathbf{x}}, \bar{\mathbf{u}})$ is non-singular (regular) for each $(\bar{t}, \bar{\mathbf{x}}, \bar{\mathbf{u}}) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ then, setting $\mathbf{y} = (\mathbf{x}, \mathbf{u})^T$, we arrive at the ordinary differential equation

$$\dot{\mathbf{y}}(t) = \tilde{\mathbf{f}}(t, \mathbf{y}(t)),$$

where, for each $(t, \mathbf{y})^T := (t, \mathbf{x}, \mathbf{u})^T \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$,

$$\tilde{\mathbf{f}}(t, \mathbf{y}) := \begin{pmatrix} \mathbf{f}(t, \mathbf{x}, \mathbf{u}) \\ -\left[\nabla_{\mathbf{u}} \mathbf{g}(t, \mathbf{x}, \mathbf{u})\right]^{-1} \left(\nabla_{\mathbf{x}} \mathbf{g}(t, \mathbf{x}, \mathbf{u})\, \mathbf{f}(t, \mathbf{x}, \mathbf{u}) + \nabla_t \mathbf{g}(t, \mathbf{x}, \mathbf{u})\right) \end{pmatrix}.$$

We completely eliminated the algebraic constraint and therefore such a DAE has the index 1.

Having an implicit function theorem in mind, the key idea emerges immediately. Given $(\bar{t}, \bar{\mathbf{x}}, \bar{\mathbf{u}}) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$, suppose that $\nabla_{\mathbf{u}} \mathbf{g}(\bar{t}, \bar{\mathbf{x}}, \bar{\mathbf{u}})$ is non-singular. Then there is $r > 0$ and a differentiable function $\mathbf{s}$ from $W := \mathbb{B}(\bar{t}, r) \times \mathbb{B}(\bar{\mathbf{x}}, r)$ to $\mathbb{R}^m$ such that

$$\mathbf{0} = \mathbf{g}(t, \mathbf{x}, \mathbf{s}(t, \mathbf{x})) \quad \text{whenever} \quad (t, \mathbf{x}) \in W.$$

Hence (2.1) can be (locally) converted to an ODE of the form

$$\dot{\mathbf{x}}(t) = \tilde{\mathbf{f}}\big(t, \mathbf{x}(t)\big) := \mathbf{f}\big(t, \mathbf{x}(t), \mathbf{s}(t, \mathbf{x}(t))\big) \qquad \text{for all } t \text{ close to } \bar{t}.$$

Clearly, if $\mathbf{f}$ is Lipschitz continuous, then so is $\tilde{\mathbf{f}}$ in a vicinity of the reference point. An easy but not simple question arises: *Is there an analogue of the implicit function theorem for inequalities (inclusions) which ensures the existence of a function* $\mathbf{s}$ *which is at least locally Lipschitz continuous a neighborhood of the reference point?* Fortunately, there is a positive answer as the Section 2.4 shows. Under quite strong monotonicity assumptions, it is possible to reduce a DVI to an ODE even globally. Let us address this issue first.

2.2. **Global Reduction.** Recall several well-known but useful properties of the set of solutions to a non-parametric variational inequality (for a comprehensive reading on this topic see [15]).

Proposition 2.2. *Consider the solution set*

$$S := \{\mathbf{u} \in \mathbb{R}^m : \ 0 \in \mathbf{h}(\mathbf{u}) + N_K(\mathbf{u})\},$$

where $\mathbf{h} : \mathbb{R}^m \to \mathbb{R}^m$ *is defined on a non-empty closed convex subset* K *of* $\mathbb{R}^m$.

(i) *If* $\mathbf{h}$ *is continuous on* K *then* S *is closed (possibly empty);*

(ii) *If* $\mathbf{h}$ *is continuous and monotone on* K*, i.e.*

$$\langle \mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{y}), \mathbf{x} - \mathbf{y}\rangle \ \geq \ 0 \quad \textit{whenever} \quad \mathbf{x}, \mathbf{y} \in K,$$

then S *is convex (possibly empty);*

(iii) *If* $\mathbf{h}$ *is strictly monotone on* K*, i.e.*

$$\langle \mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{y}), \mathbf{x} - \mathbf{y}\rangle \ > \ 0 \quad \textit{for any distinct} \quad \mathbf{x}, \mathbf{y} \in K,$$

then S *is at most singleton;*

(iv) *If* $\mathbf{h}$ *is both continuous and semi-coercive on* K*, i.e. there is* $\bar{\mathbf{u}} \in K$ *along with* $r > 0$ *such that*

$$\langle \mathbf{h}(\mathbf{w}), \mathbf{w} - \bar{\mathbf{u}}\rangle > 0 \quad \textit{for each} \quad \mathbf{w} \in K \ \textit{with} \ \|\mathbf{w}\| > r,$$

then S *is a non-empty subset of* $\mathbb{B}[\mathbf{0}, r]$.

Proof. Note that (i)–(iii) are trivial if S is empty. Until the proof of (iv) assume that this is not the case. The set S contains those $\mathbf{u} \in K$ such that

$$\langle \mathbf{h}(\mathbf{u}), \mathbf{w} - \mathbf{u}\rangle \geq 0 \quad \text{for each} \quad \mathbf{w} \in K. \tag{2.2}$$

(i) Let $(\mathbf{u}_n)_{n\in\mathbb{N}}$ be a sequence in S convergent to some $\mathbf{u} \in \mathbb{R}^m$. The continuity of $\mathbf{h}$ implies that

$$0 \underset{(2.2)}{\leq} \lim_{n\to+\infty} \langle \mathbf{h}(\mathbf{u}_n), \mathbf{w} - \mathbf{u}_n\rangle = \langle \mathbf{h}(\mathbf{u}), \mathbf{w} - \mathbf{u}\rangle \quad \text{for each} \quad \mathbf{w} \in K.$$

As K is closed, $\mathbf{u} \in K$ and thus $\mathbf{u} \in S$.

(ii) To see the convexity, pick any $\mathbf{u}$, $\mathbf{v} \in S$ and any $\lambda \in (0, 1)$. Then

$$\langle \mathbf{h}(\mathbf{u}), \mathbf{w} - \mathbf{u}\rangle \geq 0 \quad \text{and} \quad \langle \mathbf{h}(\mathbf{v}), \mathbf{w} - \mathbf{v}\rangle \geq 0 \quad \text{for each} \quad \mathbf{w} \in K. \tag{2.3}$$

Let $\bar{\mathbf{x}} := (1-\lambda)\mathbf{u}+\lambda\mathbf{v}$. Then $\bar{\mathbf{x}} \in K$ by convexity. Fix any $\bar{\mathbf{w}} \in K$. We have to show that $\langle \mathbf{h}(\bar{\mathbf{x}}), \bar{\mathbf{w}} - \bar{\mathbf{x}}\rangle \geq 0$. Taking $t \in (0, 1)$ as a parameter, let $\mathbf{w}(t) := \bar{\mathbf{x}} + t(\bar{\mathbf{w}} - \bar{\mathbf{x}})$. As K is convex, it contains any $\mathbf{w}(t)$. The monotonicity of $\mathbf{h}$ on K and the first inequality in (2.3) imply that

$$\begin{aligned} 0 \ &\leq \ \langle \mathbf{h}(\mathbf{w}(t)) - \mathbf{h}(\mathbf{u}), \mathbf{w}(t) - \mathbf{u}\rangle + \langle \mathbf{h}(\mathbf{u}), \mathbf{w}(t) - \mathbf{u}\rangle \\ &= \ \langle \mathbf{h}(\mathbf{w}(t)), \mathbf{w}(t) - \mathbf{u}\rangle. \end{aligned}$$

Similarly, $0 \le \langle \mathbf{h}(\mathbf{w}(t)), \mathbf{w}(t) - \mathbf{v} \rangle$. Therefore

$$\begin{aligned} 0 &\le (1-\lambda)\langle \mathbf{h}(\mathbf{w}(t)), \mathbf{w}(t) - \mathbf{u} \rangle + \lambda \langle \mathbf{h}(\mathbf{w}(t)), \mathbf{w}(t) - \mathbf{v} \rangle \\ &= \langle \mathbf{h}(\mathbf{w}(t)), \mathbf{w}(t) - (1-\lambda)\mathbf{u} - \lambda \mathbf{v} \rangle = \langle \mathbf{h}(\mathbf{w}(t)), \mathbf{w}(t) - \bar{\mathbf{x}} \rangle \\ &= t \langle \mathbf{h}(\mathbf{w}(t)), \bar{\mathbf{w}} - \bar{\mathbf{x}} \rangle. \end{aligned}$$

As $\mathbf{w}(t) \to \bar{\mathbf{x}}$ as $t \downarrow 0$, dividing by t and using the continuity of $\mathbf{h}$, we arrive at $\langle \mathbf{h}(\bar{\mathbf{x}}), \bar{\mathbf{w}} - \bar{\mathbf{x}} \rangle \ge 0$, as required.

(iii) Pick any two distinct $\mathbf{u}_1$, $\mathbf{u}_2 \in S$. Since both $\mathbf{u}_1$ and $\mathbf{u}_2$ are in K, writing down (2.2) for them (with $\mathbf{w} := \mathbf{u}_2$ and $\mathbf{w} := \mathbf{u}_1$ respectively), one infers that

$$\begin{aligned} 0 &< \langle \mathbf{h}(\mathbf{u}_1) - \mathbf{h}(\mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2 \rangle \\ &= -\langle \mathbf{h}(\mathbf{u}_1), \mathbf{u}_2 - \mathbf{u}_1 \rangle - \langle \mathbf{h}(\mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2 \rangle \le 0 + 0 = 0, \end{aligned}$$

which is impossible.

(iv) Suppose that there would be some $\mathbf{u} \in S$ with $\|\mathbf{u}\| > r$. As $\bar{\mathbf{u}} \in K$, the very definition of a solution and the semi-coerciveness with $\mathbf{w} := \mathbf{u}$ yield that $0 \le \langle \mathbf{h}(\mathbf{u}), \bar{\mathbf{u}} - \mathbf{u} \rangle < 0$, a contradiction. Therefore $S \subset \mathbb{B}[\mathbf{0}, r]$.

To prove the non-emptiness, suppose that K is bounded first. Lemma 1.2 (i) says that the projection mapping $\mathbf{p}_K : \mathbb{R}^m \to K$ is well-defined and that

$$\langle \mathbf{z} - \mathbf{p}_K(\mathbf{u}), \mathbf{u} - \mathbf{p}_K(\mathbf{u}) \rangle \le 0 \quad \text{whenever} \quad \mathbf{z} \in K \text{ and } \mathbf{u} \in \mathbb{R}^m.$$

Fix any two distinct $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^m$. As both $\mathbf{z}_1 := \mathbf{p}_K(\mathbf{u}_1)$ and $\mathbf{z}_2 := \mathbf{p}_K(\mathbf{u}_2)$ are in K, the above fact and the Cauchy-Schwarz inequality imply that

$$\begin{aligned} \|\mathbf{z}_1 - \mathbf{z}_2\|^2 &= \langle \mathbf{z}_1 - \mathbf{z}_2, \mathbf{z}_1 - \mathbf{z}_2 \rangle \\ &= \langle \mathbf{z}_1 - \mathbf{z}_2, \mathbf{u}_2 - \mathbf{z}_2 \rangle + \langle \mathbf{z}_1 - \mathbf{z}_2, \mathbf{u}_1 - \mathbf{u}_2 \rangle + \langle \mathbf{z}_2 - \mathbf{z}_1, \mathbf{u}_1 - \mathbf{z}_1 \rangle \\ &\le 0 + \|\mathbf{z}_1 - \mathbf{z}_2\| \, \|\mathbf{u}_1 - \mathbf{u}_2\| + 0 = \|\mathbf{z}_1 - \mathbf{z}_2\| \, \|\mathbf{u}_1 - \mathbf{u}_2\|. \end{aligned}$$

This means that $\mathbf{p}_K$ is Lipschitz continuous on the whole of $\mathbb{R}^m$. The mapping $\mathbf{x} \mapsto \mathbf{p}_K(\mathbf{x} - \mathbf{h}(\mathbf{x}))$ maps K continuously into itself as it is a composition of a continuous function from K into $\mathbb{R}^m$ with a Lipschitz function from $\mathbb{R}^m$ into K. By Brouwer's fixed-point theorem it has a fixed point, $\mathbf{u}$ say. Lemma 1.2 (iii) reveals that

$$\mathbf{p}_K(\mathbf{u} - \mathbf{h}(\mathbf{u})) = \mathbf{u} \quad \Longleftrightarrow \quad -\mathbf{h}(\mathbf{u}) \in N_K(\mathbf{u}) \quad \Longleftrightarrow \quad \mathbf{u} \in S.$$

Second, assume that K is unbounded. Clearly, an intersection of K with any closed ball centered at the origin is a compact convex set which is also non-empty if the radius is sufficiently large. The first part of the proof implies that one can find an infinite subset N of $\mathbb{N}$ in such a way that, for each $n \in N$, there is $\mathbf{u}_n \in K$ verifying

$$\langle \mathbf{h}(\mathbf{u}_n), \mathbf{v} - \mathbf{u}_n \rangle \ge 0 \quad \text{for each } \mathbf{v} \in K \text{ with } \|\mathbf{v}\| \le n. \tag{2.4}$$

Then $(\mathbf{u}_n)_{n \in N}$ has to be bounded. Indeed, suppose, on the contrary, that there is an index $n \in N$ such that both $\|\mathbf{u}_n\| > r$ and $n > \|\bar{\mathbf{u}}\|$. Then semi-coerciveness with $\mathbf{w} := \mathbf{u}_n$ and (2.4) with $\mathbf{v} := \bar{\mathbf{u}}$ would yield that $0 \le \langle \mathbf{h}(\mathbf{u}_n), \bar{\mathbf{u}} - \mathbf{u}_n \rangle < 0$, a contradiction. Let $c > 0$ be such that $\|\mathbf{u}_n\| < c$ for each $n \in N$. Pick $n > c$. We will show, that $\mathbf{u}_n \in S$. In view of (2.4), it suffices to show that for any $\mathbf{w} \in K$ with $\|\mathbf{w}\| > n$ we have $\langle \mathbf{h}(\mathbf{u}_n), \mathbf{w} - \mathbf{u}_n \rangle \ge 0$. Fix any such a point $\mathbf{w}$. Find $\lambda \in (0, 1)$

such that $\mathbf{v} := \mathbf{u}_n + \lambda(\mathbf{w} - \mathbf{u}_n)$ has the norm less than n. As both $\mathbf{w}$ and $\mathbf{u}_n$ are in K so is $\mathbf{v}$ thanks to the convexity. Therefore (2.4) reveals that

$$0 \leq \langle \mathbf{h}(\mathbf{u}_n), \mathbf{u}_n + \lambda(\mathbf{w} - \mathbf{u}_n) - \mathbf{u}_n \rangle = \lambda \langle \mathbf{h}(\mathbf{u}_n), \mathbf{w} - \mathbf{u}_n \rangle.$$

The proof is finished. $\square$

Given a non-empty closed convex subset K of $\mathbb{R}^m$ and $\mathbf{h} : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^m$, consider a *parametric variational inequality*:

$$\text{(2.5)} \qquad \text{For a} \quad \mathbf{p} \in \mathbb{R}^d \quad \text{find} \quad \mathbf{u} \in \mathbb{R}^m \quad \text{such that} \qquad \mathbf{0} \in \mathbf{h}(\mathbf{p}, \mathbf{u}) + N_K(\mathbf{u}).$$

Let us define the *solution mapping* $\mathbf{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ by

$$\text{(2.6)} \qquad \mathbb{R}^d \ni \mathbf{p} \mapsto \mathbf{S}(\mathbf{p}) := \{\mathbf{u} \in \mathbb{R}^m : \ \mathbf{u} \text{ solves (2.5)}\}.$$

From the previous theorem one gets sufficient conditions for the Lipschitz continuity of the solution mapping.

Theorem 2.3. *Let* $\mathbf{h} : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^m$ *be defined on a non-empty closed convex subset* K *of* $\mathbb{R}^m$ *and let* V *be a non-empty closed subset of* $\mathbb{R}^d$. *Suppose that*

(i) $\mathbf{h}$ *is continuous on* $V \times K$;
(ii) *there is* $L > 0$ *such that, for each* $\mathbf{u} \in K$, *the mapping* $\mathbf{h}(\cdot, \mathbf{u})$ *is Lipschitz continuous on* V *with the constant* L;
(iii) *there is* $\mu > 0$ *such that*

$$\langle \mathbf{h}(\mathbf{p}, \mathbf{u}) - \mathbf{h}(\mathbf{p}, \mathbf{w}), \mathbf{u} - \mathbf{w} \rangle \geq \mu \|\mathbf{u} - \mathbf{w}\|^2 \quad \textit{whenever} \quad \mathbf{u}, \mathbf{w} \in K \textit{ and } \mathbf{p} \in V.$$

Then the solution mapping $\mathbf{S}$ *in* (2.6) *is single-valued on all of* V *and Lipschitz continuous on* V *with the constant* L/μ.

Proof. Fix any $\mathbf{p} \in V$. We will apply Proposition 2.2 with $\mathbf{h} := \mathbf{h}(\mathbf{p}, \cdot)$. Clearly, the monotonicity assumption (iii) implies the one in Proposition 2.2 (iii). Hence, $\mathbf{S}(\mathbf{p})$ is at most singleton. Pick any $\bar{\mathbf{u}} \in K$ such that $r := \|\bar{\mathbf{u}}\| + \|\mathbf{h}(\mathbf{p}, \bar{\mathbf{u}})\|/\mu > 0$. Note that this causes no loss of generality, because otherwise the origin is the only point of K and $\mathbf{h}(\mathbf{p}, \mathbf{0}) = \mathbf{0}$ and we are done. Fix any $\mathbf{w} \in K$ with $\|\mathbf{w}\| > r$. Then

$$\begin{aligned}
\langle \mathbf{h}(\mathbf{p}, \mathbf{w}), \mathbf{w} - \bar{\mathbf{u}} \rangle &= \langle \mathbf{h}(\mathbf{p}, \mathbf{w}) - \mathbf{h}(\mathbf{p}, \bar{\mathbf{u}}), \mathbf{w} - \bar{\mathbf{u}} \rangle + \langle \mathbf{h}(\mathbf{p}, \bar{\mathbf{u}}), \mathbf{w} - \bar{\mathbf{u}} \rangle \\
&\geq \mu \|\mathbf{w} - \bar{\mathbf{u}}\|^2 - \|\mathbf{h}(\mathbf{p}, \bar{\mathbf{u}})\| \, \|\mathbf{w} - \bar{\mathbf{u}}\| \\
&\geq \|\mathbf{w} - \bar{\mathbf{u}}\| \big(\mu(\|\mathbf{w}\| - \|\bar{\mathbf{u}}\|) - \|\mathbf{h}(\mathbf{p}, \bar{\mathbf{u}})\| \big) \\
&> \|\mathbf{w} - \bar{\mathbf{u}}\| \big(\mu r - \mu \|\bar{\mathbf{u}}\| - \|\mathbf{h}(\mathbf{p}, \bar{\mathbf{u}})\| \big) = 0.
\end{aligned}$$

By Proposition 2.2 (iv), $\mathbf{S}(\mathbf{p})$ contains exactly one point, $\mathbf{s}(\mathbf{p})$ say.

We showed that $\mathbf{S}$ is single-valued on V. Finally, to show that $V \ni \mathbf{p} \mapsto \mathbf{s}(\mathbf{p}) \in K$ is Lipschitz continuous, fix arbitrary $\mathbf{p}_1$, $\mathbf{p}_2 \in V$. Let $\mathbf{u}_1 := \mathbf{s}(\mathbf{p}_1)$ and $\mathbf{u}_2 := \mathbf{s}(\mathbf{p}_2)$. Then

$$\langle \mathbf{h}(\mathbf{p}_1, \mathbf{u}_1), \mathbf{w} - \mathbf{u}_1 \rangle \geq 0 \quad \text{and} \quad \langle \mathbf{h}(\mathbf{p}_2, \mathbf{u}_2), \mathbf{w} - \mathbf{u}_2 \rangle \geq 0 \quad \text{for each} \quad \mathbf{w} \in K.$$

Taking $\mathbf{w} := \mathbf{u}_2$ and $\mathbf{w} := \mathbf{u}_1$ respectively, one sees that (iii) together with Cauchy-Schwarz inequality imply the following chain of estimates

$$\begin{aligned} \mu\|\mathbf{u}_1 - \mathbf{u}_2\|^2 &\leq \langle \mathbf{h}(\mathbf{p}_1, \mathbf{u}_1) - \mathbf{h}(\mathbf{p}_1, \mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2 \rangle = -\langle \mathbf{h}(\mathbf{p}_1, \mathbf{u}_1), \mathbf{u}_2 - \mathbf{u}_1 \rangle \\ &\quad + \langle \mathbf{h}(\mathbf{p}_2, \mathbf{u}_2) - \mathbf{h}(\mathbf{p}_1, \mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2 \rangle - \langle \mathbf{h}(\mathbf{p}_2, \mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2 \rangle \\ &\leq 0 + \|\mathbf{h}(\mathbf{p}_2, \mathbf{u}_2) - \mathbf{h}(\mathbf{p}_1, \mathbf{u}_2)\| \, \|\mathbf{u}_1 - \mathbf{u}_2\| + 0 \\ &\leq L\|\mathbf{p}_2 - \mathbf{p}_1\| \, \|\mathbf{u}_2 - \mathbf{u}_1\|. \end{aligned}$$

If $\mathbf{u}_2 \neq \mathbf{u}_1$ then, dividing by $\mu\|\mathbf{u}_1 - \mathbf{u}_2\| > 0$, we obtain that

$$\|\mathbf{u}_1 - \mathbf{u}_2\| \leq \frac{L}{\mu}\|\mathbf{p}_1 - \mathbf{p}_2\|.$$

As the above inequality holds trivially when $\mathbf{u}_2 = \mathbf{u}_1$, the proof is finished. □

Example 2.4. Given $a \in \mathbb{R}$, consider the problem:

$$\text{For a} \quad p \in \mathbb{R} \quad \text{find} \quad u \in \mathbb{R} \quad \text{such that} \quad p \in au + N_{\mathbb{R}_+}(u) =: \Phi(u).$$

Then this is a parametric variational inequality with $h(p, u) := au - p$ and $K := \mathbb{R}_+$. Then $L = 1$ and $\mu = a$ provided that $a > 0$ (see Figure 7).

We are going to apply the previous statement to an autonomous DVI:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \\ 0 &\leq \langle \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)), \mathbf{v} - \mathbf{u}(t) \rangle \quad \text{whenever} \quad \mathbf{v} \in K, \\ \mathbf{u}(t) &\in K. \end{aligned}$$

Assume that

(A) $\mathbf{g}$ is Lipschitz continuous with respect to the first variable on a closed subset Ω of $\mathbb{R}^n$ uniformly in the latter one, i.e. there is $L_{\mathbf{g}} > 0$ such that

$$\|\mathbf{g}(\mathbf{x}_1, \mathbf{u}) - \mathbf{g}(\mathbf{x}_2, \mathbf{u})\| \leq L_{\mathbf{g}}\|\mathbf{x}_1 - \mathbf{x}_2\|$$

whenever $(\mathbf{x}_1, \mathbf{u}), (\mathbf{x}_2, \mathbf{u}) \in \Omega \times K$;

(B) there is $\mu > 0$ such that, for each $\mathbf{u}_1$, $\mathbf{u}_2 \in K$ and each $\mathbf{x} \in \Omega$, one has

$$\langle \mathbf{g}(\mathbf{x}, \mathbf{u}_1) - \mathbf{g}(\mathbf{x}, \mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2 \rangle \geq \mu\|\mathbf{u}_1 - \mathbf{u}_2\|^2;$$

(C) $\mathbf{f}$ is Lipschitz continuous on $\Omega \times K$, i.e. there are positive $L_{\mathbf{x}}$ and $L_{\mathbf{u}}$ such that

$$\|\mathbf{f}(\mathbf{x}_1, \mathbf{u}_1) - \mathbf{f}(\mathbf{x}_2, \mathbf{u}_2)\| \leq L_{\mathbf{x}}\|\mathbf{x}_1 - \mathbf{x}_2\| + L_{\mathbf{u}}\|\mathbf{u}_1 - \mathbf{u}_2\|$$

for each $(\mathbf{x}_1, \mathbf{u}_1)$, $(\mathbf{x}_2, \mathbf{u}_2) \in \Omega \times K$.

Theorem 2.3, implies that there is a function $\mathbf{s} : \Omega \ni \mathbf{x} \mapsto \mathbf{s}(\mathbf{x}) \in SOL\big(K, \mathbf{g}(\mathbf{x}, \cdot)\big)$ which is Lipschitz continuous on Ω with the constant $L_{\mathbf{g}}/\mu$. Let

$$\tilde{\mathbf{f}}(\mathbf{x}) := \mathbf{f}\big(\mathbf{x}, \mathbf{s}(\mathbf{x})\big), \quad \mathbf{x} \in \Omega.$$

Then, for any $\mathbf{x}_1$, $\mathbf{x}_2 \in \Omega$, we have that

$$\begin{aligned} \|\tilde{\mathbf{f}}(\mathbf{x}_1) - \tilde{\mathbf{f}}(\mathbf{x}_2)\| &= \|\mathbf{f}\big(\mathbf{x}_1, \mathbf{s}(\mathbf{x}_1)\big) - \mathbf{f}\big(\mathbf{x}_2, \mathbf{s}(\mathbf{x}_2)\big)\| \\ &\leq L_{\mathbf{x}}\, \|\mathbf{x}_1 - \mathbf{x}_2\| + L_{\mathbf{u}}\, \|\mathbf{s}(\mathbf{x}_1) - \mathbf{s}(\mathbf{x}_2)\| \\ &\leq \big(L_{\mathbf{x}} + L_{\mathbf{u}} L_{\mathbf{g}}/\mu\big)\, \|\mathbf{x}_1 - \mathbf{x}_2\|. \end{aligned}$$

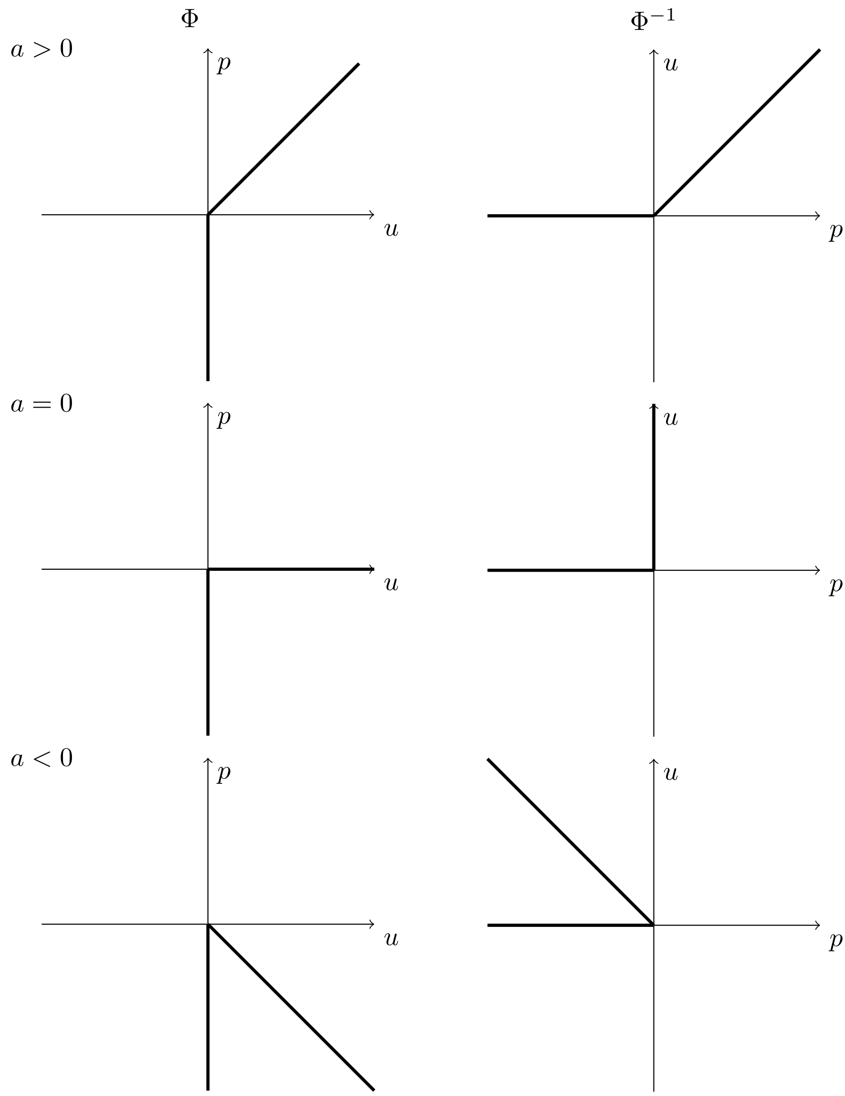

FIGURE 7. Mappings from Example 2.4.

We arrived at $\dot{\mathbf{x}} = \tilde{\mathbf{f}}(\mathbf{x})$, which is an ODE with the Lipschitz continuous right-hand side. Hence, the classical theory may be applied.

The same can be done for a general non-autonomous DVI in (1.5) – (1.7). Fix $\delta > 0$ and let $V := [a, b] \times \mathbb{B}[\mathbf{0}, \delta]$. Instead of (A) – (C) suppose that

(A') there is an integrable function $l_{\mathbf{g}} : [a, b] \to (0, \infty)$ such that

$$\|\mathbf{g}(t, \mathbf{x}_1, \mathbf{u}) - \mathbf{g}(t, \mathbf{x}_2, \mathbf{u})\| \leq l_{\mathbf{g}}(t)\, \|\mathbf{x}_1 - \mathbf{x}_2\|$$

whenever $(t, \mathbf{x}_1, \mathbf{u}), (t, \mathbf{x}_2, \mathbf{u}) \in V \times K$;

(B') there is $\mu > 0$ such that, for each $\mathbf{u}_1$, $\mathbf{u}_2 \in K$ and each $(t, \mathbf{x}) \in V$, one has

$$\langle \mathbf{g}(t, \mathbf{x}, \mathbf{u}_1) - \mathbf{g}(t, \mathbf{x}, \mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2 \rangle \geq \mu \|\mathbf{u}_1 - \mathbf{u}_2\|^2;$$

(C') there is an integrable function $l_{\mathbf{x}} : [a, b] \to (0, \infty)$ along with $L_{\mathbf{u}} > 0$ such that

$$\|\mathbf{f}(t, \mathbf{x}_1, \mathbf{u}_1) - \mathbf{f}(t, \mathbf{x}_2, \mathbf{u}_2)\| \leq l_{\mathbf{x}}(t)\|\mathbf{x}_1 - \mathbf{x}_2\| + L_{\mathbf{u}}\|\mathbf{u}_1 - \mathbf{u}_2\|$$

for each $(t, \mathbf{x}_1, \mathbf{u}_1)$, $(t, \mathbf{x}_2, \mathbf{u}_2) \in V \times K$;

(D) there are integrable functions φ_1, $\varphi_2 : [a, b] \to (0, \infty)$ along with $\bar{\mathbf{u}} \in K$ such that

$$\|\mathbf{g}(t, \mathbf{0}, \bar{\mathbf{u}})\| \leq \varphi_1(t) \quad \text{and} \quad \|\mathbf{f}(t, \mathbf{0}, \mathbf{0})\| \leq \varphi_2(t).$$

As in the proof of Theorem 2.3, there is a function $\mathbf{s} : V \ni (t, \mathbf{x}) \mapsto \mathbf{s}(t, \mathbf{x}) \in SOL\big(K, \mathbf{g}(t, \mathbf{x}, \cdot)\big)$ such that

$$\|\mathbf{s}(t, \mathbf{x}_1) - \mathbf{s}(t, \mathbf{x}_2)\| \leq \frac{l_{\mathbf{g}}(t)}{\mu}\|\mathbf{x}_1 - \mathbf{x}_2\| \quad \text{whenever} \quad (t, \mathbf{x}_1), (t, \mathbf{x}_2) \in V.$$

Let

$$\tilde{\mathbf{f}}(t, \mathbf{x}) := \mathbf{f}\big(t, \mathbf{x}, \mathbf{s}(t, \mathbf{x})\big), \quad (t, \mathbf{x}) \in V.$$

Then the function $l_{\tilde{\mathbf{f}}}(t) := l_{\mathbf{x}}(t) + l_{\mathbf{g}}(t)\, L_{\mathbf{u}}/\mu$, $t \in [a, b]$, is integrable on $[a, b]$. Moreover, for any $(t, \mathbf{x}_1)$, $(t, \mathbf{x}_2) \in V$, we have that

$$\begin{aligned}
\|\tilde{\mathbf{f}}(t, \mathbf{x}_1) - \tilde{\mathbf{f}}(t, \mathbf{x}_2)\| &= \|\mathbf{f}\big(t, \mathbf{x}_1, \mathbf{s}(t, \mathbf{x}_1)\big) - \mathbf{f}\big(t, \mathbf{x}_2, \mathbf{s}(t, \mathbf{x}_2)\big)\| \\
&\leq l_{\mathbf{x}}(t)\, \|\mathbf{x}_1 - \mathbf{x}_2\| + L_{\mathbf{u}}\, \|\mathbf{s}(t, \mathbf{x}_1) - \mathbf{s}(t, \mathbf{x}_2)\| \\
&\leq \big(l_{\mathbf{x}}(t) + l_{\mathbf{g}}(t)\, L_{\mathbf{u}}/\mu\big)\, \|\mathbf{x}_1 - \mathbf{x}_2\| = l_{\tilde{\mathbf{f}}}(t)\, \|\mathbf{x}_1 - \mathbf{x}_2\|.
\end{aligned}$$

Finally, fix arbitrary $(t, \mathbf{x}) \in V$. Then

$$\begin{aligned}
r &:= \|\bar{\mathbf{u}}\| + \frac{1}{\mu}\|\mathbf{g}(t, \mathbf{x}, \bar{\mathbf{u}})\| \leq \|\bar{\mathbf{u}}\| + \frac{1}{\mu}(\|\mathbf{g}(t, \mathbf{x}, \bar{\mathbf{u}}) - \mathbf{g}(t, \mathbf{0}, \bar{\mathbf{u}})\| + \|\mathbf{g}(t, \mathbf{0}, \bar{\mathbf{u}})\|) \\
&\leq \|\bar{\mathbf{u}}\| + \frac{1}{\mu}\big(l_{\mathbf{g}}(t)\, \|\mathbf{x}\| + \varphi_1(t)\big) \leq \|\bar{\mathbf{u}}\| + \frac{1}{\mu}\big(\delta l_{\mathbf{g}}(t) + \varphi_1(t)\big).
\end{aligned}$$

As in the proof of Theorem 2.3 we get that the assumptions in Proposition 2.2 (iv) hold. Using the estimate for the norm of the solutions therein, one infers that

$$\begin{aligned}
\|\tilde{\mathbf{f}}(t, \mathbf{x})\| &\leq \|\mathbf{f}\big(t, \mathbf{x}, \mathbf{s}(t, \mathbf{x})\big) - \mathbf{f}(t, \mathbf{0}, \mathbf{0})\| + \|\mathbf{f}(t, \mathbf{0}, \mathbf{0})\| \\
&\leq l_{\mathbf{x}}(t)\|\mathbf{x}\| + L_{\mathbf{u}}\, \|\mathbf{s}(t, \mathbf{x})\| + \varphi_2(t) \\
&\leq \delta l_{\mathbf{x}}(t) + L_{\mathbf{u}}\|\bar{\mathbf{u}}\| + \frac{L_{\mathbf{u}}}{\mu}\big(\delta l_{\mathbf{g}}(t) + \varphi_1(t)\big) + \varphi_2(t) := \varphi(t).
\end{aligned}$$

We arrived at $\dot{\mathbf{x}}(t) = \tilde{\mathbf{f}}(t, \mathbf{x}(t))$. Since φ is integrable on $[a, b]$, theory of the Carathéodory differential equations may be applied.

Now, let us discuss a higher-dimensional version of Example 2.4. We will need an easy lemma from linear algebra.

Lemma 2.5. *Let $\mathbf{A} \in \mathbb{R}^{m \times m}$. If $\langle \mathbf{A}\mathbf{h}, \mathbf{h} \rangle > 0$ for any non-zero $\mathbf{h} \in \mathbb{R}^m$, then there is $\mu > 0$ such that $\langle \mathbf{A}\mathbf{h}, \mathbf{h} \rangle \geq \mu\|\mathbf{h}\|^2$ whenever $\mathbf{h} \in \mathbb{R}^m$.*

Proof. Fix any non-zero $\mathbf{h} \in \mathbb{R}^m$. The matrix $\mathbf{A}$ is the sum of its symmetric part $\mathbf{A}_s := \left(\mathbf{A} + \mathbf{A}^T\right)/2$ and its anti-symmetric part $\mathbf{A}_a := \left(\mathbf{A} - \mathbf{A}^T\right)/2$. Then

$$2\langle \mathbf{A}_a\mathbf{h}, \mathbf{h}\rangle = \langle \mathbf{A}\mathbf{h}, \mathbf{h}\rangle - \langle \mathbf{A}^T\mathbf{h}, \mathbf{h}\rangle = \langle \mathbf{h}, \mathbf{A}\mathbf{h}\rangle - \langle \mathbf{A}^T\mathbf{h}, \mathbf{h}\rangle = \langle \mathbf{A}^T\mathbf{h}, \mathbf{h}\rangle - \langle \mathbf{A}^T\mathbf{h}, \mathbf{h}\rangle = 0,$$

which means that $\langle \mathbf{A}\mathbf{h}, \mathbf{h}\rangle = \langle \mathbf{A}_s\mathbf{h}, \mathbf{h}\rangle + \langle \mathbf{A}_a\mathbf{h}, \mathbf{h}\rangle = \langle \mathbf{A}_s\mathbf{h}, \mathbf{h}\rangle$. By the assumption, $\mathbf{A}_s$ is not only symmetric but also positive definite, let μ be its least eigenvalue. Then properties of the Rayleigh's quotient yield that

$$\frac{\langle \mathbf{A}\mathbf{h}, \mathbf{h}\rangle}{\|\mathbf{h}\|^2} = \frac{\langle \mathbf{A}_s\mathbf{h}, \mathbf{h}\rangle}{\|\mathbf{h}\|^2} \geq \mu > 0.$$

□

Example 2.6. Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be fixed. Consider the problem:

$$\text{Given} \quad \mathbf{p} \in \mathbb{R}^m \quad \text{find} \quad \mathbf{u} \in \mathbb{R}^m \quad \text{such that} \quad \mathbf{p} \in \mathbf{A}\mathbf{u} + N_{\mathbb{R}^m_+}(\mathbf{u}).$$

Let $\mathbf{h}(\mathbf{p}, \mathbf{u}) := \mathbf{A}\mathbf{u} - \mathbf{p}$ for $\mathbf{p}$, $\mathbf{u} \in \mathbb{R}^m$. The first two conditions of Theorem 2.3 hold (with $L := 1$ and $V := \mathbb{R}^m$), and the last one requests the existence of $\mu > 0$ such that

$$\langle \mathbf{A}(\mathbf{u} - \mathbf{w}), \mathbf{u} - \mathbf{w}\rangle \geq \mu\|\mathbf{u} - \mathbf{w}\|^2 \quad \text{whenever} \quad \mathbf{u}, \mathbf{w} \in \mathbb{R}^m_+.$$

In view of Lemma 2.5, this condition holds provided that $\langle \mathbf{A}\mathbf{h}, \mathbf{h}\rangle > 0$ for each non-zero $\mathbf{h} \in \mathbb{R}^m$. In particular, for any positive definite matrix $\mathbf{A}$ which restricts ourselves on the class of symmetric matrices.

The above derived condition is unnecessarily strong especially when the local reduction is considered as we will see later.

2.3. **Problems without Friction in Mechanics.** One of main goals of the classical mechanics is to describe how things move. The motion of a system is determined by the coordinates of all its constituent particles as functions of time. For a single point particle moving in three-dimensional space, we want to know its *position* (coordinates) as a function of time, i.e. we want to find a function $\mathbf{x} : \mathbb{R} \to \mathbb{R}^3$ which is called the *trajectory* of the system. In case of n particles, the motion is described by a set of functions $\mathbf{x}_i : \mathbb{R} \to \mathbb{R}^3$, where $i \in \{1, 2, \ldots, n\}$ labels which particle we are talking about. Roughly spoken, we are able to predict where a particle will be at any given instant of time. Knowing the trajectory, we can compute its derivative and obtain a *velocity*

$$\mathbf{v}(t) := \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \dot{\mathbf{x}}(t)$$

at any time t as well. Taking the second derivative of the trajectory, we obtain an *acceleration*

$$\mathbf{a}(t) := \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) = \ddot{\mathbf{x}}(t).$$

The complete motion is encoded in a system of differential equations, called the *equations of motion.* Newton's three Laws of Motion may be stated as follows:

1. A body remains in uniform motion unless acted on by a force;

2. Force equals the rate of change of momentum $\mathbf{p}$, defined by $\mathbf{p}(t) = m\mathbf{v}(t)$, $t \in \mathbb{R}$, where $m > 0$ is particle's mass. If we suppose that the mass does not depend on time (which is true in case of low velocities compared to the speed of light), this law can be written as an equality
$$\mathbf{f}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}(t)) = m\dot{\mathbf{v}}(t) = m\mathbf{a}(t);$$
3. Any two bodies exert equal and opposite forces on each other.

To convert the second Newton's law into a meaningful equation, one has to know a *force law* describing how the force $\mathbf{f}$ depends on the coordinates or velocities themselves. This is an empirical law given by physicists which approximates well the reality in a particular situation. For example, I. Newton deduced the gravitational force law, which says that the force $\mathbf{f}_{ij}$ exerted by a particle i by another particle j is

$$\mathbf{f}_{ij} = -Gm_i m_j \frac{\mathbf{x}_i - \mathbf{x}_j}{\|\mathbf{x}_i - \mathbf{x}_j\|^3}, \tag{2.7}$$

where $G := (6.6726 \pm 0.0008) \times 10^{-11}$ Nm^2/kg^2 is the *Cavendish constant.* In particular, for a particle of mass m near the surface of the Earth with the mass m_e and the radius r_e, taking $m_i = m$ and $m_j = m_e$, with $\mathbf{x}_i - \mathbf{x}_j := -r_e\mathbf{r}$, we obtain
$$\mathbf{f} = -mg\mathbf{r} \equiv -m\mathbf{g},$$
where $\mathbf{r}$ is a radial unit vector pointing from the Earth's center and $g := Gm_e/r_e^2 \approx 9.81$ m/s^2 is the acceleration due to gravity at (near) the Earth's surface. Newton's second law says that $\mathbf{a} = -\mathbf{g}$, i.e. objects accelerate as they fall to the Earth. Hence, if we want to describe the motion of a particle of mass m near the surface of the Earth, we may reduce the original tree-dimensional problem to one-dimensional one and assume a uniform gravitational field, with $f = -mg$.

As in electronics there are several basic elements (with appropriate force laws) used in mechanics (see Figure 8 for graphical representation). Denote f and x the force and the displacement, respectively. Then for the *spring* we have $f_S(t) = -kx(t)$, where $k > 0$ is a given *stiffness.* Similarly, the *damper* can be described by $f_D(t) = -c\dot{x}(t)$, where $c > 0$ is a given viscous damping coefficient (often denoted also by b).

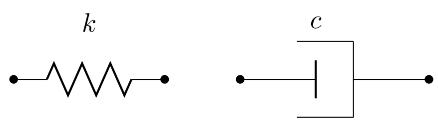

FIGURE 8. Representation of spring and damper.

Example 2.7. Consider a rigid ball of mass m and radius r falling downward onto a rigid table due to the gravitational acceleration g. Assume that there is no air resistance and denote the height of the center of the ball above the table at time $t \geq 0$ by $y(t)$ (see Figure 9). Let $v(t) := \dot{y}(t)$ be the velocity of the ball at a given time. Sooner or later we must face the hard constraint that $y(t) - r \geq 0$. When

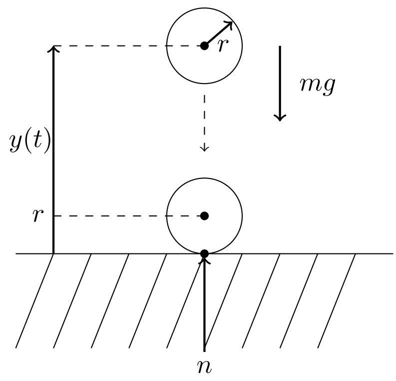

FIGURE 9. A bouncing ball.

$y(t) = r$, there has to be a reaction force $n(t)$ to prevent penetration of the table. This and the Newton's second law of motion imply that

$$m\dot{v}(t) = -mg + n(t) \quad \text{and} \quad 0 \leq y(t) - r \perp n(t) \geq 0 \quad \text{for all} \quad t \geq 0.$$

Unfortunately, these conditions together with initial conditions do not determine the trajectory uniquely. Even in this simple case one needs an extra condition, usually given in terms of a *coefficient of restitution* $0 \leq e \leq 1$. Its value determines "bouncing" after a collision, e.g. if $y(t) = r$ then the relationship between pre-impact and post-impact velocities is given by

$$v(t+) = -e\, v(t-).$$

Again this is only a model of impact and the value of e is not easy to determine. Moreover, as v has a discontinuity at the time when the impact occurs, the reaction force $n(\cdot)$ must contain a Dirac-δ function, or impulse, at this time. Instead, a common approach is to use *normal compliance*, which assumes that there is a slight interpenetration of the ball and the surface. The contact is represented by a stiff spring applying no force when there is no interpenetration. But when there is interpenetration, the force in the spring is proportional to the depth of interpenetration (see Figure 10). Let $k > 0$. Consider the following model

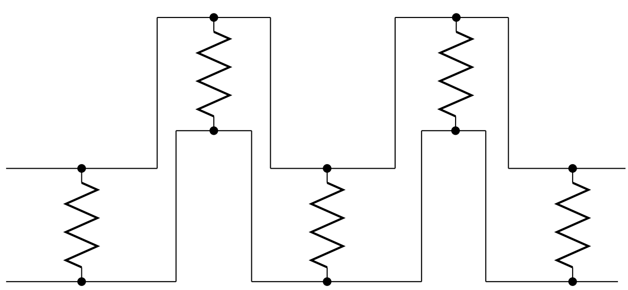

FIGURE 10. Normal compliance approach

$$m\dot{v}(t) = -mg + \lambda(t) \quad \text{and} \quad 0 \le y(t) - r + \lambda(t)/k \perp \lambda(t) \ge 0 \quad \text{for all} \quad t \ge 0.$$

As in Example 1.6, then $\lambda(t) = k\big[y(t) - r\big]^- = k \max\{r - y(t), 0\}$. Proposition 1.5 implies that

$$-y(t) + r - \lambda(t)/k \in N_{\mathbb{R}_+}\big(\lambda(t)\big) \quad \text{whenever} \quad t \ge 0.$$

Clearly, the function $g(y, v, \lambda) := y - r + \lambda/k$ is strongly monotone in the third variable uniformly with respect to the first two ones with the constant $\mu := 1/k$.

In general, the state of a rigid body, at time $t \in [a, b]$, can be represented by a vector $\mathbf{q}(t) \in \mathbb{R}^m$ of the so-called *generalized coordinates* (angles, positions of centers of mass, angular and ordinary velocities, ...). Then m is the number of degrees of freedom. Let $\mathbf{v}(t) = \dot{\mathbf{q}}(t)$. Friction-less impact problems contain inequality constraints on the generalized coordinates:

$$h_i(\mathbf{q}(t)) \ge 0, \quad i \in \{1, 2, \ldots, m\},\ t \in [a, b], \tag{2.8}$$

where $h_i : \mathbb{R}^m \to \mathbb{R}$ are given. Then the motion of the system can be described by a system of ODEs:

$$\begin{array}{rcll} \dot{\mathbf{q}}(t) & = & \mathbf{v}(t), & \\ \mathbf{M}\dot{\mathbf{v}}(t) & = & -\mathbf{C}\mathbf{v}(t) - \Pi'(\mathbf{q}(t)) + \big[\mathbf{h}'\big(\mathbf{q}(t)\big)\big]^T \mathbf{u}(t), & \\ \mathbf{0} \preceq \mathbf{u}(t) & \perp & \mathbf{h}(\mathbf{q}(t)) \succeq \mathbf{0}, & t \in [a, b], \end{array}$$

where

- $\mathbf{M}$ is the mass matrix (which may depend on $\mathbf{q}$ in general);
- $\mathbf{C}$ is the viscous damping matrix;
- Π represents the potential energy of the system. Often, we assume that $\Pi(\mathbf{x}) = \frac{1}{2}\langle \mathbf{K}\mathbf{x}, \mathbf{x}\rangle$, $\mathbf{x} \in \mathbb{R}^m$, where $\mathbf{K}$ is the symmetric stiffness matrix (so $\Pi'(\mathbf{x}) = \mathbf{K}\mathbf{x}$);
- $\mathbf{h}(\mathbf{q}) := \big(h_1(\mathbf{q}), h_2(\mathbf{q}), \ldots, h_m(\mathbf{q})\big)^T$, $\mathbf{q} \in \mathbb{R}^m$; and
- $\mathbf{u}(t) \in \mathbb{R}^m$ is a vector of Lagrange multipliers.

Using the normal compliance, one may approximate the above system by:

$$\begin{array}{rcll} \dot{\mathbf{q}}(t) & = & \mathbf{v}(t), & \\ \mathbf{M}\dot{\mathbf{v}}(t) & = & -\mathbf{C}\mathbf{v}(t) - \Pi'(\mathbf{q}(t)) + \big[\mathbf{h}'\big(\mathbf{q}(t)\big)\big]^T \mathbf{u}(t), & \\ \mathbf{0} \preceq \mathbf{u}(t) & \perp & \mathbf{h}(\mathbf{q}(t)) + k^{-1}\mathbf{u}(t) \succeq \mathbf{0}, & t \in [a, b], \end{array}$$

In this case, $\mathbf{g}(\mathbf{q}, \mathbf{v}, \mathbf{u}) := \mathbf{h}(\mathbf{q}) + k^{-1}\mathbf{u}$ and the reaction forces are given by

$$k\left[\mathbf{h}'(\mathbf{q})\right]^T [\mathbf{h}(\mathbf{q})]^-.$$

Main advantage of the normal compliance approach is that the equations of motion are just ordinary differential equations. However, most bodies are stiff, which means that k is large. The question what happens when $k \to +\infty$ we leave open here. This together with a more general model can be found in [32].

In Example 2.7 we have seen that when modeling an impact, one arrives at the following DVI:

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{f}(t,\mathbf{x}(t)) + \mathbf{u}(t),\\
0 &\le \langle \mathbf{g}(t,\mathbf{x}(t),\mathbf{u}(t)), \mathbf{v}-\mathbf{u}(t)\rangle \quad \text{whenever} \quad \mathbf{v}\in K,\\
\mathbf{u}(t) &\in K.
\end{aligned}$$

Without using a normal compliance approach, we need to allow $\mathbf{u}(\cdot)$ containing Dirac-δ functions or more general distributions. Assuming that $\mathbf{u}(\cdot)$ is a distributional derivative of some measurable function $\mathbf{p}(\cdot)$ which is bounded on each finite interval, we have that $\mathbf{p}(\cdot)$ is an integral (Perron, Denjoy, Denjoy-Khintchine) of $\mathbf{u}(\cdot)$. Then via a substitution $\mathbf{x} = \mathbf{y} + \mathbf{p}$, one can transform the above differential equation to the Carathéodory equation

$$\dot{\mathbf{y}}(t) = \mathbf{f}(t, \mathbf{y}(t) + \mathbf{p}(t)).$$

The inequality constraint, can be replaced by

$$0 \le \int_a^b \langle \mathbf{g}(t,\mathbf{x}(t),\mathbf{u}(t)), \mathbf{v}(t)-\mathbf{u}(t)\rangle \,\mathrm{d}t \quad \text{whenever} \quad \mathbf{v}(\cdot)\in\mathcal{C}^\infty([a,b],K),$$

which is under certain integrability condition equivalent to the original one, see [32, Lemma 3.1]. The last remaining question is what means that $\mathbf{u}(t) \in K$ *for (almost) all* $t \in [a,b]$ as the point-wise values are meaningless in general. This can be interpreted as

$$\frac{\int_{-\infty}^{+\infty}\phi(t)\mathbf{u}(t)\,\mathrm{d}t}{\int_{-\infty}^{+\infty}\phi(t)\,\mathrm{d}t} \in K \quad \text{for all non-negative} \quad \phi(\cdot)\in\mathcal{C}_0^\infty(\mathbb{R})\setminus\{0\}.$$

2.4. **Local Reduction.** Suppose that $\mathbf{h} : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^m$ and $\mathbf{H} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ are given. The *parametric generalized equation* is a problem:

$$\text{For} \quad \mathbf{p}\in\mathbb{R}^d \quad \text{find} \quad \mathbf{u}\in\mathbb{R}^m \quad \text{such that} \quad \mathbf{0}\in\mathbf{h}(\mathbf{p},\mathbf{u})+\mathbf{H}(\mathbf{u}). \tag{2.9}$$

Since we are interested in "local continuity properties" of the solution with respect the parameter around a fixed reference point, we need the following definition (see Figure 11).

Definition 2.8. Given a set-valued mapping $\mathbf{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ and $(\bar{\mathbf{p}}, \bar{\mathbf{u}}) \in \operatorname{gph}\mathbf{S}$, a *(local) selection for* $\mathbf{S}$ *around* $\bar{\mathbf{p}}$ *for* $\bar{\mathbf{u}}$ is any single-valued mapping $\mathbf{s} : \mathbb{R}^d \to \mathbb{R}^m$ defined on a neighborhood V of $\bar{\mathbf{p}}$ such that

$$\mathbf{s}(\bar{\mathbf{p}}) = \bar{\mathbf{u}} \quad \text{and} \quad \mathbf{s}(\mathbf{p})\in\mathbf{S}(\mathbf{p}) \quad \text{for each} \quad \mathbf{p}\in V.$$

A *(graphical) localization of* $\mathbf{S}$ *around* $\bar{\mathbf{p}}$ *for* $\bar{\mathbf{u}}$ is a set-valued mapping $\widetilde{\mathbf{S}} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ such that for some neighborhoods U of $\bar{\mathbf{u}}$ and V of $\bar{\mathbf{p}}$ we have

$$\widetilde{\mathbf{S}}(\mathbf{p}) = \begin{cases} \mathbf{S}(\mathbf{p})\cap U & \text{if } \ \mathbf{p}\in V,\\ \emptyset & \text{otherwise.}\end{cases}$$

The existence of a localization, which is single-valued and Lipschitz continuous in a vicinity of the reference point, is also known as the *strong metric regularity of* $\mathbf{S}^{-1}$. This notion was introduced by S. M. Robinson in [29]. Clearly, for any

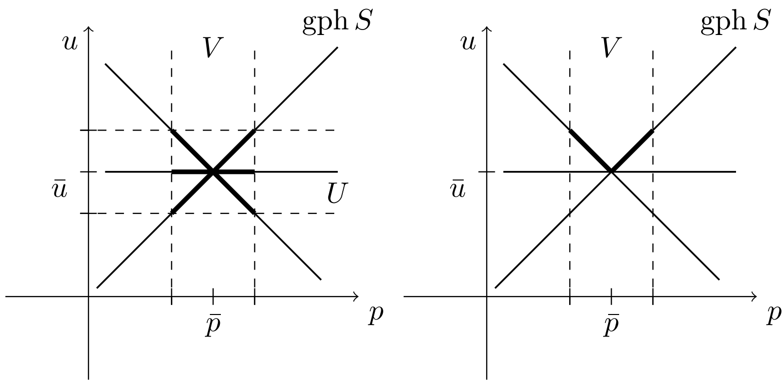

FIGURE 11. Difference between a selection and a localization.

$\mathbf{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ a graphical localization of $\mathbf{S}$ around $\bar{\mathbf{p}}$ for $\bar{\mathbf{u}}$, which is both single-valued and Lipschitz continuous, is a Lipschitz continuous local selection for $\mathbf{S}$ around $\bar{\mathbf{p}}$ for $\bar{\mathbf{u}}$. The converse is not true in general.

Example 2.9. Let $S : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined by $S(p) = \{-p, 0, p\}$, $p \in \mathbb{R}$. Then $s(p) := 0$, $p \in \mathbb{R}$, is one possible (global) selection for S around 0 for 0 which is Lipschitz continuous (on whole of $\mathbb{R}$). However, there is no localization of S around 0 for 0 being single-valued.

The equivalence holds true if $\mathbf{S} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is *locally monotone* at $(\bar{\mathbf{p}}, \bar{\mathbf{u}}) \in \operatorname{gph} \mathbf{S}$, that is, there is a neighborhood W of $(\bar{\mathbf{p}}, \bar{\mathbf{u}})$ such that

$$\langle \hat{\mathbf{u}} - \tilde{\mathbf{u}}, \hat{\mathbf{p}} - \tilde{\mathbf{p}} \rangle \geq 0 \quad \text{whenever} \quad (\hat{\mathbf{p}}, \hat{\mathbf{u}}), (\tilde{\mathbf{p}}, \tilde{\mathbf{u}}) \in \operatorname{gph} \mathbf{S} \cap W. \tag{2.10}$$

If $W = \mathbb{R}^m \times \mathbb{R}^m$, then $\mathbf{S}$ is called (globally) *monotone*. Clearly, $\mathbf{S}$ is (locally) monotone at $(\bar{\mathbf{p}}, \bar{\mathbf{u}})$ if and only if so is $\mathbf{S}^{-1}$ at $(\bar{\mathbf{u}}, \bar{\mathbf{p}})$.

Example 2.10. The local monotonicity of a continuous function $s : \mathbb{R} \to \mathbb{R}$ at $(\bar{p}, s(\bar{p}))$ means that there is $\tau > 0$ such that

$$(s(\hat{p}) - s(\tilde{p})).(\hat{p} - \tilde{p}) \geq 0 \quad \text{for each} \quad \hat{p}, \tilde{p} \in (\bar{p} - \tau, \bar{p} + \tau),$$

so s is increasing on a neighborhood of the reference point.

Example 2.11. Let $\mathbf{A} \in \mathbb{R}^{m \times m}$ be positive semi-definite and K be a closed convex subset of $\mathbb{R}^m$. Then

$$\mathbf{S}(\mathbf{p}) := \mathbf{A}\mathbf{p} + N_K(\mathbf{p}), \quad \mathbf{p} \in \mathbb{R}^m,$$

is (globally) monotone. Indeed, let $\hat{\mathbf{u}} \in \mathbf{S}(\hat{\mathbf{p}})$ and $\tilde{\mathbf{u}} \in \mathbf{S}(\tilde{\mathbf{p}})$ be arbitrary. Find $\hat{\mathbf{w}} \in N_K(\hat{\mathbf{p}})$ and $\tilde{\mathbf{w}} \in N_K(\tilde{\mathbf{p}})$ such that $\hat{\mathbf{u}} = \mathbf{A}\hat{\mathbf{p}} + \hat{\mathbf{w}}$ and $\tilde{\mathbf{u}} = \mathbf{A}\tilde{\mathbf{p}} + \tilde{\mathbf{w}}$. As both $\hat{\mathbf{p}}$ and $\tilde{\mathbf{p}}$ lie in K, the definition of the normal cone reveals that $\langle \hat{\mathbf{w}}, \tilde{\mathbf{p}} - \hat{\mathbf{p}} \rangle \leq 0$ and $\langle \tilde{\mathbf{w}}, \hat{\mathbf{p}} - \tilde{\mathbf{p}} \rangle \leq 0$. Since $\mathbf{A}$ is positive semi-definite, we have

$$\begin{aligned} \langle \hat{\mathbf{u}} - \tilde{\mathbf{u}}, \hat{\mathbf{p}} - \tilde{\mathbf{p}} \rangle &= \langle \mathbf{A}(\hat{\mathbf{p}} - \tilde{\mathbf{p}}), \hat{\mathbf{p}} - \tilde{\mathbf{p}} \rangle + \langle \hat{\mathbf{w}} - \tilde{\mathbf{w}}, \hat{\mathbf{p}} - \tilde{\mathbf{p}} \rangle \\ &= \langle \mathbf{A}(\hat{\mathbf{p}} - \tilde{\mathbf{p}}), \hat{\mathbf{p}} - \tilde{\mathbf{p}} \rangle + \langle \hat{\mathbf{w}}, \hat{\mathbf{p}} - \tilde{\mathbf{p}} \rangle + \langle \tilde{\mathbf{w}}, \tilde{\mathbf{p}} - \hat{\mathbf{p}} \rangle \geq 0. \end{aligned}$$

Theorem 2.12. *A set-valued mapping* $\mathbf{S} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, *which is locally monotone at* $(\bar{\mathbf{p}}, \bar{\mathbf{u}}) \in \operatorname{gph} \mathbf{S}$, *has a localization around* $\bar{\mathbf{p}}$ *for* $\bar{\mathbf{u}}$ *which is both single-valued and Lipschitz continuous if and only if it has a local selection around* $\bar{\mathbf{p}}$ *for* $\bar{\mathbf{u}}$ *which is Lipschitz continuous.*

Proof. We shall imitate the proof of [13, Theorem 3G.5]. Find W such that (2.10) holds. Let $\mathbf{s}$ be a local selection for $\mathbf{S}$ which is both defined and Lipschitz continuous on $\mathbb{B}(\bar{\mathbf{p}}, r)$ for some $r > 0$ such that $\mathbb{B}(\bar{\mathbf{p}}, r) \times \mathbb{B}(\bar{\mathbf{u}}, \kappa r) \subset W$, where $\kappa > 0$ is the Lipschitz constant. Fix any $\mathbf{p} \in \mathbb{B}(\bar{\mathbf{p}}, r)$. As $\mathbf{s}(\bar{\mathbf{p}}) = \bar{\mathbf{u}}$, we have $\mathbf{s}(\mathbf{p}) \in \mathbb{B}(\bar{\mathbf{u}}, \kappa r)$. Therefore, the point $\mathbf{s}(\mathbf{p})$ lies in $\mathbf{S}(\mathbf{p}) \cap \mathbb{B}(\bar{\mathbf{u}}, \kappa r)$. It suffices to show that the latter set is singleton. Suppose that this is not the case. Find $\mathbf{u} \in \mathbb{R}^m$ such that

$$\mathbf{u} \in \mathbf{S}(\mathbf{p}) \cap \mathbb{B}(\bar{\mathbf{u}}, \kappa r) \quad \text{with} \quad \mathbf{u} \neq \mathbf{s}(\mathbf{p}).$$

Let $b := \|\mathbf{u} - \mathbf{s}(\mathbf{p})\|$ and $\mathbf{c} := (\mathbf{u} - \mathbf{s}(\mathbf{p}))/b$, which means that

$$b > 0, \quad \|\mathbf{c}\| = 1, \quad \text{and} \quad \langle \mathbf{u}, \mathbf{c} \rangle = b + \langle \mathbf{s}(\mathbf{p}), \mathbf{c} \rangle. \tag{2.11}$$

Find $\tau > 0$ such that $\kappa\tau < b$ and that $\mathbf{p} + \tau\mathbf{c} \in \mathbb{B}(\bar{\mathbf{p}}, r)$. Since $\|\mathbf{c}\| = 1$, the Cauchy-Schwarz inequality and the Lipschitz continuity of $\mathbf{s}$ imply that

$$\langle \mathbf{s}(\mathbf{p} + \tau\mathbf{c}) - \mathbf{s}(\mathbf{p}), \mathbf{c} \rangle \;\le\; \|\mathbf{s}(\mathbf{p} + \tau\mathbf{c}) - \mathbf{s}(\mathbf{p})\| \, \|\mathbf{c}\| \;\le\; \kappa\tau. \tag{2.12}$$

Since $(\mathbf{p} + \tau\mathbf{c}, \mathbf{s}(\mathbf{p} + \tau\mathbf{c}))$ and $(\mathbf{p}, \mathbf{u})$ are in $\operatorname{gph} \mathbf{S} \cap W$, (2.10) reveals that

$$0 \;\le\; \langle \mathbf{s}(\mathbf{p} + \tau\mathbf{c}) - \mathbf{u}, \mathbf{p} + \tau\mathbf{c} - \mathbf{p} \rangle \;=\; \tau \langle \mathbf{s}(\mathbf{p} + \tau\mathbf{c}) - \mathbf{u}, \mathbf{c} \rangle. \tag{2.13}$$

Now, we may estimate

$$b + \langle \mathbf{s}(\mathbf{p}), \mathbf{c} \rangle \underset{(2.11)}{=} \langle \mathbf{u}, \mathbf{c} \rangle \underset{(2.13)}{\le} \langle \mathbf{s}(\mathbf{p} + \tau\mathbf{c}), \mathbf{c} \rangle \underset{(2.12)}{\le} \langle \mathbf{s}(\mathbf{p}), \mathbf{c} \rangle + \kappa\tau < \langle \mathbf{s}(\mathbf{p}), \mathbf{c} \rangle + b.$$

We arrived at a contradiction, therefore $\mathbf{S}(\mathbf{p}) \cap \mathbb{B}(\bar{\mathbf{u}}, \kappa r) = \{\mathbf{s}(\mathbf{p})\}$ for each $\mathbf{p} \in \mathbb{B}(\bar{\mathbf{p}}, r)$. The opposite direction is trivial. $\square$

We will investigate the existence of a Lipschitz continuous selection for the *solution mapping* $\mathbf{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ corresponding to (2.9) defined by

$$\mathbf{S}(\mathbf{p}) := \big\{ \mathbf{u} \in \mathbb{R}^m : \; \mathbf{0} \in \mathbf{h}(\mathbf{p}, \mathbf{u}) + \mathbf{H}(\mathbf{u}) \big\}, \qquad \mathbf{p} \in \mathbb{R}^d. \tag{2.14}$$

First, we reduce this problem to a “linear" one.

Theorem 2.13. *Given* $\mathbf{H} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ *and* $\mathbf{h} : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^m$, *let* $\mathbf{S}$ *be a solution mapping defined in* (2.14) *with* $\bar{\mathbf{u}} \in \mathbf{S}(\bar{\mathbf{p}})$. *For a given mapping* $\mathbf{l} : \mathbb{R}^m \to \mathbb{R}^m$, *define* $\mathbf{e} : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^m$ *by* $\mathbf{e}(\mathbf{p}, \mathbf{u}) = \mathbf{h}(\mathbf{p}, \mathbf{u}) - \mathbf{l}(\mathbf{u})$. *Suppose that*

(*i*) $\mathbf{e}(\bar{\mathbf{p}}, \bar{\mathbf{u}}) = \mathbf{0}$;

(*ii*) $(\mathbf{l} + \mathbf{H})^{-1}$ *has a* `selection` *around* $\mathbf{0}$ *for* $\bar{\mathbf{u}}$ *which is Lipschitz continuous with a constant* $\kappa > 0$;

(*iii*) *there are* $\mu \in (0, 1/\kappa)$, $\alpha > 0$ *and* $\tau > 0$ *such that*

$$\|\mathbf{e}(\mathbf{p}, \mathbf{u}) - \mathbf{e}(\mathbf{p}, \mathbf{v})\| \le \mu \, \|\mathbf{u} - \mathbf{v}\| \quad \textit{whenever} \quad \mathbf{u}, \mathbf{v} \in \mathbb{B}(\bar{\mathbf{u}}, \alpha), \; \mathbf{p} \in \mathbb{B}(\bar{\mathbf{p}}, \tau);$$

(*iv*) *there is* $\nu > 0$ *such that*

$$\|\mathbf{e}(\hat{\mathbf{p}}, \mathbf{u}) - \mathbf{e}(\tilde{\mathbf{p}}, \mathbf{u})\| \le \nu \, \|\hat{\mathbf{p}} - \tilde{\mathbf{p}}\| \quad \textit{whenever} \quad \hat{\mathbf{p}}, \tilde{\mathbf{p}} \in \mathbb{B}(\bar{\mathbf{p}}, \tau), \; \mathbf{u} \in \mathbb{B}(\bar{\mathbf{u}}, \alpha).$$

Then $\mathbf{S}$ *has a* `selection` *around* $\bar{\mathbf{p}}$ *for* $\bar{\mathbf{u}}$ *which is Lipschitz continuous.*

Proof. We shall imitate the proof of [13, Theorem 5F.4]. Use (ii) to find $\alpha_1 > 0$ and a function $\tilde{\mathbf{s}}$ which is Lipschitz continuous on $\mathbb{B}(\mathbf{0}, \alpha_1)$ with the constant κ and such that

$$\tilde{\mathbf{s}}(\mathbf{0}) = \bar{\mathbf{u}} \quad \text{and} \quad \tilde{\mathbf{s}}(\mathbf{y}) \in (\mathbf{l} + \mathbf{H})^{-1}(\mathbf{y}) \quad \text{for each } \mathbf{y} \in \mathbb{B}(\mathbf{0}, \alpha_1). \tag{2.15}$$

Fix $a > 0$ such that

$$a < \min\{\alpha, \kappa\alpha_1\}. \tag{2.16}$$

By (iv), the mapping $\mathbf{h}(\cdot, \bar{\mathbf{u}})$ is continuous at $\bar{\mathbf{p}}$. As $\mu\kappa < 1$, we can make τ smaller, if necessary, to have

$$\|\mathbf{h}(\mathbf{p}, \bar{\mathbf{u}}) - \mathbf{h}(\bar{\mathbf{p}}, \bar{\mathbf{u}})\| < a(1 - \mu\kappa)/\kappa \qquad \text{for each } \mathbf{p} \in \mathbb{B}(\bar{\mathbf{p}}, \tau). \tag{2.17}$$

We `claim` that $\|\mathbf{e}(\mathbf{p}, \mathbf{u})\| < a/\kappa$ for any $\mathbf{u} \in \mathbb{B}[\bar{\mathbf{u}}, a]$ and any $\mathbf{p} \in \mathbb{B}(\bar{\mathbf{p}}, \tau)$. Indeed, fix any such $\mathbf{u}$ and $\mathbf{p}$. Note that $\mathbb{B}[\bar{\mathbf{u}}, a] \subset \mathbb{B}(\bar{\mathbf{u}}, \alpha)$ by (2.16). Thus

$$\begin{aligned}
\|\mathbf{e}(\mathbf{p}, \mathbf{u})\| &\underset{(i)}{=} \|\mathbf{e}(\mathbf{p}, \mathbf{u}) - \mathbf{e}(\mathbf{p}, \bar{\mathbf{u}}) + \big(\mathbf{e}(\mathbf{p}, \bar{\mathbf{u}}) - \mathbf{e}(\bar{\mathbf{p}}, \bar{\mathbf{u}})\big)\| \\
&\underset{(iii)}{\leq} \mu\|\mathbf{u} - \bar{\mathbf{u}}\| + \|\mathbf{h}(\mathbf{p}, \bar{\mathbf{u}}) - \mathbf{l}(\bar{\mathbf{u}}) - \big(\mathbf{h}(\bar{\mathbf{p}}, \bar{\mathbf{u}}) - \mathbf{l}(\bar{\mathbf{u}})\big)\| \\
&\underset{(2.17)}{<} \mu a + \frac{a(1 - \mu\kappa)}{\kappa} = \frac{a}{\kappa}.
\end{aligned}$$

The claim is proved. Now, fix $\mathbf{p} \in \mathbb{B}(\bar{\mathbf{p}}, \tau)$ and consider a mapping

$$\Phi_{\mathbf{p}} : \mathbb{B}[\bar{\mathbf{u}}, a] \ni \mathbf{u} \longmapsto \tilde{\mathbf{s}}(-\mathbf{e}(\mathbf{p}, \mathbf{u})).$$

Pick any $\mathbf{u} \in \mathbb{B}[\bar{\mathbf{u}}, a]$. Then $\Phi_{\mathbf{p}}(\mathbf{u})$ is well-defined, because the claim together with (2.16) implies that $-\mathbf{e}(\mathbf{p}, \mathbf{u}) \in \mathbb{B}(\mathbf{0}, \alpha_1)$. Using the claim again, one gets

$$\|\bar{\mathbf{u}} - \Phi_{\mathbf{p}}(\mathbf{u})\| \underset{(2.15)}{=} \|\tilde{\mathbf{s}}(\mathbf{0}) - \tilde{\mathbf{s}}(-\mathbf{e}(\mathbf{p}, \mathbf{u}))\| \leq \kappa\|\mathbf{e}(\mathbf{p}, \mathbf{u})\| < a.$$

Therefore $\Phi_{\mathbf{p}}$ maps $\mathbb{B}[\bar{\mathbf{u}}, a]$ into itself.
Finally, pick any $\mathbf{u}, \mathbf{v} \in \mathbb{B}[\bar{\mathbf{u}}, a]$. Then $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{B}(\bar{\mathbf{u}}, \alpha)$ thanks to (2.16). Moreover, the claim implies that both $-\mathbf{e}(\mathbf{p}, \mathbf{u})$ and $-\mathbf{e}(\mathbf{p}, \mathbf{v})$ lie in $\mathbb{B}(\mathbf{0}, \alpha_1)$, hence we get that

$$\begin{aligned}
\|\Phi_{\mathbf{p}}(\mathbf{u}) - \Phi_{\mathbf{p}}(\mathbf{v})\| &= \|\tilde{\mathbf{s}}(-\mathbf{e}(\mathbf{p}, \mathbf{u})) - \tilde{\mathbf{s}}(-\mathbf{e}(\mathbf{p}, \mathbf{v}))\| \leq \kappa\,\|\mathbf{e}(\mathbf{p}, \mathbf{u}) - \mathbf{e}(\mathbf{p}, \mathbf{v})\| \\
&\underset{(iii)}{\leq} \mu\kappa\,\|\mathbf{u} - \mathbf{v}\|.
\end{aligned}$$

This reveals that $\Phi_{\mathbf{p}}$ is a contraction from $\mathbb{B}[\bar{\mathbf{u}}, a]$ into itself, so it has a unique fixed point.
For any $\mathbf{p} \in \mathbb{B}(\bar{\mathbf{p}}, \tau)$ denote by $\mathbf{s}(\mathbf{p})$ the (unique) point in $\mathbb{B}[\bar{\mathbf{u}}, a]$ such that $\mathbf{s}(\mathbf{p}) = \Phi_{\mathbf{p}}(\mathbf{s}(\mathbf{p}))$. Since

$$\Phi_{\bar{\mathbf{p}}}(\bar{\mathbf{u}}) = \tilde{\mathbf{s}}(-\mathbf{e}(\bar{\mathbf{p}}, \bar{\mathbf{u}})) \overset{(i)}{=} \tilde{\mathbf{s}}(\mathbf{0}) \overset{(2.15)}{=} \bar{\mathbf{u}},$$

the uniqueness of the fixed point implies that $\mathbf{s}(\bar{\mathbf{p}}) = \bar{\mathbf{u}}$. Also, note that

$$\begin{aligned}
\mathbf{s}(\mathbf{p}) = \Phi_{\mathbf{p}}(\mathbf{s}(\mathbf{p})) &\underset{(2.15)}{\Longrightarrow} \mathbf{l}(\mathbf{s}(\mathbf{p})) + \mathbf{H}(\mathbf{s}(\mathbf{p})) \ni -\mathbf{e}(\mathbf{p}, \mathbf{s}(\mathbf{p})) = \mathbf{l}(\mathbf{s}(\mathbf{p})) - \mathbf{h}(\mathbf{p}, \mathbf{s}(\mathbf{p})) \\
&\Longleftrightarrow \mathbf{0} \in \mathbf{h}(\mathbf{p}, \mathbf{s}(\mathbf{p})) + \mathbf{H}(\mathbf{s}(\mathbf{p})) \Longleftrightarrow \mathbf{s}(\mathbf{p}) \in \mathbf{S}(\mathbf{p}).
\end{aligned}$$

Therefore $\mathbf{s}$ is a local selection for $\mathbf{S}$ around $\bar{\mathbf{p}}$ for $\bar{\mathbf{u}}$. To prove the Lipschitz continuity, fix any $\hat{\mathbf{p}}$, $\tilde{\mathbf{p}} \in \mathbb{B}(\bar{\mathbf{p}}, \tau)$. Note that $\mathbf{s}(\hat{\mathbf{p}})$ and $\mathbf{s}(\tilde{\mathbf{p}})$ lie in $\mathbb{B}(\bar{\mathbf{u}}, \alpha)$ because of (2.16). The claim combined with (2.16), implies that $\mathbf{e}(\hat{\mathbf{p}}, \mathbf{s}(\hat{\mathbf{p}}))$, $\mathbf{e}(\tilde{\mathbf{p}}, \mathbf{s}(\tilde{\mathbf{p}}))$, and $\mathbf{e}(\tilde{\mathbf{p}}, \mathbf{s}(\hat{\mathbf{p}}))$ are in $\mathbb{B}(\mathbf{0}, \alpha_1)$. Thus

$$\begin{aligned}
\|\mathbf{s}(\hat{\mathbf{p}}) - \mathbf{s}(\tilde{\mathbf{p}})\| &= \|\tilde{\mathbf{s}}(-\mathbf{e}(\hat{\mathbf{p}}, \mathbf{s}(\hat{\mathbf{p}}))) - \tilde{\mathbf{s}}(-\mathbf{e}(\tilde{\mathbf{p}}, \mathbf{s}(\hat{\mathbf{p}})))\\
&\quad + \tilde{\mathbf{s}}(-\mathbf{e}(\tilde{\mathbf{p}}, \mathbf{s}(\hat{\mathbf{p}}))) - \tilde{\mathbf{s}}(-\mathbf{e}(\tilde{\mathbf{p}}, \mathbf{s}(\tilde{\mathbf{p}})))\|\\
&\leq \kappa \|\mathbf{e}(\hat{\mathbf{p}}, \mathbf{s}(\hat{\mathbf{p}})) - \mathbf{e}(\tilde{\mathbf{p}}, \mathbf{s}(\hat{\mathbf{p}}))\| + \kappa \|\mathbf{e}(\tilde{\mathbf{p}}, \mathbf{s}(\hat{\mathbf{p}})) - \mathbf{e}(\tilde{\mathbf{p}}, \mathbf{s}(\tilde{\mathbf{p}}))\|\\
&\underset{(iii)}{\leq} \kappa \|\mathbf{e}(\hat{\mathbf{p}}, \mathbf{s}(\hat{\mathbf{p}})) - \mathbf{e}(\tilde{\mathbf{p}}, \mathbf{s}(\hat{\mathbf{p}}))\| + \mu\kappa \|\mathbf{s}(\hat{\mathbf{p}}) - \mathbf{s}(\tilde{\mathbf{p}})\|\\
&\underset{(iv)}{\leq} \kappa\nu \|\hat{\mathbf{p}} - \tilde{\mathbf{p}}\| + \mu\kappa \|\mathbf{s}(\hat{\mathbf{p}}) - \mathbf{s}(\tilde{\mathbf{p}})\|.
\end{aligned}$$

So $\mathbf{s}$ is Lipschitz continuous on $\mathbb{B}(\bar{\mathbf{p}}, \tau)$ with the constant $\kappa\nu/(1 - \kappa\mu)$. □

Example 2.14. Suppose that $\mathbf{h} : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable at $(\bar{\mathbf{p}}, \bar{\mathbf{u}}) \in \operatorname{gph} \mathbf{S}$. Let

$$\mathbf{l}(\mathbf{u}) := \mathbf{h}(\bar{\mathbf{p}}, \bar{\mathbf{u}}) + \nabla_{\mathbf{u}} \mathbf{h}(\bar{\mathbf{p}}, \bar{\mathbf{u}})(\mathbf{u} - \bar{\mathbf{u}}), \quad \mathbf{u} \in \mathbb{R}^m.$$

Then the conditions (i), (iii) and (iv) in Theorem 2.13 are satisfied. The first one is trivial. To show the rest, let us prove a (stronger) `claim` that for any $\mu > 0$ there is $\delta > 0$ such that for each $\mathbf{u}, \mathbf{v} \in \mathbb{B}(\bar{\mathbf{u}}, \delta)$ and each $\mathbf{p} \in \mathbb{B}(\bar{\mathbf{p}}, \delta)$ we have

$$\|\mathbf{h}(\mathbf{p}, \mathbf{u}) - \mathbf{h}(\mathbf{p}, \mathbf{v}) - \nabla_{\mathbf{u}} \mathbf{h}(\bar{\mathbf{p}}, \bar{\mathbf{u}})(\mathbf{u} - \mathbf{v})\| \leq \mu \|\mathbf{u} - \mathbf{v}\|. \tag{2.18}$$

Fix any $\mu > 0$ and then find $\delta > 0$ such that

$$\|\nabla_{\mathbf{u}} \mathbf{h}(\mathbf{p}, \mathbf{w}) - \nabla_{\mathbf{u}} \mathbf{h}(\bar{\mathbf{p}}, \bar{\mathbf{u}})\| \leq \mu \quad \text{for each} \quad (\mathbf{p}, \mathbf{w}) \in \mathbb{B}(\bar{\mathbf{p}}, \delta) \times \mathbb{B}(\bar{\mathbf{u}}, \delta).$$

Pick any $\mathbf{u}, \mathbf{v} \in \mathbb{B}(\bar{\mathbf{u}}, \delta)$ and any $\mathbf{p} \in \mathbb{B}(\bar{\mathbf{p}}, \delta)$. Put

$$\mathbf{c} = \mathbf{h}(\mathbf{p}, \mathbf{u}) - \mathbf{h}(\mathbf{p}, \mathbf{v}) - \nabla_{\mathbf{u}} \mathbf{h}(\bar{\mathbf{p}}, \bar{\mathbf{u}})(\mathbf{u} - \mathbf{v}).$$

If $\mathbf{c} = \mathbf{0}$ then there is nothing to prove. Assume that $\|\mathbf{c}\| > 0$. Consider the function $\varphi(t) := \langle \mathbf{c}, \mathbf{h}(\mathbf{p}, t\mathbf{u} + (1 - t)\mathbf{v}) \rangle$, $t \in [0, 1]$. Then $\varphi(0) = \langle \mathbf{c}, \mathbf{h}(\mathbf{p}, \mathbf{v}) \rangle$, $\varphi(1) = \langle \mathbf{c}, \mathbf{h}(\mathbf{p}, \mathbf{u}) \rangle$, and

$$\varphi'(t) = \langle \mathbf{c}, \nabla_{\mathbf{u}} \mathbf{h}(\mathbf{p}, t\mathbf{u} + (1 - t)\mathbf{v})(\mathbf{u} - \mathbf{v}) \rangle \quad \text{for each} \quad t \in (0, 1).$$

By the Mean Value Theorem, there is $\tau \in (0, 1)$ such that $\varphi(1) - \varphi(0) = \varphi'(\tau)$, in other words

$$\langle \mathbf{c}, \mathbf{h}(\mathbf{p}, \mathbf{u}) - \mathbf{h}(\mathbf{p}, \mathbf{v}) - \nabla_{\mathbf{u}} \mathbf{h}(\mathbf{p}, \tau\mathbf{u} + (1 - \tau)\mathbf{v})(\mathbf{u} - \mathbf{v}) \rangle = 0.$$

Set $\mathbf{w} = \tau\mathbf{u} + (1 - \tau)\mathbf{v}$. Then $\mathbf{w} \in \mathbb{B}(\bar{\mathbf{u}}, \delta)$. Taking into account the definition of $\mathbf{c}$, we get

$$\begin{aligned}
0 &< \|\mathbf{c}\|^2 = \langle \mathbf{c}, \mathbf{c} \rangle = \langle \mathbf{c}, \mathbf{h}(\mathbf{p}, \mathbf{u}) - \mathbf{h}(\mathbf{p}, \mathbf{v}) - \nabla_{\mathbf{u}} \mathbf{h}(\mathbf{p}, \mathbf{w})(\mathbf{u} - \mathbf{v}) \rangle\\
&\quad + \langle \mathbf{c}, (\nabla_{\mathbf{u}} \mathbf{h}(\mathbf{p}, \mathbf{w}) - \nabla_{\mathbf{u}} \mathbf{h}(\bar{\mathbf{p}}, \bar{\mathbf{u}}))(\mathbf{u} - \mathbf{v}) \rangle\\
&= \langle \mathbf{c}, (\nabla_{\mathbf{u}} \mathbf{h}(\mathbf{p}, \mathbf{w}) - \nabla_{\mathbf{u}} \mathbf{h}(\bar{\mathbf{p}}, \bar{\mathbf{u}}))(\mathbf{u} - \mathbf{v}) \rangle \leq \mu \|\mathbf{c}\| \, \|\mathbf{u} - \mathbf{v}\|.
\end{aligned}$$

Dividing this inequality by $\|\mathbf{c}\|$, we arrive at (2.18).

To end the proof, fix $\nu > \|\nabla_{\mathbf{p}}\mathbf{h}(\bar{\mathbf{p}}, \bar{\mathbf{u}})\| =: \beta$. As in the above claim, there is $r > 0$ such that for each $\hat{\mathbf{p}}, \tilde{\mathbf{p}} \in \mathbb{B}(\bar{\mathbf{p}}, r)$ and each $\mathbf{u} \in \mathbb{B}(\bar{\mathbf{u}}, r)$ we have

$$\|\mathbf{h}(\hat{\mathbf{p}}, \mathbf{u}) - \mathbf{h}(\tilde{\mathbf{p}}, \mathbf{u}) - \nabla_{\mathbf{p}}\mathbf{h}(\bar{\mathbf{p}}, \bar{\mathbf{u}})(\hat{\mathbf{p}} - \tilde{\mathbf{p}})\| \leq \big(\nu - \beta\big)\|\hat{\mathbf{p}} - \tilde{\mathbf{p}}\|.$$

Hence for such points, the triangle inequality yields that

$$\|\mathbf{h}(\hat{\mathbf{p}}, \mathbf{u}) - \mathbf{h}(\tilde{\mathbf{p}}, \mathbf{u})\| \quad \leq \quad \big(\nu - \beta\big)\, \|\hat{\mathbf{p}} - \tilde{\mathbf{p}}\| + \beta\|\hat{\mathbf{p}} - \tilde{\mathbf{p}}\| = \nu\, \|\hat{\mathbf{p}} - \tilde{\mathbf{p}}\|.$$

Then the conditions *(iii)* and *(iv)* in Theorem 2.13 are satisfied with $\alpha := \min\{\delta, r\}$ and $\tau := \alpha$.

A mapping satisfying (2.18) is called *strictly differentiable* with respect to the second variable uniformly in the first one at the reference point. Inspecting the proof, one sees that Theorem 2.13 is valid when the word `selection` is replaced by `localization`. This is [13, Theorem 5F.4] extending a pioneering work by S. M. Robinson [29], where $\mathbf{H}$ is the normal cone mapping to a closed convex set K. The whole problem boils down to a problem how to check that the solution mapping of a linear problem has a locally Lipschitz selection [localization] around the reference point. Let us investigate in detail the case when $K := \mathbb{R}^m_+$. We have already touched this issue in Section 2.2, but here we want to profit from the local properties only.

As before, consider an autonomous DVI:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \\ 0 &\leq \langle \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)), \mathbf{v} - \mathbf{u}(t)\rangle \quad \text{whenever} \quad \mathbf{v} \in \mathbb{R}^m_+, \\ \mathbf{u}(t) &\in \mathbb{R}^m_+, \quad t \in [a, b], \end{aligned}$$

where $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable. Let us prescribe initial conditions $\mathbf{x}(a) = \bar{\mathbf{x}}$ and $\mathbf{u}(a) = \bar{\mathbf{u}}$ with $-\mathbf{g}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) \in N_{\mathbb{R}^m_+}(\bar{\mathbf{u}})$. We want to show that there is a neighborhood $\Omega \times U$ of $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ along with a Lipschitz continuous function $\mathbf{s} : \Omega \ni \mathbf{x} \mapsto \mathbf{s}(\mathbf{x}) \in SOL\big(\mathbb{R}^m_+, \mathbf{g}(\mathbf{x}, \cdot)\big) \cap U$. In view of Theorem 2.13, it suffices to show that the inverse of the mapping

$$\mathbf{G} : \mathbf{u} \mapsto \mathbf{g}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) + \nabla_{\mathbf{u}}\mathbf{g}(\bar{\mathbf{x}}, \bar{\mathbf{u}})(\mathbf{u} - \bar{\mathbf{u}}) + N_{\mathbb{R}^m_+}(\mathbf{u})$$

has a Lipschitz continuous selection [localization] around $\mathbf{0}$ for $\bar{\mathbf{u}}$. Note that $(\bar{\mathbf{u}}, \mathbf{0}) \in \operatorname{gph} \mathbf{G}$, since

$$\mathbf{G}(\bar{\mathbf{u}}) = \mathbf{g}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) + N_{\mathbb{R}^m_+}(\bar{\mathbf{u}}) \ni \mathbf{g}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) + (-\mathbf{g}(\bar{\mathbf{x}}, \bar{\mathbf{u}})) = \mathbf{0}.$$

Let $\mathbf{A} := \nabla_{\mathbf{u}}\mathbf{g}(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ and $\mathbf{a} := \mathbf{g}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) - \nabla_{\mathbf{u}}\mathbf{g}(\bar{\mathbf{x}}, \bar{\mathbf{u}})\bar{\mathbf{u}}$. We have to consider a "linear" generalized equation:

$$\text{Given } \mathbf{y} \text{ close to } \mathbf{0} \text{ find } \mathbf{u} \text{ close to } \bar{\mathbf{u}} \text{ such that} \quad \mathbf{y} \in \mathbf{A}\mathbf{u} + \mathbf{a} + N_{\mathbb{R}^m_+}(\mathbf{u}).$$

Reorder the indices, if necessary, to find $j,\, s \in \mathbb{N}$ with $j + s \leq m$ such that

$$\begin{aligned} (\mathbf{A}\bar{\mathbf{u}} + \mathbf{a})_i &= 0 \quad \text{and} \quad \bar{u}_i > 0 \quad \text{for} \quad i = 1, \ldots, j, \\ (\mathbf{A}\bar{\mathbf{u}} + \mathbf{a})_i &= 0 \quad \text{and} \quad \bar{u}_i = 0 \quad \text{for} \quad i = j+1, \ldots, j+s, \\ (\mathbf{A}\bar{\mathbf{u}} + \mathbf{a})_i &> 0 \quad \text{and} \quad \bar{u}_i = 0 \quad \text{for} \quad i = j+s+1, \ldots, m. \end{aligned}$$

Partition $\mathbf{u} \in \mathbb{R}^m$ as $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ with $\mathbf{u}_1 \in \mathbb{R}^j, \mathbf{u}_2 \in \mathbb{R}^s, \mathbf{u}_3 \in \mathbb{R}^{m-j-s}$ and partition $\mathbf{A}$, in the same way, as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} \end{pmatrix}.$$

By Proposition 1.7, we have

$$N_{\mathbb{R}^m_+}(\mathbf{u}) = \prod_{i=1}^{m} N_{\mathbb{R}_+}(u_i) = N_{\mathbb{R}^j_+}(\mathbf{u}_1) \times N_{\mathbb{R}^s_+}(\mathbf{u}_2) \times N_{\mathbb{R}^{m-j-s}_+}(\mathbf{u}_3).$$

Therefore the above generalized equation falls into three pieces

$$\begin{aligned} \mathbf{y}_1 &\in \mathbf{A}_{11}\mathbf{u}_1 + \mathbf{A}_{12}\mathbf{u}_2 + \mathbf{A}_{13}\mathbf{u}_3 + \mathbf{a}_1 + N_{\mathbb{R}^j_+}(\mathbf{u}_1), \\ \mathbf{y}_2 &\in \mathbf{A}_{21}\mathbf{u}_1 + \mathbf{A}_{22}\mathbf{u}_2 + \mathbf{A}_{23}\mathbf{u}_3 + \mathbf{a}_2 + N_{\mathbb{R}^s_+}(\mathbf{u}_2), \\ \mathbf{y}_3 &\in \mathbf{A}_{31}\mathbf{u}_1 + \mathbf{A}_{32}\mathbf{u}_2 + \mathbf{A}_{33}\mathbf{u}_3 + \mathbf{a}_3 + N_{\mathbb{R}^{m-j-s}_+}(\mathbf{u}_3). \end{aligned}$$

Fix any $i \in \{j+s+1, \ldots, m\}$. As $(\mathbf{A}\bar{\mathbf{u}}+\mathbf{a})_i$ is positive, so is $(\mathbf{A}\mathbf{u}+\mathbf{a}-\mathbf{y})_i$ provided that $(\mathbf{u}, \mathbf{y})$ is sufficiently close to $(\bar{\mathbf{u}}, \mathbf{0})$. Therefore $u_i = 0$. So $\mathbf{u}_3 = \mathbf{0}_{m-j-s}$ and the last inclusion holds trivially. Similarly, fixing arbitrary $i \in \{1, \ldots, j\}$ and using that $\bar{u}_i > 0$, we get $u_i > 0$. This means that $N_{\mathbb{R}^j_+}(\mathbf{u}_1) = \{\mathbf{0}_j\}$. Therefore the above system reduces to

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{A}_{11}\mathbf{u}_1 + \mathbf{A}_{12}\mathbf{u}_2 + \mathbf{a}_1 \\ \mathbf{y}_2 &\in \mathbf{A}_{21}\mathbf{u}_1 + \mathbf{A}_{22}\mathbf{u}_2 + \mathbf{a}_2 + N_{\mathbb{R}^s_+}(\mathbf{u}_2). \end{aligned}$$

Since both $\bar{\mathbf{u}}_2$ and $\bar{\mathbf{u}}_3$ are zero vectors and $(\mathbf{A}\bar{\mathbf{u}}+\mathbf{a})_i = 0$ for each $i \in \{1, \ldots, j+s\}$, one has

$$\mathbf{A}\bar{\mathbf{u}} + \mathbf{a} = \begin{pmatrix} \mathbf{0}_j \\ \mathbf{0}_s \\ \mathbf{A}_{31}\bar{\mathbf{u}}_1 + \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{11} \\ \mathbf{A}_{21} \\ \mathbf{A}_{31} \end{pmatrix} \bar{\mathbf{u}}_1 + \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix}.$$

Using the first two rows and $\bar{\mathbf{u}}_2 = \mathbf{0}_s$, we infer that $\mathbf{a}_1 = -\mathbf{A}_{11}\bar{\mathbf{u}}_1 - \mathbf{A}_{12}\bar{\mathbf{u}}_2$ and $\mathbf{a}_2 = -\mathbf{A}_{21}\bar{\mathbf{u}}_1 - \mathbf{A}_{22}\bar{\mathbf{u}}_2$. Plugging this into the reduced system of generalized equations and setting $\mathbf{w} = (\mathbf{u}_1 - \bar{\mathbf{u}}_1, \mathbf{u}_2 - \bar{\mathbf{u}}_2)^T$, we arrive at

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{A}_{11}\mathbf{w}_1 + \mathbf{A}_{12}\mathbf{w}_2 \\ \mathbf{y}_2 &\in \mathbf{A}_{21}\mathbf{w}_1 + \mathbf{A}_{22}\mathbf{w}_2 + N_{\mathbb{R}^s_+}(\mathbf{w}_2). \end{aligned}$$

From now on, assume that $\mathbf{A}_{11}$ is non-singular and denote its Schur's complement in $\mathbf{A}$ by

$$\mathbf{B} := \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \in \mathbb{R}^{s \times s}.$$

Then solving the first equation, one gets

$$\mathbf{y}_2 \in \mathbf{A}_{21}\mathbf{A}_{11}^{-1}(\mathbf{y}_1 - \mathbf{A}_{12}\mathbf{w}_2) + \mathbf{A}_{22}\mathbf{w}_2 + N_{\mathbb{R}^s_+}(\mathbf{w}_2).$$

For $\mathbf{z} := \mathbf{y}_2 - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{y}_1 \in \mathbb{R}^s$, we obtain a reduced problem in its final form

$$\mathbf{z} \in \mathbf{B}\mathbf{w}_2 + N_{\mathbb{R}^s_+}(\mathbf{w}_2).$$

Note that if the inverse of the mapping

$$\mathbf{G}_2 : \mathbf{w}_2 \mapsto \mathbf{B}\mathbf{w}_2 + N_{\mathbb{R}^s_+}(\mathbf{w}_2)$$

has a Lipschitz continuous selection [localization] $\mathbf{s}_2$ around $\mathbf{0}_s$ for $\mathbf{0}_s$ (with a constant $L_2 > 0$ and a neighborhood V_2, say) then so does the original mapping $\mathbf{G}$ around $\mathbf{0}$ for $\bar{\mathbf{u}}$. Indeed, choose a neighborhood V of $\mathbf{0}$ such that $\mathbf{y}_2 - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{y}_1 \in V_2$ for any $\mathbf{y} \in V$. Fix any $\hat{\mathbf{y}}$, $\tilde{\mathbf{y}} \in V$. Then both $\hat{\mathbf{z}} = \hat{\mathbf{y}}_2 - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\hat{\mathbf{y}}_1$ and $\tilde{\mathbf{z}} = \tilde{\mathbf{y}}_2 - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\tilde{\mathbf{y}}_1$ are in V_2, hence let $\hat{\mathbf{w}}_2 := \mathbf{s}_2(\hat{\mathbf{z}})$ and $\tilde{\mathbf{w}}_2 := \mathbf{s}_2(\tilde{\mathbf{z}})$, and then $\hat{\mathbf{w}}_1 := \mathbf{A}_{11}^{-1}(\hat{\mathbf{y}}_1 - \mathbf{A}_{12}\hat{\mathbf{w}}_2)$ and $\tilde{\mathbf{w}}_1 := \mathbf{A}_{11}^{-1}(\tilde{\mathbf{y}}_1 - \mathbf{A}_{12}\tilde{\mathbf{w}}_2)$. Thus

$$\hat{\mathbf{u}} := (\hat{\mathbf{w}}_1 + \bar{\mathbf{u}}_1, \hat{\mathbf{w}}_2 + \bar{\mathbf{u}}_2, \mathbf{0}_{m-j-s}) \in \mathbf{G}^{-1}(\hat{\mathbf{y}})$$

and

$$\tilde{\mathbf{u}} := (\tilde{\mathbf{w}}_1 + \bar{\mathbf{u}}_1, \tilde{\mathbf{w}}_2 + \bar{\mathbf{u}}_2, \mathbf{0}_{m-j-s}) \in \mathbf{G}^{-1}(\tilde{\mathbf{y}}).$$

Moreover, for $K := L_2\big(1 + \|\mathbf{A}_{11}^{-1}\|\|\mathbf{A}_{12}\|\big)\|$ one gets that

$$\begin{aligned}
\|\hat{\mathbf{u}} - \tilde{\mathbf{u}}\| &\leq \|\hat{\mathbf{w}}_1 - \tilde{\mathbf{w}}_1\| + \|\hat{\mathbf{w}}_2 - \tilde{\mathbf{w}}_2\| \leq \|\hat{\mathbf{w}}_1 - \tilde{\mathbf{w}}_1\| + L_2\|\hat{\mathbf{z}} - \tilde{\mathbf{z}}\| \\
&\leq \|\mathbf{A}_{11}^{-1}\|\big(\|\hat{\mathbf{y}}_1 - \tilde{\mathbf{y}}_1\| + \|\mathbf{A}_{12}\|\|\hat{\mathbf{w}}_2 - \tilde{\mathbf{w}}_2\|\big) + L_2\|\hat{\mathbf{z}} - \tilde{\mathbf{z}}\| \\
&= \|\mathbf{A}_{11}^{-1}\|\|\hat{\mathbf{y}}_1 - \tilde{\mathbf{y}}_1\| + L_2\big(1 + \|\mathbf{A}_{11}^{-1}\|\|\mathbf{A}_{12}\|\big)\|\hat{\mathbf{z}} - \tilde{\mathbf{z}}\| \\
&\leq \big(\|\mathbf{A}_{11}^{-1}\| + K\|\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\|\big)\|\hat{\mathbf{y}}_1 - \tilde{\mathbf{y}}_1\| + K\|\hat{\mathbf{y}}_2 - \tilde{\mathbf{y}}_2\|.
\end{aligned}$$

We have already seen in Example 2.6 that a sufficient condition for the existence of $\mathbf{s}_2$ is that

$$\langle \mathbf{B}\mathbf{h}, \mathbf{h} \rangle > 0 \quad \text{for each non-zero} \quad \mathbf{h} \in \mathbb{R}^s.$$

In fact, this means that $\mathbf{G}_2^{-1}$ is a globally Lipschitz continuous function.

Let us discuss a weaker notion. Recall that a matrix $\mathbf{B} \in \mathbb{R}^{s\times s}$ is called the *P-matrix* provided that, for all $k \in \{1, \dots, s\}$, any k-by-k principal minor (the determinant of the matrix obtained by deleting $s-k$ rows and the $s-k$ columns with the same numbers) is positive. It is well-known, that $\mathbf{B}$ is a P-matrix if and only if for any non-zero $\mathbf{h} \in \mathbb{R}^s$ there is $j \in \{1, \dots, s\}$ such that $h_j(\mathbf{B}\mathbf{h})_j > 0$. It is obvious that any positive definite matrix is a P-matrix. On the other hand, a symmetric P-matrix is positive definite.

The following statement is well-known in the theory of linear complementarity problems.

Proposition 2.15. *A matrix $\mathbf{B} \in \mathbb{R}^{s\times s}$ is a P-matrix if and only if for each $\mathbf{q} \in \mathbb{R}^s$ there is a unique $\mathbf{u}(\mathbf{q}) \in \mathbb{R}^s$ such that*

$$\mathbf{0} \preceq \mathbf{u}(\mathbf{q}) \perp \mathbf{B}\mathbf{u}(\mathbf{q}) + \mathbf{q} \succeq \mathbf{0}.$$

Moreover, the mapping $\mathbf{s} : \mathbb{R}^s \ni \mathbf{q} \mapsto \mathbf{u}(\mathbf{q}) \in \mathbb{R}^s$ is piece-wise linear, i.e. it is continuous on whole of $\mathbb{R}^s$ and there is a finite set of linear mappings $\{\mathbf{l}_1, \dots, \mathbf{l}_k\}$ acting from $\mathbb{R}^s$ into itself such that

$$\mathbf{s}(\mathbf{x}) \in \{\mathbf{l}_1(\mathbf{x}), \dots, \mathbf{l}_k(\mathbf{x})\} \quad \textit{for each} \quad \mathbf{x} \in \mathbb{R}^s.$$

Clearly, a mapping $\mathbf{s}$ in the above proposition is Lipschitz continuous globally. To sum up, we are able to reduce (uniquely) the autonomous DVI to an ODE around the reference point (determined by the initial conditions) provided that $\mathbf{A}_{11}$ is non-singular and $\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$ is a P-matrix. It can be shown that these conditions are also necessary for $\mathbf{G}_2^{-1}$ to be a Lipschitz single-valued mapping defined on all of $\mathbb{R}^s$ [29, Theorem 3.1]. Moreover, [13, Theorem 2E.6] shows that the same consideration goes through when K is any polyhedron in $\mathbb{R}^m$.

Defining a suitable "derivative" of a set-valued mapping, one obtains conditions for checking that $(\mathbf{l}+\mathbf{H})^{-1}$ in Theorem 2.13 has a localization around $\mathbf{0}$ for $\bar{\mathbf{u}}$ which is Lipschitz continuous [13, Theorem 4D.1].

Robinson's theorem [29] mentioned above can be stated as follows:

Given $\mathbf{H} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ and a continuously differentiable $\mathbf{h} : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^m$, let $\mathbf{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ be defined by

$$\mathbf{S}(\mathbf{p}) := \big\{ \mathbf{u} \in \mathbb{R}^m : \ \mathbf{0} \in \mathbf{h}(\mathbf{p}, \mathbf{u}) + \mathbf{H}(\mathbf{u}) \big\}, \qquad \mathbf{p} \in \mathbb{R}^d,$$

and let $\bar{\mathbf{u}} \in \mathbf{S}(\bar{\mathbf{p}})$. Suppose that the inverse of the mapping

$$\mathbf{h}(\bar{\mathbf{p}}, \bar{\mathbf{u}}) + \nabla_{\mathbf{u}} \mathbf{h}(\bar{\mathbf{p}}, \bar{\mathbf{u}})(\cdot - \bar{\mathbf{u}}) + \mathbf{H}$$

has a localization around $\mathbf{0}$ for $\bar{\mathbf{u}}$ which is Lipschitz continuous. Then $\mathbf{S}$ has a localization around $\bar{\mathbf{p}}$ for $\bar{\mathbf{u}}$ which is Lipschitz continuous.

The same was obtained for a non-smooth mapping $\mathbf{h}$ by A. F. Izmailov [18] in finite dimension and in [9] in general Banach spaces. For simplicity, let us state a non-parametric version.

Theorem 2.16. *Let X and Y be Banach spaces, let $h : X \to Y$ be continuous at $\bar{x} \in X$, let $\bar{y} \in Y$, and let $H : X \rightrightarrows Y$. Suppose that there is a compact convex $\mathcal{A} \subset \mathcal{L}(X, Y)$ and $c > 0$ such that*

(i) *there is $r > 0$ such that for each u, $v \in \mathbb{B}(\bar{x}, r)$ one can find $A \in \mathcal{A}$ such that*
$$\|h(v) - h(u) - A(v - u)\| \leq c\|v - u\|;$$
(ii) *for every $A \in \mathcal{A}$, the inverse of the mapping*
$$h(\bar{x}) + A(\cdot - \bar{x}) + H$$
has a localization around $\bar{y}$ for $\bar{x}$ which is Lipschitz continuous with the constant strictly less than $1/c$.

Then $h + H$ has a localization around $\bar{y}$ for $\bar{x}$ which is Lipschitz continuous.

Consider a locally Lipschitz continuous $\mathbf{h} : \mathbb{R}^m \to \mathbb{R}^d$, that is, for any $\bar{\mathbf{u}} \in \mathbb{R}^m$ there is a neighborhood U of $\bar{\mathbf{u}}$ along with a constant $L_{\bar{\mathbf{u}}} > 0$ such that

$$\|\mathbf{h}(\hat{\mathbf{u}}) - \mathbf{h}(\tilde{\mathbf{u}})\| \leq L_{\bar{\mathbf{u}}} \|\hat{\mathbf{u}} - \tilde{\mathbf{u}}\| \quad \text{whenever} \quad \hat{\mathbf{u}}, \tilde{\mathbf{u}} \in U.$$

Rademacher's theorem says that there is a dense set D of points $\mathbf{u} \in U$ where $\mathbf{h}$ is differentiable. Hence there exists a sequence $(\mathbf{u}_n)_{n\in\mathbb{N}}$ in D converging to $\bar{\mathbf{u}}$ such that the corresponding sequence $(\|\nabla \mathbf{h}(\mathbf{u}_n)\|)_{n\in\mathbb{N}}$ is bounded. Thus $(\nabla \mathbf{h}(\mathbf{u}_n))_{n\in\mathbb{N}}$ has at least one cluster point. This leads to the definition of the *Bouligand's limiting Jacobian of* $\mathbf{h}$ *at* $\bar{\mathbf{u}}$, which is the set $\partial_B \mathbf{h}(\bar{\mathbf{u}})$ consisting of all matrices $\mathbf{A} \in \mathbb{R}^{d\times m}$ for which there is a sequence $(\mathbf{u}_n)_{n\in\mathbb{N}}$ converging to $\bar{\mathbf{u}}$ such that $\mathbf{h}$ is differentiable at each $\mathbf{u}_n$ and $\nabla \mathbf{h}(\mathbf{u}_n) \to \mathbf{A}$ as $n \to +\infty$. The *Clarke's generalized Jacobian of* $\mathbf{h}$ *at* $\bar{\mathbf{u}}$, denoted by $\partial_C \mathbf{h}(\bar{\mathbf{u}})$, is the convex hull of $\partial_B \mathbf{h}(\bar{\mathbf{u}})$. It is well-known that $\mathcal{A} := \partial_C \mathbf{h}(\bar{\mathbf{u}})$ satisfies the condition (i) of the preceding statement while $\mathcal{A} := \partial_B \mathbf{h}(\bar{\mathbf{u}})$ not. We will investigate the properties of these objects later in the last chapter. Very often, the authors call $\partial_B \mathbf{h}(\bar{\mathbf{u}})$ the *Bouligand (sub)differential* or *B-(sub)differential* in

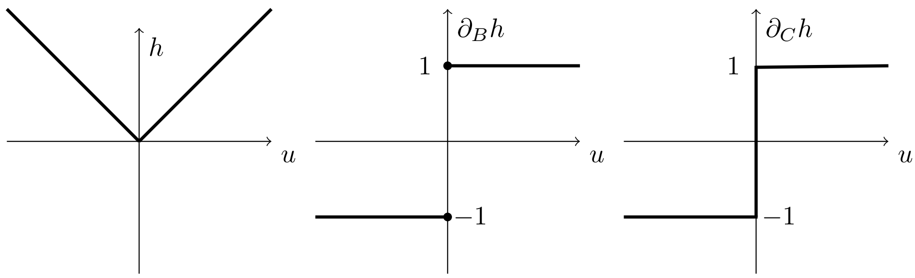

FIGURE 12. Clarke's and Bouligand's subdifferential of $h := |\cdot|$.

short. We argue that this terminology is misleading since we work with vector-valued mappings in general and the notion "(sub)differential" should be used only when $d = 1$, that is, when one works with real-valued functions $h : \mathbb{R}^m \to \mathbb{R}$.

Example 2.17. Let $h(u) := |u|$, $u \in \mathbb{R}$. Then (see also Figure 12) we have

$$\partial_B h(0) = \{-1, 1\} \quad \text{and} \quad \partial_C h(0) = [-1, 1].$$

A non-smooth non-parametric Izmailov-Robinson theorem can be stated as follows.

Theorem 2.18. *Let* $\mathbf{h} : \mathbb{R}^m \to \mathbb{R}^m$ *be locally Lipschitz continuous at* $\bar{\mathbf{u}} \in \mathbb{R}^m$ *and let* $\mathbf{H} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$. *If for every* $\mathbf{A} \in \partial_C \mathbf{h}(\bar{\mathbf{u}})$, *the inverse of the mapping*

$$\mathbf{h}(\bar{\mathbf{u}}) + \mathbf{A}(\cdot - \bar{\mathbf{u}}) + \mathbf{H}$$

has a localization around $\mathbf{0}$ *for* $\bar{\mathbf{u}}$ *which is Lipschitz continuous, then so does the inverse of* $\mathbf{h} + \mathbf{H}$.

For $\mathbf{H} \equiv \mathbf{0}$, the above result was proved by F. H. Clarke [12] and the above assumption means nothing else but that all the matrices in $\partial_C \mathbf{h}(\bar{\mathbf{u}})$ are non-singular.

To conclude this section, let us comment on the existence of a Lipschitz continuous selection briefly.

Remark 2.19. Given $\mathbf{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$, assume that there is $\kappa > 0$ along with closed convex neighborhoods U of $\bar{\mathbf{u}}$ and V of $\bar{\mathbf{p}}$ such that

(i) $\mathbf{S}(\mathbf{p}) \cap U$ is closed and convex;
(ii) $\mathbf{S}(\tilde{\mathbf{p}}) \cap U \subset \mathbf{S}(\hat{\mathbf{p}}) + \kappa \|\tilde{\mathbf{p}} - \hat{\mathbf{p}}\| \, \mathbb{B}[\mathbf{0}, 1]$ for each $\tilde{\mathbf{p}}, \hat{\mathbf{p}} \in V$.

By [13, Theorem 3E.3], there there is $\kappa_1 > 0$ together with closed convex neighborhoods U_1 of $\bar{\mathbf{u}}$ and V_1 of $\bar{\mathbf{p}}$ such that

(i') $\mathbf{S}(\mathbf{p}) \cap U_1$ is closed and convex;
(ii') $\mathbf{S}(\tilde{\mathbf{p}}) \cap U_1 \subset \mathbf{S}(\hat{\mathbf{p}}) \cap U_1 + \kappa_1 \|\tilde{\mathbf{p}} - \hat{\mathbf{p}}\| \, \mathbb{B}[\mathbf{0}, 1]$ for each $\tilde{\mathbf{p}}, \hat{\mathbf{p}} \in V_1$.

Using Steiner selection, the remark following [2, Theorem 9.4.3] implies that $\mathbf{S}$ has a Lipschitz continuous selection around $\bar{\mathbf{p}}$ for $\bar{\mathbf{u}}$.

The property in (ii) is known as *Aubin/pseudo Lipschitz/Lipschitz-like continuity (property)* of $\mathbf{S}$, or equivalently, as the *metric regularity* of $\mathbf{S}^{-1}$ at the reference point.

3.1. **Existence and Uniqueness Results on DIs.** Given an open subset D of $\mathbb{R}^{n+1}$ containing a point $(a, \mathbf{x}_a)$ and a set-valued mapping $\mathbf{F} : \mathbb{R}^{n+1} \rightrightarrows \mathbb{R}^n$ the *differential inclusion (DI)* is given by

$$
\left\{ \begin{array}{lcl} \dot{\mathbf{x}}(t) & \in & \mathbf{F}(t, \mathbf{x}(t)) \quad \text{for} \quad t > a, \\ \mathbf{x}(a) & = & \mathbf{x}_a. \end{array} \right. \tag{3.1}
$$

A function $\bar{\mathbf{x}}(\cdot)$ is called a *solution* of (3.1) if there is $\sigma > 0$ such that

(i) $\bar{\mathbf{x}}(\cdot)$ is absolutely continuous on $[a, a + \sigma]$;
(ii) $\dot{\bar{\mathbf{x}}}(t) \in \mathbf{F}(t, \bar{\mathbf{x}}(t))$ for almost all $t \in (a, a + \sigma)$;
(iii) $\bar{\mathbf{x}}(a) = \mathbf{x}_a$.

The following theorem, due to A. F. Filippov [17], provides sufficient conditions for local existence and uniqueness of a solution. It will be presented with a sketch of the proof because it gives an insight to the numerical methods discussed in the last chapter.

Theorem 3.1. *Let D be an open convex subset of $\mathbb{R}^{n+1}$ which contains a point $(a, \mathbf{x}_a) \in \mathbb{R}^{n+1}$. Suppose that $\mathbf{F} : \mathbb{R}^{n+1} \rightrightarrows \mathbb{R}^n$ satisfies for each $(t, \mathbf{x}) \in D$ the following conditions:*

(i) *the set $\mathbf{F}(t, \mathbf{x})$ is non-empty, bounded, closed, and convex;*
(ii) *$\mathbf{F}$ is* Pompeiu-Hausdorff outer semi-continuous *at $(t, \mathbf{x})$ meaning that for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\|(s, \mathbf{y}) - (t, \mathbf{x})\| < \delta \quad \textit{implies that} \quad \mathbf{F}(s, \mathbf{y}) \subset \mathbf{F}(t, \mathbf{x}) + \mathbb{B}(\mathbf{0}, \varepsilon).$$

Then differential inclusion (3.1) *has a solution.*

If, in addition, $\mathbf{F}$ satisfies one-sided Lipschitz condition *in D, that is, there is a non-negative Lebesgue integrable function $l : \mathbb{R} \to \mathbb{R}$ such that*

$$\langle \mathbf{u} - \mathbf{v}, \mathbf{x} - \mathbf{y} \rangle \ \leq \ l(t)\|\mathbf{x} - \mathbf{y}\|^2$$

whenever $(t, \mathbf{x})$, $(t, \mathbf{y}) \in D$, $\mathbf{u} \in \mathbf{F}(t, \mathbf{x})$, $\mathbf{v} \in \mathbf{F}(t, \mathbf{y})$, then the solution is unique.

Proof. As $(a, \mathbf{x}_a)$ is an interior point of D, one may find the constants $c > 0$ and $d > 0$ such that

$$Z := [a, a + c] \times \mathbb{B}[\mathbf{x}_a, d] \subset D.$$

We divide the proof into several steps.

Step 1: *There is $m > 0$ such that for each $(t, \mathbf{x}) \in Z$ and each $\mathbf{y} \in \mathbf{F}(t, \mathbf{x})$ we have $\|\mathbf{y}\| \leq m$.*

Suppose on the contrary that there is a sequence $(\mathbf{z}_k)_{k \in \mathbb{N}}$ in Z and a sequence $(\mathbf{y}_k)_{k \in \mathbb{N}}$ such that

$$\mathbf{y}_k \in \mathbf{F}(\mathbf{z}_k) \text{ for each } k \in \mathbb{N} \quad \text{and} \quad \|\mathbf{y}_k\| \to +\infty \text{ as } k \to +\infty.$$

Since Z is compact, we may choose a convergent sub-sequence $(\mathbf{z}_{k_i})_{i \in \mathbb{N}}$ of $(\mathbf{z}_k)_{k \in \mathbb{N}}$, with the limit $\mathbf{z} \in Z$ say. As $\mathbf{F}(\mathbf{z})$ is bounded, there is $\alpha > 0$ such that for each $\mathbf{u} \in \mathbf{F}(\mathbf{z})$ we have $\|\mathbf{u}\| \leq \alpha$. By the outer semi-continuity of $\mathbf{F}$ at $\mathbf{z}$, we may find $i_0 \in \mathbb{N}$ such that

$$\mathbf{F}(\mathbf{z}_{k_i}) \subset \mathbf{F}(\mathbf{z}) + \mathbb{B}(\mathbf{0}, \alpha) \quad \text{whenever} \quad i > i_0.$$

So, for any fixed $i > i_0$, find $\mathbf{u}_i \in \mathbf{F}(\mathbf{z})$ and $\mathbf{b}_i \in \mathbb{B}(\mathbf{0}, \alpha)$ such that $\mathbf{y}_{k_i} = \mathbf{u}_i + \mathbf{b}_i$. Hence

$$\|\mathbf{y}_{k_i}\| \leq \|\mathbf{u}_i\| + \|\mathbf{b}_i\| \leq \alpha + \alpha = 2\alpha \quad \text{whenever} \quad i > i_0.$$

We arrived at a contradiction with the assumption that $\|\mathbf{y}_{k_i}\| \to +\infty$ as $i \to +\infty$.

Step 2: Put $\sigma := \min\{c, d/m\}$. For any fixed $k \in \mathbb{N}$ we will construct a broken line $\mathbf{x}_k(\cdot)$ defined on $[a, a+\sigma]$ by an inductive process (see Figure 13). Let

$$h_k := \frac{\sigma}{k}, \quad t_{k,i} := a + ih_k \quad \text{for} \quad i = 0, 1, \ldots, k \quad \text{and} \quad \mathbf{x}_k(t_{k,0}) := \mathbf{x}_a.$$

Assume that for some $i \geq 0$ the value $\mathbf{x}_{k,i} := \mathbf{x}_k(t_{k,i})$ has already been defined and

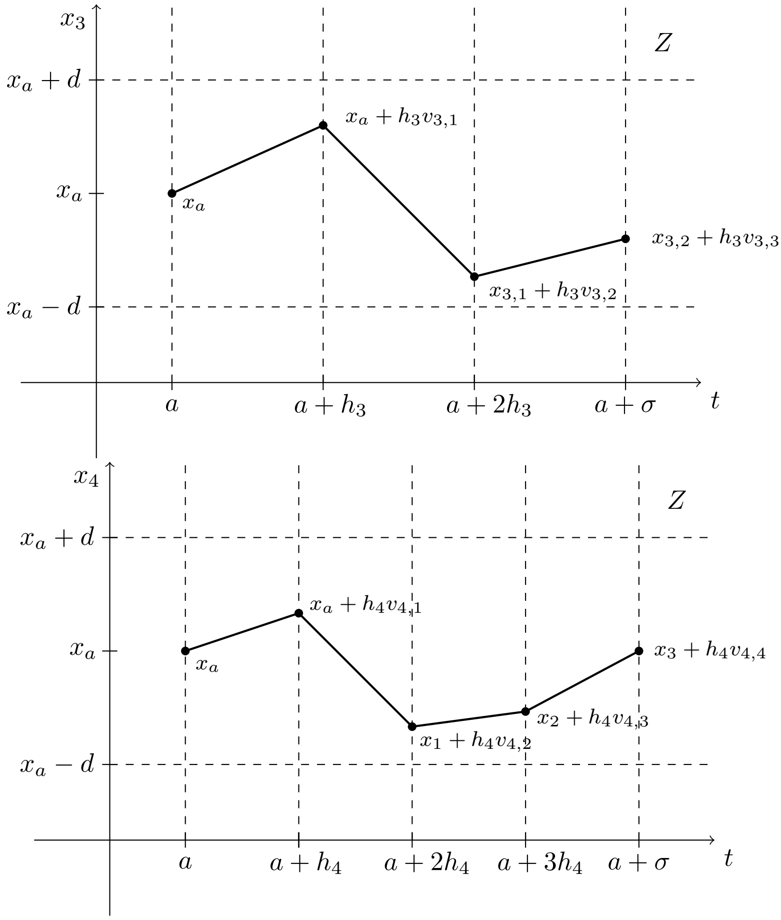

Figure 13. Two broken lines from Step 2 for $k = 3, 4$.

satisfies

$$\|\mathbf{x}_{k,i} - \mathbf{x}_a\| \leq m|t_{k,i} - a|. \tag{3.2}$$

As $(t_{k,i}, \mathbf{x}_{k,i}) \in Z$, take any $\mathbf{v}_{k,i} \in \mathbf{F}(t_{k,i}, \mathbf{x}_{k,i})$, and define

$$\mathbf{x}_k(t) = \mathbf{x}_{k,i} + (t - t_{k,i})\mathbf{v}_{k,i} \quad \text{for} \quad t \in [t_{k,i}, t_{k,i+1}].$$

Since $\|\mathbf{v}_{k,i}\| \leq m$, (3.2) implies that for each $t \in [t_{k,i}, t_{k,i+1}]$ we have

$$\|\mathbf{x}_k(t) - \mathbf{x}_a\| \leq m|t_{k,i} - a| + m|t - t_{k,i}| = m|t - a|. \tag{3.3}$$

Therefore $\mathbf{x}_{k,i+1} := \mathbf{x}_k(t_{k,i+1})$ satisfies (3.2) with i replaced by $i+1$. We constructed $\mathbf{x}_k(\cdot)$ successively on the whole interval $[a, a+\sigma]$.

Step 3: *A suitable sub-sequence of* $(\mathbf{x}_k(\cdot))_{k\in\mathbb{N}}$ *converges to an absolutely continuous function* $\bar{\mathbf{x}}(\cdot)$ *on* $[a, a+\sigma]$ *the graph of which lies in* Z.
Indeed, fix any $k \in \mathbb{N}$. By the very definition, $\|\dot{\mathbf{x}}_k(t)\| \leq m$ for almost all $t \in (a, a+\sigma)$. Hence, for each t_1, $t_2 \in [a, a+\sigma]$, one has

$$\|\mathbf{x}_k(t_1) - \mathbf{x}_k(t_2)\| = \left\| \int_{t_1}^{t_2} \dot{\mathbf{x}}_k(t)\, \mathrm{d}t \right\| \leq m|t_1 - t_2|.$$

Taking into account this and (3.3), one sees that $(\mathbf{x}_k(\cdot))_{k\in\mathbb{N}}$ is a sequence of uniformly bounded and equi-continuous functions. Therefore, using Arzelà-Ascoli Theorem, one can select a uniformly convergent sub-sequence of it. Denote its limit by $\bar{\mathbf{x}}(\cdot)$. As all the functions $\mathbf{x}_k(\cdot)$ are Lipschitz continuous on $[a, a+\sigma]$ and their graphs lie in Z, the limit $\bar{\mathbf{x}}(\cdot)$ has the same properties. In particular, it is absolutely continuous.

Step 4: Clearly, $\bar{\mathbf{x}}(a) = \mathbf{x}_a$. So, it remains to prove that

$$\dot{\bar{\mathbf{x}}}(\bar{t}) \in \mathbf{F}(\bar{t}, \bar{\mathbf{x}}(\bar{t})) \quad \textit{whenever } \bar{t} \in (a, a+\sigma) \textit{ is such that } \dot{\bar{\mathbf{x}}}(\bar{t}) \textit{ exists.}$$

To do so, pick any such a point $\bar{t}$ and put $\bar{\mathbf{x}} = \bar{\mathbf{x}}(\bar{t})$. Let $\varepsilon > 0$ be arbitrary. As $\mathbf{F}$ is outer semi-continuous, there is $\eta > 0$ such that

$$\mathbf{F}(t, \mathbf{x}) \subset \mathbf{F}(\bar{t}, \bar{\mathbf{x}}) + \mathbb{B}[\mathbf{0}, \varepsilon] =: \Omega \quad \text{whenever} \quad (t, \mathbf{x}) \in [\bar{t} - \eta, \bar{t} + \eta] \times \mathbb{B}[\bar{\mathbf{x}}, \eta].$$

Being the sum of two closed, convex and bounded sets, the set Ω is closed, convex and bounded. By the very definition, there is $k_0 \in \mathbb{N}$ such that the derivative $\dot{\mathbf{x}}_k(t)$ (whenever it exists) is in Ω for each $k > k_0$. Therefore

$$\frac{\mathbf{x}_k(\bar{t} + h) - \mathbf{x}_k(\bar{t})}{h} = \frac{1}{h} \int_{\bar{t}}^{\bar{t}+h} \dot{\mathbf{x}}_k(\tau)\, \mathrm{d}\tau \in \Omega \quad \text{for} \quad h \in (-\eta, \eta),$$

because this mean value can be viewed as a limit of a convex combination of function values $\dot{\mathbf{x}}_k(\tau_i)$ at appropriately chosen points $\tau_i \in (\bar{t} - h, \bar{t} + h)$ corresponding to the partition of the interval $[\bar{t} - h, \bar{t} + h]$. Hence

$$\dot{\bar{\mathbf{x}}}(\bar{t}) = \lim_{k\to+\infty} \lim_{h\to 0} \frac{\mathbf{x}_k(\bar{t} + h) - \mathbf{x}_k(\bar{t})}{h} \in \Omega = \mathbf{F}(\bar{t}, \bar{\mathbf{x}}) + \mathbb{B}[\mathbf{0}, \varepsilon].$$

Since $\varepsilon > 0$ was arbitrary, we get $\dot{\bar{\mathbf{x}}}(\bar{t}) \in \mathbf{F}(\bar{t}, \bar{\mathbf{x}}(\bar{t}))$.

This establishes the existence part. To prove the rest, suppose that the one-sided Lipschitz condition holds true.
Step 5: *The solution is unique.*

Suppose that there is $\sigma > 0$ such that there are two solutions $\mathbf{x}(\cdot)$ and $\mathbf{y}(\cdot)$ of (3.1) on the interval $[a, a+\sigma]$. Put $\mathbf{z}(\cdot) := \mathbf{x}(\cdot) - \mathbf{y}(\cdot)$. Define a non-negative function

$$V(t) := \|\mathbf{z}(t)\|^2 e^{-2\int_a^t l(\tau)\mathrm{d}\tau}, \quad t \in [a, a+\sigma].$$

As $\mathbf{z}(\cdot)$ is absolutely continuous, so is V. Hence for almost all $t \in (a, a+\sigma)$, we have

$$\begin{aligned} \dot{V}(t) &= 2\langle \mathbf{z}(t), \dot{\mathbf{z}}(t)\rangle\, e^{-2\int_a^t l(\tau)\mathrm{d}\tau} - 2l(t)\|\mathbf{z}(t)\|^2 e^{-2\int_a^t l(\tau)\mathrm{d}\tau} \\ &= 2e^{-2\int_a^t l(\tau)\mathrm{d}\tau}\Big(\langle \mathbf{x}(t) - \mathbf{y}(t), \dot{\mathbf{x}}(t) - \dot{\mathbf{y}}(t)\rangle \;-\; l(t)\|\mathbf{x}(t) - \mathbf{y}(t)\|^2\Big). \end{aligned}$$

Moreover, for almost all $t \in (a, a+\sigma)$, we have $\dot{\mathbf{x}}(t) \in \mathbf{F}(t, \mathbf{x}(t))$ and $\dot{\mathbf{y}}(t) \in \mathbf{F}(t, \mathbf{y}(t))$, the one-sided Lipschitz condition yields that

$$\dot{V}(t) \le 0 \quad \text{for almost all} \quad t \in (a, a+\sigma).$$

Now, $\mathbf{x}(a) = \mathbf{y}(a) = \mathbf{x}_a$ yields that $V(a) = 0$. Therefore, for each $t \in [a, a+\sigma]$ one has

$$0 \le V(t) = V(a) + \int_a^t \dot{V}(\tau)\mathrm{d}\tau \;\le\; 0.$$

Hence, $V(t) = 0$ for each $t \in [a, a+\sigma]$, which means that $\mathbf{x}(\cdot) = \mathbf{y}(\cdot)$. □

The Pompeiu-Hausdorff outer semi-continuity is often called just outer semi-continuity or upper semi-continuity in the literature. Note that this property does not imply that the mapping in question has closed graph. Similarly, if a mapping has closed graph, then it does not need to be outer semi-continuous (see Figure 14).

Lemma 3.2. (i) *Any outer semi-continuous set-valued mapping $\mathbf{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ with closed domain and closed values has closed graph;*

(ii) *Any locally bounded set-valued mapping $\mathbf{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, meaning that for each $\mathbf{x} \in \operatorname{dom} \mathbf{F}$ there is $r > 0$ such that $\mathbf{F}(\mathbb{B}(\mathbf{x}, r))$ is bounded, having a closed graph is outer semi-continuous.*

Proof. (i) Take any sequence $(\mathbf{x}_k, \mathbf{y}_k)$ in $\operatorname{gph} \mathbf{F}$ which converges to some $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m \times \mathbb{R}^n$. As $\mathbf{x}_k \to \mathbf{x}$ for $k \to +\infty$ and $\operatorname{dom} \mathbf{F}$ is closed, one has $\mathbf{x} \in \operatorname{dom} \mathbf{F}$, so $\mathbf{F}(\mathbf{x}) \neq \emptyset$. Moreover, since $\mathbf{y}_k \in \mathbf{F}(\mathbf{x}_k)$ for each $k \in \mathbb{N}$, by outer semi-continuity and passing to a sub-sequence, if necessary, we infer that

$$\mathbf{y}_k \in \mathbf{F}(\mathbf{x}) + \mathbb{B}(\mathbf{0}, 1/k) \quad \text{for each} \quad k \in \mathbb{N}.$$

So, $d(\mathbf{y}, \mathbf{F}(\mathbf{x})) = \lim\limits_{k \to +\infty} d(\mathbf{y}_k, \mathbf{F}(\mathbf{x})) = 0$. As $\mathbf{F}(\mathbf{x})$ is closed, one gets that $\mathbf{y} \in \mathbf{F}(\mathbf{x})$. Therefore $(\mathbf{x}, \mathbf{y}) \in \operatorname{gph} \mathbf{F}$.

(ii) Let $\mathbf{x} \in \operatorname{dom} \mathbf{F}$ be arbitrary. Suppose on the contrary that there is $\varepsilon > 0$ along with sequences $(\mathbf{x}_k)_{k \in \mathbb{N}}$ in $\mathbb{R}^m$ converging to $\mathbf{x}$ and $(\mathbf{v}_k)_{k \in \mathbb{N}}$ in $\mathbb{R}^n$ such that

$$\mathbf{v}_k \in \mathbf{F}(\mathbf{x}_k) \quad \text{and} \quad d(\mathbf{v}_k, \mathbf{F}(\mathbf{x})) \ge \varepsilon \quad \text{for each} \quad k \in \mathbb{N}.$$

As $(\mathbf{x}_k)_{k \in \mathbb{N}}$ converges to $\mathbf{x}$ and $\mathbf{F}$ is locally bounded, $(\mathbf{v}_k)_{k \in \mathbb{N}}$ is bounded. Find a sub-sequence $(\mathbf{v}_{k_i})_{i \in \mathbb{N}}$ of it which converges, to a point $\mathbf{v} \in \mathbb{R}^n$ say. Then

$$d(\mathbf{v}, \mathbf{F}(\mathbf{x})) = \lim_{i \to +\infty} d(\mathbf{v}_{k_i}, \mathbf{F}(\mathbf{x})) \ge \varepsilon > 0.$$

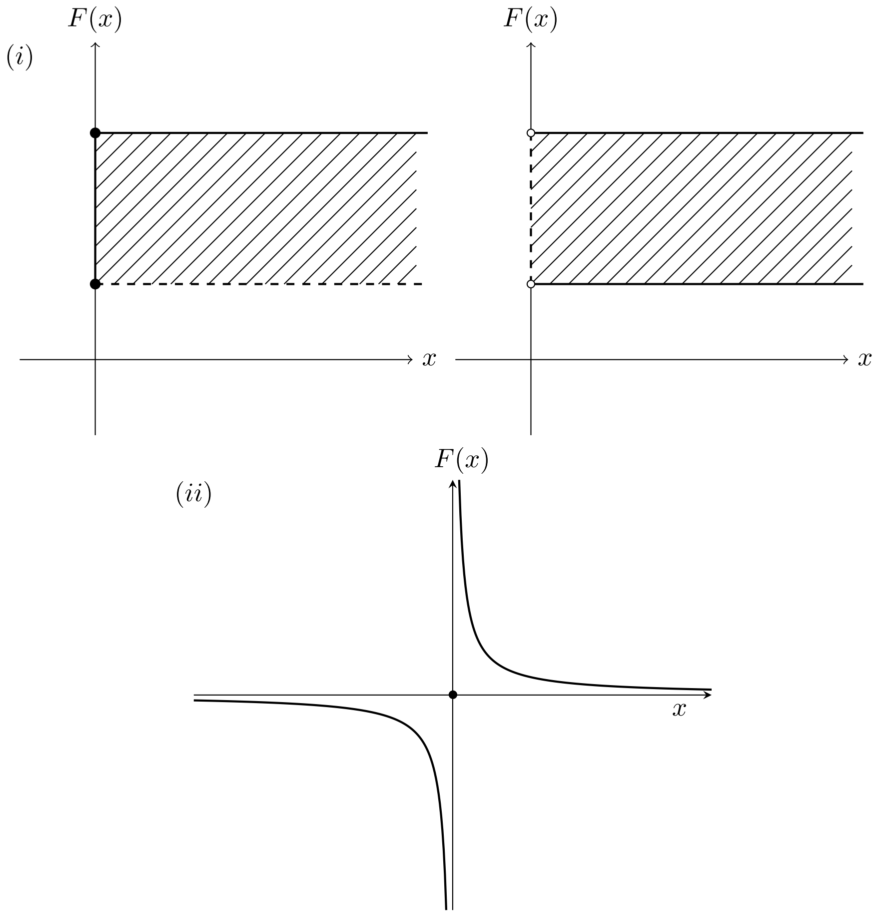

FIGURE 14. Counterexamples illustrating Lemma 3.2.

Since $\mathbf{F}(\mathbf{x})$ is closed, $\mathbf{v} \notin \mathbf{F}(\mathbf{x})$. On the other hand, $\mathbf{v}_{k_i} \in \mathbf{F}(\mathbf{x}_{k_i})$ for each $i \in \mathbb{N}$, hence the closeness of the graph of $\mathbf{F}$ implies that $\mathbf{v} \in \mathbf{F}(\mathbf{x})$, a contradiction. $\square$

Example 3.3. Let $D = (a, b) \times \Omega$ be a domain in $\mathbb{R} \times \mathbb{R}^n$, let $\mathbf{f} : D \to \mathbb{R}^n$ be a mapping for which there is a Lebesgue integrable function $l : (a, b) \to \mathbb{R}$ such that

$$\|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})\| \leq l(t)\|\mathbf{x} - \mathbf{y}\| \quad \text{whenever} \quad (t, \mathbf{x}), (t, \mathbf{y}) \in D,$$

and let $\mathbf{G} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a monotone mapping with $\Omega \subset \operatorname{dom} \mathbf{G}$. Then

$$\mathbf{F}(t, \mathbf{x}) := \mathbf{f}(t, \mathbf{x}) - \mathbf{G}(\mathbf{x}), \quad (t, \mathbf{x}) \in D,$$

satisfies one-sided Lipschitz condition in D. Indeed, take any $(t, \mathbf{x})$, $(t, \mathbf{y}) \in D$, $\mathbf{u} \in \mathbf{F}(t, \mathbf{x})$, and $\mathbf{v} \in \mathbf{F}(t, \mathbf{y})$. Find $\mathbf{p}_\mathbf{x} \in \mathbf{G}(\mathbf{x})$ and $\mathbf{p}_\mathbf{y} \in \mathbf{G}(\mathbf{y})$ such that $\mathbf{u} = \mathbf{f}(t, \mathbf{x}) - \mathbf{p}_\mathbf{x}$ and $\mathbf{v} = \mathbf{f}(t, \mathbf{y}) - \mathbf{p}_\mathbf{y}$. The monotonicity of $\mathbf{G}$ and the Cauchy-Schwarz inequality yield that

$$\begin{aligned}\langle \mathbf{u}-\mathbf{v}, \mathbf{x}-\mathbf{y}\rangle &= \langle \mathbf{f}(t,\mathbf{x})-\mathbf{f}(t,\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle - \langle \mathbf{p}_{\mathbf{x}}-\mathbf{p}_{\mathbf{y}}, \mathbf{x}-\mathbf{y}\rangle \\ &\le \langle \mathbf{f}(t,\mathbf{x})-\mathbf{f}(t,\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle \le \|\mathbf{f}(t,\mathbf{x})-\mathbf{f}(t,\mathbf{y})\|\,\|\mathbf{x}-\mathbf{y}\| \\ &\le l(t)\|\mathbf{x}-\mathbf{y}\|^2.\end{aligned}$$

Recall that a monotone mapping $\mathbf{G}:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called *maximal* if there does not exist other monotone mapping whose graph strictly contains the graph of $\mathbf{G}$. This means that for each $(\mathbf{x},\mathbf{u}) \in \mathbb{R}^n \times \mathbb{R}^n$ we have

$$(\mathbf{x},\mathbf{u}) \in \operatorname{gph}\mathbf{G} \quad \Leftrightarrow \quad \langle \mathbf{u}-\mathbf{v}, \mathbf{x}-\mathbf{y}\rangle \ge 0 \;\text{ for each }\; (\mathbf{y},\mathbf{v}) \in \operatorname{gph}\mathbf{G}. \tag{3.4}$$

We have the following well-known result, the last statement of which was proved by R. T. Rockafellar.

Lemma 3.4. *If $\mathbf{G}:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone then*
(i) *$\mathbf{G}(\mathbf{x})$ is closed and convex for each $\mathbf{x} \in \mathbb{R}^n$;*
(ii) *the graph of $\mathbf{G}$ is closed;*
(iii) *$\mathbf{G}$ is locally bounded at each interior point of its domain.*

Proof. (i) An empty set is both closed and convex. So take any $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{G}(\mathbf{x}) \neq \emptyset$. Then (3.4) reveals that $\mathbf{G}(\mathbf{x})$ is equal to

$$\bigcap_{(\mathbf{y},\mathbf{v})\in\operatorname{gph}\mathbf{G}} \{\mathbf{u} \in \mathbb{R}^n : \; \langle \mathbf{u}-\mathbf{v}, \mathbf{x}-\mathbf{y}\rangle \ge 0\},$$

(an intersection of closed convex sets), hence it is closed and convex as well.

(ii) Take any sequence $\big((\mathbf{x}_k,\mathbf{u}_k)\big)_{k\in\mathbb{N}}$ in $\operatorname{gph}\mathbf{G}$ which converges to some $(\mathbf{x},\mathbf{u}) \in \mathbb{R}^n \times \mathbb{R}^n$. Pick any $(\mathbf{y},\mathbf{v}) \in \operatorname{gph}\mathbf{G}$. Then

$$\langle \mathbf{u}-\mathbf{v}, \mathbf{x}-\mathbf{y}\rangle = \lim_{k\to+\infty} \langle \mathbf{u}_k-\mathbf{v}, \mathbf{x}_k-\mathbf{y}\rangle \ge 0.$$

As $(\mathbf{y},\mathbf{v}) \in \operatorname{gph}\mathbf{G}$ was arbitrary, using (3.4), one infers that $(\mathbf{x},\mathbf{u}) \in \operatorname{gph}\mathbf{G}$.

(iii) See any standard book on convex analysis, e.g. [30]. □

Remark 3.5. Let $h:\mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function. In Proposition 4.5, we will see that Bouligand's sub-differential mapping $\mathbf{x} \rightrightarrows \partial_B h(\mathbf{x})$ and the Clarke's sub-differential mapping $\mathbf{x} \rightrightarrows \partial_C h(\mathbf{x})$, defined in the previous chapter, are non-empty-valued and locally bounded on whole of $\mathbb{R}^n$; and have closed graphs. In particular, they are both Pompeiu-Hausdorff outer semi-continuous. The latter mapping has even closed convex values. None of these mappings is (maximal) monotone in general (unless h is a continuous convex function).

Very often, one has to address the existence and uniqueness of a solution for a DI on the whole prescribed time interval which may even be unbounded. This is the case, for example, when a question of (asymptotic) stability is on stage. The proof of the next statement follows the same pattern as the one of Theorem 3.1.

Theorem 3.6. *Given $(a,\mathbf{x}_a) \in \mathbb{R}\times\mathbb{R}^n$ and $b > a$, let $D := [a,b]\times\mathbb{R}^n$. Suppose that $\mathbf{F}: D \rightrightarrows \mathbb{R}^n$ satisfies the following conditions:*
(i) *$\mathbf{F}(t,\mathbf{x})$ is non-empty, closed, and convex for each $(t,\mathbf{x}) \in D$;*
(ii) *$\mathbf{F}$ is Pompeiu-Hausdorff outer semi-continuous at each $(t,\mathbf{x}) \in D$;*

(iii) $\mathbf{F}$ *has a linear growth on* D*, that is, there are* $\alpha > 0$ *and* $\beta > 0$ *such that*

$$\|\mathbf{z}\| \leq \alpha\|\mathbf{x}\| + \beta \quad \textit{whenever} \quad \mathbf{z} \in \mathbf{F}(t,\mathbf{x}) \ \textit{and} \ (t,\mathbf{x}) \in D.$$

Then there is $\mathbf{x} : [a,b] \to \mathbb{R}^n$ *such that*

(i) $\mathbf{x}(\cdot)$ *is absolutely continuous on* $[a,b]$*;*
(ii) $\dot{\mathbf{x}}(t) \in \mathbf{F}(t,\mathbf{x}(t))$ *for almost all* $t \in (a,b)$*;*
(iii) $\mathbf{x}(a) = \mathbf{x}_a$.

If $\mathbf{F}$ *satisfies the one-sided Lipschitz condition in* D*, then the solution is unique.*

Remark 3.7. As any set-valued mapping with a linear growth is locally bounded, one can assume that $\mathbf{F}$ has closed graph instead of its outer semi-continuity.

3.2. **From a DVI to a DI and Back.** Since the conditions in Theorems 3.1 and 3.6 are very restrictive, the same will be the case for differential variational inequalities which can be studied via a transformation to a differential inclusion. Consider the following DVI:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t,\mathbf{x}(t)) + \mathbf{B}(t,\mathbf{x}(t))\,\mathbf{u}(t), \tag{3.5}$$

$$0 \leq \langle \mathbf{g}(t,\mathbf{x}(t)) + \mathbf{h}(\mathbf{u}(t)), \mathbf{v} - \mathbf{u}(t)\rangle \quad \text{whenever} \quad \mathbf{v} \in K, \tag{3.6}$$

$$\mathbf{u}(t) \in K, \tag{3.7}$$

where $K \subset \mathbb{R}^m$ is closed and convex, $\mathbf{f} : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{B} : [a,b] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $\mathbf{g} : [a,b] \times \mathbb{R}^n \to \mathbb{R}^m$, and $\mathbf{h} : \mathbb{R}^m \to \mathbb{R}^m$ are given functions. Set $D := [a,b] \times \mathbb{R}^n$. From now on, assume that

(A1) both $\mathbf{f}$ and $\mathbf{g}$ are continuous and have a linear growth on D meaning that there are positive constants α and β such that

$$\|\mathbf{f}(t,\mathbf{x})\| \leq \alpha\|\mathbf{x}\| + \beta \quad \text{and} \quad \|\mathbf{g}(t,\mathbf{x})\| \leq \alpha\|\mathbf{x}\| + \beta \quad \text{whenever} \quad (t,\mathbf{x}) \in D;$$

(A2) $\mathbf{B}$ is continuous and bounded on D with

$$\sigma := \sup_{(t,\mathbf{x})\in D} \|\mathbf{B}(t,\mathbf{x})\| < +\infty;$$

(A3) $\mathbf{h}$ is both continuous and monotone on K, and there is $\bar{\mathbf{u}} \in K$ such that

$$\liminf_{\mathbf{w}\in K, \|\mathbf{w}\|\to+\infty} \frac{\langle \mathbf{h}(\mathbf{w}), \mathbf{w} - \bar{\mathbf{u}}\rangle}{\|\mathbf{w}\|^2} > 0.$$

Define the solution mapping $\mathbf{S} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ corresponding to (3.6) by

$$\mathbf{S}(\mathbf{p}) := \{\mathbf{u} \in K : \ \langle \mathbf{p} + \mathbf{h}(\mathbf{u}), \mathbf{v} - \mathbf{u}\rangle \geq 0 \text{ for each } \mathbf{v} \in K\}, \quad \mathbf{p} \in \mathbb{R}^m.$$

Consider $\mathbf{F} : D \rightrightarrows \mathbb{R}^n$ defined by

$$\mathbf{F}(t,\mathbf{x}) = \big\{\mathbf{f}(t,\mathbf{x}) + \mathbf{B}(t,\mathbf{x})\,\mathbf{u} : \ \mathbf{u} \in \mathbf{S}(\mathbf{g}(t,\mathbf{x}))\big\}, \quad (t,\mathbf{x}) \in D. \tag{3.8}$$

We have to address two issues. First, how are solutions for DVI (3.5)–(3.7) related to solutions for a DI in the form

$$\dot{\mathbf{x}}(t) \in \mathbf{F}(t,\mathbf{x}(t)) \quad \text{for almost all} \quad t \in [a,b]. \tag{3.9}$$

Second, does this $\mathbf{F}$ satisfy the conditions in Filippov's theorems? Let us start with the latter question.

Proposition 3.8. *Suppose that* $\mathbf{h}$ *verifies (A3). Then* $\mathbf{S}$ *has non-empty closed convex values, its domain is whole of* $\mathbb{R}^m$*, and there is* $\varrho > 0$ *such that*

$$\|\mathbf{u}\| \le \varrho(1 + \|\mathbf{p}\|) \qquad \textit{whenever} \quad \mathbf{u} \in \mathbf{S}(\mathbf{p}) \quad \textit{and} \quad \mathbf{p} \in \mathbb{R}^m. \tag{3.10}$$

Proof. Fix any $\mathbf{p} \in \mathbb{R}^m$. We are going to use Proposition 2.2 with $\mathbf{h}(\cdot) := \mathbf{h}(\cdot) + \mathbf{p}$. Indeed, as $\mathbf{h}(\cdot) + \mathbf{p}$ is continuous and monotone on K, the set $\mathbf{S}(\mathbf{p})$ is closed and convex (possibly empty). To prove its non-emptiness we have to show that $\mathbf{h}(\cdot) + \mathbf{p}$ is semi-coercive on K. Let $\bar{\mathbf{u}}$ be as in (A3). We find $r > 0$ such that

$$\langle \mathbf{h}(\mathbf{w}) + \mathbf{p}, \mathbf{w} - \bar{\mathbf{u}} \rangle > 0 \quad \text{for each} \quad \mathbf{w} \in K \text{ with } \|\mathbf{w}\| > r.$$

Observing that

$$\left| \frac{\|\mathbf{w} - \bar{\mathbf{u}}\|}{\|\mathbf{w}\|} - 1 \right| = \frac{\big| \|\mathbf{w} - \bar{\mathbf{u}}\| - \|\mathbf{w}\| \big|}{\|\mathbf{w}\|} \le \frac{\|\bar{\mathbf{u}}\|}{\|\mathbf{w}\|} \to 0 \quad \text{as} \quad \|\mathbf{w}\| \to +\infty, \tag{3.11}$$

one infers that

$$\liminf_{\mathbf{w} \in K, \|\mathbf{w}\| \to +\infty} \frac{\langle \mathbf{h}(\mathbf{w}), \mathbf{w} - \bar{\mathbf{u}} \rangle}{\|\mathbf{w} - \bar{\mathbf{u}}\|} = \liminf_{\mathbf{w} \in K, \|\mathbf{w}\| \to +\infty} \frac{\langle \mathbf{h}(\mathbf{w}), \mathbf{w} - \bar{\mathbf{u}} \rangle}{\|\mathbf{w}\|^2} \frac{\|\mathbf{w}\|}{\|\mathbf{w} - \bar{\mathbf{u}}\|} \|\mathbf{w}\| = +\infty.$$

Let $r > \|\bar{\mathbf{u}}\|$ be such that

$$\frac{\langle \mathbf{h}(\mathbf{w}), \mathbf{w} - \bar{\mathbf{u}} \rangle}{\|\mathbf{w} - \bar{\mathbf{u}}\|} > \|\mathbf{p}\| \quad \text{for each} \quad \mathbf{w} \in K \quad \text{with} \quad \|\mathbf{w}\| > r.$$

For any such $\mathbf{w}$, we have

$$\begin{aligned} \langle \mathbf{h}(\mathbf{w}) + \mathbf{p}, \mathbf{w} - \bar{\mathbf{u}} \rangle &= \langle \mathbf{h}(\mathbf{w}), \mathbf{w} - \bar{\mathbf{u}} \rangle + \langle \mathbf{p}, \mathbf{w} - \bar{\mathbf{u}} \rangle \\ &> \|\mathbf{p}\| \, \|\mathbf{w} - \bar{\mathbf{u}}\| - \|\mathbf{p}\| \, \|\mathbf{w} - \bar{\mathbf{u}}\| = 0. \end{aligned}$$

Hence, $\mathbf{S}(\mathbf{p})$ is non-empty.

To prove the rest, suppose that there is no $\varrho > 0$ such that (3.10) is valid. Find sequences $(\mathbf{p}_k)_{k \in \mathbb{N}}$ and $(\mathbf{u}_k)_{k \in \mathbb{N}}$ in $\mathbb{R}^m$ such that

$$\|\mathbf{u}_k\| > k(1 + \|\mathbf{p}_k\|) \quad \text{and} \quad \mathbf{u}_k \in \mathbf{S}(\mathbf{p}_k) \quad \text{for each} \quad k \in \mathbb{N}.$$

Then $\|\mathbf{u}_k\| \to +\infty$ as $k \to +\infty$. Since all $\mathbf{u}_k$ together with $\bar{\mathbf{u}}$ lie in K, we have

$$\langle \mathbf{p}_k + \mathbf{h}(\mathbf{u}_k), \bar{\mathbf{u}} - \mathbf{u}_k \rangle \ge 0 \quad \text{for each} \quad k \in \mathbb{N}.$$

Since $\|\mathbf{p}_k\| / \|\mathbf{u}_k\| \to 0$ as $k \to +\infty$, the above inequality and (3.11) imply that

$$\limsup_{k \to +\infty} \frac{\langle \mathbf{h}(\mathbf{u}_k), \mathbf{u}_k - \bar{\mathbf{u}} \rangle}{\|\mathbf{u}_k\|^2} \le \limsup_{k \to +\infty} \frac{\langle \mathbf{p}_k, \bar{\mathbf{u}} - \mathbf{u}_k \rangle}{\|\mathbf{u}_k\|^2} \le \limsup_{k \to +\infty} \frac{\|\mathbf{p}_k\|}{\|\mathbf{u}_k\|} \frac{\|\mathbf{u}_k - \bar{\mathbf{u}}\|}{\|\mathbf{u}_k\|} = 0,$$

a contradiction. □

Theorem 3.9. *Consider a DVI* (3.5) – (3.7) *under the assumptions (A1) – (A3) with* $D := [a, b] \times \mathbb{R}^n$*. Given* $(a, \mathbf{x}_a) \in D$*, the mapping* $\mathbf{F}$ *defined in* (3.8) *satisfies all assumptions of Theorem 3.6.*

Proof. Fix any $(t, \mathbf{x}) \in D$. Then Proposition 3.8 implies that $\mathbf{S}(\mathbf{g}(t, \mathbf{x}))$ is non-empty, closed, and convex; hence so is the set $\mathbf{F}(t, \mathbf{x})$ because $\mathbf{B}(t, \mathbf{x})$ is a fixed matrix in $\mathbb{R}^{n \times m}$. Let $\mathbf{v} \in \mathbf{F}(t, \mathbf{x})$ be arbitrary. Then

$$\mathbf{v} = \mathbf{f}(t, \mathbf{x}) + \mathbf{B}(t, \mathbf{x}) \mathbf{u} \quad \text{for some} \quad \mathbf{u} \in \mathbf{S}(\mathbf{g}(t, \mathbf{x})).$$

This, (A1), (A2), and (3.10) reveal that, setting $\tilde{\beta} = \beta + \sigma\varrho(1+\beta)$ and $\tilde{\alpha} = \alpha(1+\sigma\varrho)$, one has

$$\begin{aligned} \|\mathbf{v}\| &\leq \|\mathbf{f}(t,\mathbf{x})\| + \|\mathbf{B}(t,\mathbf{x})\|\|\mathbf{u}\| \leq \alpha\|\mathbf{x}\| + \beta + \sigma\|\mathbf{u}\| \\ &\leq \alpha\|\mathbf{x}\| + \beta + \sigma\varrho(1 + \|\mathbf{g}(t,\mathbf{x})\|) \leq \alpha\|\mathbf{x}\| + \beta + \sigma\varrho(1 + \alpha\|\mathbf{x}\| + \beta) \\ &= \tilde{\alpha}\|\mathbf{x}\| + \tilde{\beta}. \end{aligned}$$

In particular, $\mathbf{F}$ has a linear growth on D. In view of Remark 3.7, to prove outer semi-continuity of $\mathbf{F}$, it suffices to show that $\operatorname{gph}\mathbf{F}$ is closed. Let $((t_k, \mathbf{x}_k))_{k\in\mathbb{N}}$ be a sequence in D converging to $(t,\mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n$ and $(\mathbf{v}_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^n$ converging to $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{v}_k \in \mathbf{F}(t_k,\mathbf{x}_k)$ for each $k \in \mathbb{N}$. We have to show that $\mathbf{v} \in \mathbf{F}(t,\mathbf{x})$. As D is closed, it contains $(t,\mathbf{x})$. Let $(\mathbf{u}_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^m$ such that

$$\mathbf{v}_k = \mathbf{f}(t_k,\mathbf{x}_k) + \mathbf{B}(t_k,\mathbf{x}_k)\mathbf{u}_k \quad \text{and} \quad \mathbf{u}_k \in \mathbf{S}(\mathbf{g}(t_k,\mathbf{x}_k)) \quad \text{for each } k \in \mathbb{N}. \tag{3.12}$$

Combining (3.10) and (A2), for each $k \in \mathbb{N}$, one has

$$\|\mathbf{u}_k\| \leq \varrho(1 + \|\mathbf{g}(t_k,\mathbf{x}_k)\|) \leq \varrho(1 + \alpha\|\mathbf{x}_k\| + \beta).$$

As $(\mathbf{x}_k)_{k\in\mathbb{N}}$ is bounded, so is $(\mathbf{u}_k)_{k\in\mathbb{N}}$. Find an infinite subset N of $\mathbb{N}$ such that $(\mathbf{u}_k)_{k\in N}$ converges, to $\mathbf{u} \in \mathbb{R}^m$ say. Fix any $\mathbf{w} \in K$. Note that

$$\langle \mathbf{g}(t_k,\mathbf{x}_k) + \mathbf{h}(\mathbf{u}_k), \mathbf{w} - \mathbf{u}_k\rangle \geq 0 \quad \text{for each} \quad k \in \mathbb{N}.$$

Passing to the limit as $N \ni k \to +\infty$ and using the continuity of both $\mathbf{g}$ and $\mathbf{h}$, we arrive at

$$\langle \mathbf{g}(t,\mathbf{x}) + \mathbf{h}(\mathbf{u}), \mathbf{w} - \mathbf{u}\rangle \geq 0.$$

Since $\mathbf{w}$ was an arbitrary element of K, we conclude that $\mathbf{u} \in \mathbf{S}(\mathbf{g}(t,\mathbf{x}))$. By the equality in (3.12), we have $\mathbf{v} = \mathbf{f}(t,\mathbf{x}) + \mathbf{B}(t,\mathbf{x})\mathbf{u}$ thanks to the continuity of $\mathbf{f}$ and $\mathbf{B}$. This means that $\mathbf{v} \in \mathbf{F}(t,\mathbf{x})$. □

Taking into account the above statement, we proved that, for any $\mathbf{x}_a \in \mathbb{R}^n$, there is a solution $\mathbf{x} : [a,b] \to \mathbb{R}^n$ of DI (3.9) with $\mathbf{F}$ defined in (3.8), which means that

(i) $\mathbf{x}(\cdot)$ is absolutely continuous on $[a,b]$;
(ii) $\dot{\mathbf{x}}(t) \in \mathbf{F}(t,\mathbf{x}(t))$ for almost all $t \in (a,b)$;
(iii) $\mathbf{x}(a) = \mathbf{x}_a$.

Now, we have to show that there is an integrable $\mathbf{u} : [a,b] \to \mathbb{R}^n$ such that the pair $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ solves DVI (3.5) – (3.7). We need a measurable selection lemma [24, Lemma 6.3].

Lemma 3.10. *Suppose that*

(i) $\mathbf{U} : D \rightrightarrows \mathbb{R}^m$ *has closed graph and there is* $\nu > 0$ *such that*

$$\|\mathbf{u}\| \leq \nu(1 + \|\mathbf{x}\|) \qquad \textit{whenever} \quad (t,\mathbf{x}) \in D \quad \textit{and} \quad \mathbf{u} \in \mathbf{U}(t,\mathbf{x});$$

(ii) *a continuous* $\mathbf{x} : [a,b] \to \mathbb{R}^n$*, a measurable* $\mathbf{v} : [a,b] \to \mathbb{R}^n$*, and a continuous* $\mathbf{o} : D \times \mathbb{R}^m \to \mathbb{R}^n$ *are such that*

$$\mathbf{v}(t) \in \mathbf{o}\big(t, \mathbf{x}(t), \mathbf{U}(t,\mathbf{x}(t))\big) \quad \textit{for almost all} \quad t \in [a,b].$$

Then there is a measurable $\mathbf{u} : [a,b] \to \mathbb{R}^m$ *such that*

$$\mathbf{u}(t) \in \mathbf{U}(t,\mathbf{x}(t)) \quad \textit{and} \quad \mathbf{v}(t) = \mathbf{o}\big(t, \mathbf{x}(t), \mathbf{u}(t)\big) \quad \textit{for almost all} \quad t \in [a,b].$$

Theorem 3.11. *Consider a DVI* (3.5) – (3.7) *under the assumptions (A1) – (A3) with* $D := [a,b] \times \mathbb{R}^n$*. Given* $(a, \mathbf{x}_a) \in D$*, there is an absolutely continuous* $\mathbf{x} : [a,b] \to \mathbb{R}^n$ *and an integrable* $\mathbf{u} : [a,b] \to \mathbb{R}^m$ *solving* (3.5) – (3.7) *with* $\mathbf{x}(a) = \mathbf{x}_a$.

Proof. Clearly, a single-valued mapping

$$D \times \mathbb{R}^m \ni (t, \mathbf{x}, \mathbf{u}) \longmapsto \mathbf{o}(t, \mathbf{x}, \mathbf{u}) := \mathbf{f}(t, \mathbf{x}) + \mathbf{B}(t, \mathbf{x})\mathbf{u} \in \mathbb{R}^n$$

is continuous. As implicitly showed in the proof of Theorem 3.9, a set-valued mapping

$$D \ni (t, \mathbf{x}) \quad \rightrightarrows \quad \mathbf{U}(t, \mathbf{x}) := \mathbf{S}(\mathbf{g}(t, \mathbf{x})) \subset \mathbb{R}^m$$

has closed graph and a linear growth in D. We already know, that there is an absolutely continuous function $\mathbf{x} : [a,b] \to \mathbb{R}^n$ such that

$$\dot{\mathbf{x}}(t) \in \mathbf{F}(t, \mathbf{x}(t)) = \mathbf{o}\big(t, \mathbf{x}, \mathbf{U}(t, \mathbf{x}(t))\big) \quad \text{for almost all} \quad t \in [a,b].$$

Let $\mathbf{v}(\cdot) := \dot{\mathbf{x}}(\cdot)$. Proposition 3.10 provides a measurable $\mathbf{u} : [a,b] \to \mathbb{R}^m$ such that

$$\mathbf{u}(t) \in \mathbf{S}(\mathbf{g}(t, \mathbf{x})) \quad \text{and} \quad \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t)) + \mathbf{B}(t, \mathbf{x}(t))\mathbf{u}(t) \quad \text{for almost all } t \in [a,b].$$

Denote by M the maximum of $\|\mathbf{x}(\cdot)\|$ on $[a,b]$ and let $\varrho > 0$ be as in (3.10). The linear growth of $\mathbf{g}$ implies that, for almost all $t \in [a,b]$, one has

$$\|\mathbf{u}(t)\| \underset{(3.10)}{\leq} \varrho(1 + \|\mathbf{g}(t, \mathbf{x}(t))\|) \leq \varrho(1 + \beta + \alpha\|\mathbf{x}(t)\|) \leq \varrho(1 + \beta + \alpha M).$$

So $\mathbf{u}$ is integrable on $[a,b]$, and hence the pair $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ solves (3.5) – (3.7). $\square$

3.3. **Mechanical Problems with Friction.** In this section, a consideration on mechanical applications is continued. Namely, we add friction in the model.

Example 3.12. Consider a body of a mass $m > 0$ being dragged across a rough surface by a time-dependent external force $f(t)$ (see Figure 15). Denote by $q(t)$ the position of the center of mass at time $t > 0$. Then the *Coulomb friction model* says that a friction force is equal to $\mu f_N \operatorname{sgn}(\dot{q}(t))$, where

- $\mu > 0$ is a friction coefficient depending on a material which the surface is made of;
- $f_N = mg$ is the normal force and $g = 9.81$ m/s^2;
- $$\operatorname{sgn} x := \begin{cases} 1 & \text{for} \quad x > 0, \\ 0 & \text{for} \quad x = 0, \\ -1 & \text{otherwise.} \end{cases}$$

Let $x(t) := \dot{q}(t)$ be the velocity at time t. The Newton's law of motion says that

$$m\dot{x}(t) = \underbrace{f(t) - mg\mu \operatorname{sgn} x(t)}_{\text{the sum of external forces}}.$$

Clearly, it may happen that $q(t)$ remains constant for t in some interval $[t_1, t_2] \subset \mathbb{R}_+$ although $f(t) \neq 0$ for each $t \in [t_1, t_2]$. Thus $\dot{x}(t) = x(t) = 0$ whenever $t \in (t_1, t_2)$. But no classical solution can satisfy this. One can handle this issue by considering a differential inclusion instead of the above differential equation in form

$$\dot{x}(t) \in \frac{1}{m} f(t) - G\big(x(t)\big), \quad t > 0, \tag{3.13}$$

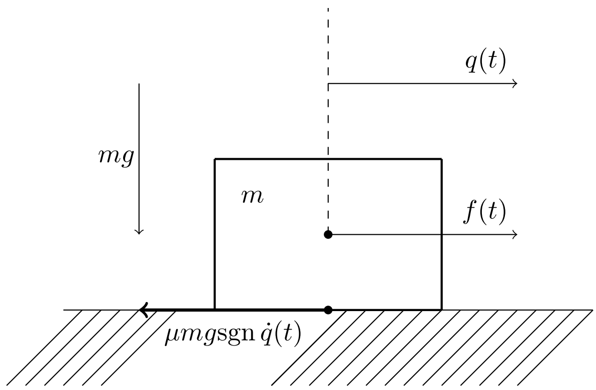

FIGURE 15. Illustration of Example 3.12.

where

$$G(x) := \begin{cases} g\mu & \text{for} \quad x > 0, \\ [-g\mu, g\mu] & \text{for} \quad x = 0, \\ -g\mu & \text{otherwise.} \end{cases}$$

Setting $F(t,x) := f(t)/m - G(x)$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, one gets

$$\begin{cases} \dot{x}(t) & \in & F(t, x(t)) \quad \text{for} \quad t > 0, \\ x(0) & = & x_0, \end{cases}$$

with $x_0 \in \mathbb{R}$ being the initial velocity. Suppose that f is continuous. As G is monotone, globally bounded, has closed graph and also non-empty closed convex bounded values, F satisfies the assumptions of Theorem 3.1.

Note that $G = \partial_C \varphi$, where $\varphi := g\mu|\cdot|$. So G is the Clarke's sub-differential of a (globally) Lispchitz function. More precisely, as φ is convex, its Clarke's sub-differential coincides with the *Fenchel-Moreau-Rockafellar* sub-differential from convex analysis. The non-smooth and, in general, non-convex function φ is called *Moreau-Panagiotopoulos super-potential* because it generalizes the notion of potential energy used in the classical physics which is usually assumed to be smooth.

There are various (equivalent) formulations of a particular problem. For example, Filippov's inclusions can be formulated as differential variational inequalities or even as differential linear complementarity problems.

Example 3.13. Consider the inclusion (3.13). First, let us reformulate this problem as a DVI. Set $K := [-\mu g, +\mu g]$. Then, for (almost) all $t > 0$, we obtain

$$\begin{aligned} \dot{x}(t) &= \frac{1}{m} f(t) + u, \\ 0 &\leq x(t)\big(v - u(t)\big) \quad \text{whenever} \quad v \in K, \\ u(t) &\in K. \end{aligned}$$

Indeed, $-u \in G(x)$ if and only if $x \in G^{-1}(-u) = -N_K(u)$.

To obtain an equivalent differential linear complementarity problem, write $x = x^+ - x^-$. Then $G(x) = g\mu - y$ with y satisfying

$$0 \leq y \perp x^+ \geq 0 \quad \text{and} \quad 0 \leq 2g\mu - y \perp x^- \geq 0.$$

When also the impacts come into play, then without a normal compliance approach, we arrive at the framework of *measure differential inclusions*.

4. Numerical Methods for DVIs

4.1. **Time-stepping Schemes.** Throughout this chapter, an interval $I := [a, b]$, a point $\mathbf{x}_a \in \mathbb{R}^n$ and $N \in \mathbb{N}$ are given. For simplicity, we consider the uniform grid with the step-size

$$h := \frac{b-a}{N} > 0$$

defined by

$$t_k = a + kh \quad \text{with} \quad k = 0, 1, 2 \ldots, N.$$

For any function $\mathbf{w} : I \to \mathbb{R}^d$, we set

$$\mathbf{w}_k = \mathbf{w}(t_k) \quad \text{for each} \quad k = 0, 1, 2 \ldots, N.$$

In the previous chapter, we focused on the possibility to reformulate a differential variational inequality as a differential inclusion. Let us start with a brief discussion on numerical schemes for solving the latter problem.

Given a set-valued mapping $\mathbf{F} : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, we want to find an absolutely continuous function $\mathbf{x}(\cdot) : I \to \mathbb{R}^n$ which solves the initial value problem

$$\begin{cases} \dot{\mathbf{x}}(t) & \in & \mathbf{F}(t, \mathbf{x}(t)) \quad \text{for almost all} \quad t \in I, \\ \mathbf{x}(a) & = & \mathbf{x}_a. \end{cases} \tag{4.1}$$

Denote by $\mathcal{X}$ the set of solutions to (4.1) which, in general, consists of more than one element or can even be empty. There are various approaches to approximating solutions $\mathbf{x}(\cdot) \in \mathcal{X}$. Using a finite difference scheme together with a suitable selection procedure, we want to obtain a sequence $(\mathbf{y}_N(\cdot))_{N\in\mathbb{N}}$ of grid functions (piece-wise linear for example) such that one can choose a sub-sequence of it which converges to a solution $\mathbf{x}(\cdot) \in \mathcal{X}$. In the proof of Theorem 3.1, we have seen the *explicit (forward) Euler scheme.*

Example 4.1. Put $\mathbf{y}_0 = \mathbf{x}_a$ and compute $\mathbf{y}_{k+1}$ from

$$\mathbf{y}_{k+1} \in \mathbf{y}_k + h\mathbf{F}(t_k, \mathbf{y}_k) \quad \text{for} \quad k = 0, 1 \ldots, N-1.$$

The above algorithm can be for $k = 0, 1, \ldots, N-1$ written in the following way

$$\begin{cases} \mathbf{y}_{k+1} - \mathbf{y}_k = h\mathbf{v}_k, \\ \mathbf{v}_k \in \mathbf{F}(t_k, \mathbf{y}_k). \end{cases}$$

At each step, one has to choose an element $\mathbf{v}_k$ in the set $\mathbf{F}(t_k, \mathbf{y}_k)$, which is called a *selection procedure.*

Another way how to proceed is an *implicit (backward) Euler scheme.*

Example 4.2. Put $\mathbf{y}_0 = \mathbf{x}_a$ and, for each $k = 0, 1 \ldots, N-1$, compute $\mathbf{y}_{k+1}$ from

$$\mathbf{y}_{k+1} \in \mathbf{y}_k + h\mathbf{F}(t_{k+1}, \mathbf{y}_{k+1})$$

or equivalently

$$\begin{cases} \mathbf{y}_{k+1} - \mathbf{y}_k = h\mathbf{v}_{k+1}, \\ \mathbf{v}_{k+1} \in \mathbf{F}(t_{k+1}, \mathbf{y}_{k+1}). \end{cases}$$

In both the cases, the corresponding piece-wise linear approximation (a broken line joining the points $\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_N$) is given by

$$\mathbf{y}_N(t) = \mathbf{y}_k + \frac{1}{h}(t - t_k)(\mathbf{y}_{k+1} - \mathbf{y}_k), \; t \in [t_k, t_{k+1}] \quad \text{for each} \quad k = 0, 1, \dots, N-1.$$

The latter scheme is more stable in sense that it prevents oscillations around discontinuity surfaces. However, in general, there is no hope to solve the implicit inclusion effectively. One can also use a mixture of the previous schemes.

We have seen that differential variational inequalities form a special subclass of differential inclusions and one can profit from their special structure. To illustrate this, consider matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{m\times n}$, $\mathbf{D} \in \mathbb{R}^{m\times m}$ along with the corresponding differential linear complementarity problem which reads as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \\ \mathbf{0} &\preceq \mathbf{y}(t) \perp \mathbf{u}(t) \succeq \mathbf{0}, \qquad \text{for} \quad t \in I, \end{aligned}$$

where $\mathbf{x}(\cdot) : \mathbb{R} \to \mathbb{R}^n$, $\mathbf{y}(\cdot) : \mathbb{R} \to \mathbb{R}^m$, and $\mathbf{u}(\cdot) : \mathbb{R} \to \mathbb{R}^m$. The backward Euler scheme reads as

$$\begin{aligned} \frac{\mathbf{x}_{k+1} - \mathbf{x}_k}{h} &= \mathbf{A}\mathbf{x}_{k+1} + \mathbf{B}\mathbf{u}_{k+1} \\ \mathbf{y}_{k+1} &= \mathbf{C}\mathbf{x}_{k+1} + \mathbf{D}\mathbf{u}_{k+1} \\ \mathbf{0} \preceq \mathbf{y}_{k+1} &\perp \mathbf{u}_{k+1} \succeq \mathbf{0}. \end{aligned}$$

For h small enough, the matrix $\mathbf{I}_n - h\mathbf{A}$ is non-singular. Computing $\mathbf{x}_{k+1}$ from the first relation and plugging it into the latter ones, one infers that the next step $\mathbf{u}_{k+1}$ has to solve a standard complementarity problem

$$\mathbf{0} \preceq \mathbf{u}_{k+1} \perp \underbrace{\mathbf{C}(\mathbf{I}_n - h\mathbf{A})^{-1}\mathbf{x}_k}_{\mathbf{q}_k} + \underbrace{(h\mathbf{C}(\mathbf{I} - h\mathbf{A})^{-1}\mathbf{B} + \mathbf{D})}_{\mathbf{M}}\mathbf{u}_{k+1} \succeq \mathbf{0}.$$

Thanks to Proposition 2.15, the necessary and sufficient condition guaranteeing the existence of a unique $\mathbf{u}_{k+1}$ is that $\mathbf{M}$ is a P-matrix.

Let $\theta \in [0, 1]$ be given. A combination of the forward and the backward Euler method is called *Moreau's time-stepping scheme* which reads as

$$\begin{aligned} \frac{\mathbf{x}_{k+1} - \mathbf{x}_k}{h} &= \mathbf{A}\mathbf{x}_{k+\theta} + \mathbf{B}\mathbf{u}_{k+1}, \\ \mathbf{y}_{k+1} &= \mathbf{C}\mathbf{x}_{k+1} + \mathbf{D}\mathbf{u}_{k+1} \\ \mathbf{0} \preceq \mathbf{y}_{k+1} &\perp \mathbf{u}_{k+1} \succeq \mathbf{0}, \end{aligned}$$

where $\mathbf{x}_{k+\theta} := \theta\mathbf{x}_{k+1} + (1-\theta)\mathbf{x}_k$. Again, for h small enough, the matrix $\mathbf{I}_n - h\theta\mathbf{A}$ is non-singular. Setting

$$\mathbf{W} = (\mathbf{I}_n - h\theta\mathbf{A})^{-1},$$

one sees that the next step $\mathbf{u}_{k+1}$ solves

$$\mathbf{0} \preceq \mathbf{u}_{k+1} \perp \underbrace{\mathbf{C}\mathbf{W}\Big(\mathbf{I}_n + h(1-\theta)\mathbf{A}\Big)\mathbf{x}_k}_{\mathbf{q}_k} + \underbrace{(h\mathbf{C}\mathbf{W}\mathbf{B} + \mathbf{D})}_{\mathbf{M}}\mathbf{u}_{k+1} \succeq \mathbf{0}.$$

Having $\mathbf{u}_{k+1}$ in hand, we compute the new state $\mathbf{x}_{k+1}$ by

$$\mathbf{x}_{k+1} = \mathbf{W}\Big(\mathbf{I}_n + h(1-\theta)\mathbf{A}\Big)\mathbf{x}_k + h\mathbf{W}\mathbf{B}\mathbf{u}_{k+1}.$$

The above discussion reveals that, at each step, one has for given $\mathbf{q} \in \mathbb{R}^m$ find a solution $\mathbf{u} \in \mathbb{R}^m$ of

$$\mathbf{0} \preceq \mathbf{u} \perp \mathbf{q} + \mathbf{M}\mathbf{u} \succeq \mathbf{0}, \tag{4.2}$$

where the matrix $\mathbf{M} \in \mathbb{R}^{m\times m}$ depends on a particular choice of the scheme. We also know, by Proposition 1.5, that this is equivalent to the generalized equation

$$\mathbf{0} \in \mathbf{M}\mathbf{u} + \mathbf{q} + N_{\mathbb{R}^m_+}(\mathbf{u}).$$

4.2. **Newton's Method for Non-smooth Equations.** In view of Lemma 1.2 (iii), using a projection mapping, we are able to transform (4.2) into an equation

$$\mathbf{h}(\mathbf{u}) = \mathbf{0}, \tag{4.3}$$

with a non-smooth $\mathbf{h}(\mathbf{u}) := \mathbf{p}_{\mathbb{R}^m_+}(\mathbf{u} - \mathbf{M}\mathbf{u} - \mathbf{q}) - \mathbf{u}$, $\mathbf{u} \in \mathbb{R}^m$. Nevertheless, we show one of many other possible choices of $\mathbf{h}$ which is frequently used in the literature. Recall, that *Fischer-Burmeister function* $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\varphi(\mathbf{x}) = \sqrt{{x_1}^2 + {x_2}^2} - x_1 - x_2 = \|\mathbf{x}\| - \langle (1,1)^T, \mathbf{x}\rangle, \quad \mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2. \tag{4.4}$$

This function is an example of the so-called *complementarity function/C-function.*

Lemma 4.3. *The function* φ *in* (4.4) *is*

(i) *continuously differentiable off the origin with*

$$\nabla\varphi(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|} - \begin{pmatrix}1\\1\end{pmatrix}, \quad \mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\};$$

(ii) *Lipschitz continuous on* $\mathbb{R}^2$*;*

(iii) $\varphi(\mathbf{x}) = 0$ *if and only if* $\quad 0 \le x_1 \perp x_2 \ge 0$.

Proof. (i) follows from elementary calculus. To prove (ii), fix any two distinct $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^2$. The triangle and Cauchy-Schwarz inequality imply that

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| = \big|\|\mathbf{x}\| - \|\mathbf{y}\| - \langle (1,1)^T, \mathbf{x} - \mathbf{y}\rangle\big| \le \|\mathbf{x} - \mathbf{y}\| + \sqrt{2}\|\mathbf{x} - \mathbf{y}\|.$$

To see (iii) we have the following chain of equivalences:

$$\begin{aligned}
\varphi(\mathbf{x}) = 0 \quad &\Leftrightarrow \quad \sqrt{{x_1}^2 + {x_2}^2} = x_1 + x_2 \\
&\Leftrightarrow \quad x_1 + x_2 \ge 0 \quad \text{and} \quad {x_1}^2 + {x_2}^2 = {x_1}^2 + 2x_1x_2 + {x_2}^2 \\
&\Leftrightarrow \quad x_1 + x_2 \ge 0 \quad \text{and} \quad x_1x_2 = 0.
\end{aligned}$$

□

This simple statement enables us write (4.2) as (4.3) with

$$\mathbf{h}(\mathbf{u}) := \begin{pmatrix} \varphi\big(u_1, (\mathbf{q} + \mathbf{M}\mathbf{u})_1\big) \\ \varphi\big(u_2, (\mathbf{q} + \mathbf{M}\mathbf{u})_2\big) \\ \vdots \\ \varphi\big(u_m, (\mathbf{q} + \mathbf{M}\mathbf{u})_m\big) \end{pmatrix}, \quad \mathbf{u} \in \mathbb{R}^m. \tag{4.5}$$

At the end of Section 2.4, a definition of the Bouligand's limiting Jacobian of a mapping $\mathbf{h} : \mathbb{R}^m \to \mathbb{R}^d$ which is locally Lipschitz continuous at $\bar{\mathbf{u}} \in \mathbb{R}^m$ was mentioned. Recall, that this is the set $\partial_B \mathbf{h}(\bar{\mathbf{u}})$ consisting of all matrices $\mathbf{A} \in \mathbb{R}^{d\times m}$ for which there is a sequence $(\mathbf{u}_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^m$ converging to $\bar{\mathbf{u}}$ such that $\mathbf{h}$ is differentiable at each $\mathbf{u}_k$ and $\nabla \mathbf{h}(\mathbf{u}_k) \to \mathbf{A}$ as $k \to +\infty$. Based on this, Clarke's generalized Jacobian $\partial_C \mathbf{h}(\bar{\mathbf{u}})$ of $\mathbf{h}$ at $\bar{\mathbf{u}}$ is the convex hull of $\partial_B \mathbf{h}(\bar{\mathbf{u}})$.

Example 4.4.

$$\partial_B \|\cdot\|(\mathbf{u}) = \begin{cases} \left\{ \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\}, & \text{if} \quad \mathbf{u} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}, \\ \mathbb{S} := \mathbb{B}[\mathbf{0}, 1] \setminus \mathbb{B}(\mathbf{0}, 1), & \text{if} \quad \mathbf{u} = \mathbf{0}, \end{cases}$$

and

$$\partial_C \|\cdot\|(\mathbf{u}) = \begin{cases} \left\{ \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\}, & \text{if} \quad \mathbf{u} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}, \\ \mathbb{B}[\mathbf{0}, 1], & \text{if} \quad \mathbf{u} = \mathbf{0}. \end{cases}$$

Indeed, the first expressions for $\partial_B \|\cdot\|(\mathbf{u})$ and $\partial_C \|\cdot\|(\mathbf{u})$ follow from the fact that $\nabla \|\cdot\|(\mathbf{u}) = \mathbf{u}/\|\mathbf{u}\|$ at any non-zero $\mathbf{u} \in \mathbb{R}^2$. To see the latter ones, note that $\partial_B \|\cdot\|(\mathbf{0}) \subset \mathbb{S}$ because $\nabla \|\cdot\|(\mathbf{u}) \in \mathbb{S}$ whenever $\mathbf{u} \neq \mathbf{0}$ and $\mathbb{S}$ is closed. Let $\mathbf{a} \in \mathbb{S}$ be arbitrary. Then $\mathbf{u}_k := \mathbf{a}/k$, $k \in \mathbb{N}$, is a sequence converging to $\mathbf{0}$ and

$$\nabla \|\cdot\|(\mathbf{u}_k) = \frac{\mathbf{a}/k}{\|\mathbf{a}\|/k} = \mathbf{a}, \quad \text{for each} \quad k \in \mathbb{N}.$$

Thus $\mathbf{a} \in \partial_B \|\cdot\|(\mathbf{0})$. Hence $\partial_B \|\cdot\|(\mathbf{0}) = \mathbb{S}$, and consequently $\partial_C \|\cdot\|(\mathbf{0}) = \mathbb{B}[0, 1]$.

Let us gather several fundamental properties of the objects above.

Proposition 4.5. *Suppose that a function $\mathbf{h} : \mathbb{R}^m \to \mathbb{R}^d$ is locally Lipschitz continuous and let $* \in \{B, C\}$. Then*

(i) *if $\mathbf{h}$ is continuously differentiable at $\mathbf{u} \in \mathbb{R}^m$, then $\partial_* \mathbf{h}(\mathbf{u}) = \{\nabla \mathbf{h}(\mathbf{u})\}$;*

(ii) *if $\mathbf{h} = \mathbf{h}_1 + \mathbf{h}_2$ for a continuously differentiable $\mathbf{h}_1 : \mathbb{R}^m \to \mathbb{R}^d$ and a locally Lipschitz continuous $\mathbf{h}_2 : \mathbb{R}^m \to \mathbb{R}^d$, then*

$$\partial_* \mathbf{h}(\mathbf{u}) = \nabla \mathbf{h}_1(\mathbf{u}) + \partial_* \mathbf{h}_2(\mathbf{u}) \quad \textit{for each} \quad \mathbf{u} \in \mathbb{R}^m;$$

(iii) *$\partial_* \mathbf{h}(\mathbf{u})$ is non-empty and compact for each $\mathbf{u} \in \mathbb{R}^m$;*

(iv) *$\operatorname{gph}(\partial_* \mathbf{h})$ is closed;*

(v) *$\partial_* \mathbf{h}$ is locally bounded;*

(vi) *$\partial_* \mathbf{h}$ is Pompeiu-Hausdorff outer semi-continuous.*

Proof. Fix any $\mathbf{u} \in \mathbb{R}^m$ for a longer while. By the very definition, (i) is valid. To show (ii), (iii), and (v), find positive constants $\delta_\mathbf{u}$ and $L_\mathbf{u}$ such that

$$\|\mathbf{h}(\mathbf{v}) - \mathbf{h}(\mathbf{w})\| \le L_\mathbf{u} \|\mathbf{v} - \mathbf{w}\| \quad \text{whenever} \quad \mathbf{v}, \mathbf{w} \in \mathbb{B}(\mathbf{u}, 2\delta_\mathbf{u}).$$

Let D be a set of points $\mathbf{v} \in \mathbb{B}(\mathbf{u}, 2\delta_\mathbf{u})$ at which $\mathbf{h}$ is differentiable (Rademacher's theorem says that this set is dense $\mathbb{B}(\mathbf{u}, 2\delta_\mathbf{u})$).

First, we `claim` that

$$\|\nabla \mathbf{h}(\mathbf{v})\| \le 1 + L_\mathbf{u} \quad \text{whenever} \quad \mathbf{v} \in \mathbb{B}(\mathbf{u}, \delta_\mathbf{u}) \cap D.$$

Fix any such a point $\mathbf{v}$. Find $r \in (0, \delta_\mathbf{u})$ such that

$$\|\mathbf{h}(\mathbf{v} + \mathbf{y}) - \mathbf{h}(\mathbf{v}) - \nabla \mathbf{h}(\mathbf{v})\mathbf{y}\| \le \|\mathbf{y}\| \quad \text{for each} \quad \mathbf{y} \in \mathbb{B}[\mathbf{0}, r].$$

Since both $\mathbf{v}+\mathbf{y}$ and $\mathbf{v}$ lie in $\mathbb{B}(\mathbf{u}, 2\delta_{\mathbf{u}})$, for any $\mathbf{y} \in \mathbb{B}[\mathbf{0}, r]$, one has that

$$\begin{aligned} \|\nabla\mathbf{h}(\mathbf{v})\mathbf{y}\| &\le \|\mathbf{h}(\mathbf{v}+\mathbf{y}) - \mathbf{h}(\mathbf{v}) - \nabla\mathbf{h}(\mathbf{v})\mathbf{y}\| + \|\mathbf{h}(\mathbf{v}) - \mathbf{h}(\mathbf{v}+\mathbf{y})\| \\ &\le \|\mathbf{y}\| + L_{\mathbf{u}}\|\mathbf{y}\|. \end{aligned}$$

Hence, for any non-zero $\mathbf{w} \in \mathbb{R}^m$, taking $\mathbf{y} := \frac{r}{\|\mathbf{w}\|}\mathbf{w}$, we get

$$\|\nabla\mathbf{h}(\mathbf{v})\mathbf{w}\| = \frac{\|\mathbf{w}\|}{r}\|\nabla\mathbf{h}(\mathbf{v})\mathbf{y}\| \le \frac{\|\mathbf{w}\|}{r}(1+L_{\mathbf{u}})\|\mathbf{y}\| = (1+L_{\mathbf{u}})\|\mathbf{w}\|.$$

This proves the claim.

To show (iii), note that there exists a sequence $(\mathbf{u}_k)_{k\in\mathbb{N}}$ in D converging to $\mathbf{u}$ such that the corresponding sequence $(\|\nabla\mathbf{h}(\mathbf{u}_k)\|)_{k\in\mathbb{N}}$ is bounded. Thus $(\nabla\mathbf{h}(\mathbf{u}_k))_{k\in\mathbb{N}}$ has at least one cluster point, which lies in $\partial_B\mathbf{h}(\mathbf{u})$ by the very definition of this set. The claim also reveals that $\partial_B\mathbf{h}(\mathbf{u}) \subset \mathbb{B}[\mathbf{0}, 1+L_{\mathbf{u}}]$. Note that taking the closed convex hull does not change anything. The justification that $\partial_B\mathbf{h}(\mathbf{u})$ and $\partial_C\mathbf{h}(\mathbf{u})$ are closed is postponed since this follows directly from (iv).

Since the sum of a continuously differentiable function and a locally Lipschitz one is again locally Lipschitz, the standard sum rule for the derivatives yields (ii).

From the claim, we also get that

$$\bigcup_{\mathbf{v}\in\mathbb{B}(\mathbf{u},\delta_{\mathbf{u}})} \partial_B\mathbf{h}(\mathbf{v}) \subset \mathbb{B}[\mathbf{0}, 1+L_{\mathbf{u}}] \quad \text{and} \quad \bigcup_{\mathbf{v}\in\mathbb{B}(\mathbf{u},\delta_{\mathbf{u}})} \partial_C\mathbf{h}(\mathbf{v}) \subset \mathbb{B}[\mathbf{0}, 1+L_{\mathbf{u}}],$$

which proves (v).

To show (iv) for $* = B$, pick any sequence $(\mathbf{u}_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^m$ converging to $\mathbf{u} \in \mathbb{R}^m$ along with a sequence $(\mathbf{A}_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^{d\times m}$ converging to $\mathbf{A} \in \mathbb{R}^{d\times m}$ such that

$$\mathbf{A}_k \in \partial_B\mathbf{h}(\mathbf{u}_k) \quad \text{for each} \quad k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, find $\mathbf{v}_k \in D$ (where D is as above) such that

$$\|\mathbf{v}_k - \mathbf{u}_k\| < 1/k \quad \text{and} \quad \|\nabla\mathbf{h}(\mathbf{v}_k) - \mathbf{A}_k\| < 1/k.$$

Then $0 \le \|\mathbf{v}_k - \mathbf{u}\| \le \|\mathbf{v}_k - \mathbf{u}_k\| + \|\mathbf{u}_k - \mathbf{u}\| \to 0$ as $k \to +\infty$, and similarly, $0 \le \|\nabla\mathbf{h}(\mathbf{v}_k) - \mathbf{A}\| \le \|\nabla\mathbf{h}(\mathbf{v}_k) - \mathbf{A}_k\| + \|\mathbf{A}_k - \mathbf{A}\| \to 0$ as $k \to +\infty$. So $\mathbf{A} \in \partial_B\mathbf{h}(\mathbf{u})$.

As any set-valued mapping with closed graph has to have closed values, we get that $\partial_B\mathbf{h}$ has closed values. In view of the above consideration, the values of $\partial_B\mathbf{h}$ are compact. Since the convex hull of a compact set is always closed, we fully established (iii).

Let $* = B$. Then (vi) follows from (iv) and (v) by Lemma 3.2 (ii). Summarizing, we proved the whole statement for $* = B$ and (i), (ii), (iii), (v) for $* = C$. From now on assume that $* = C$.

Suppose that (vi) fails. Find $\mathbf{u} \in \mathbb{R}^m$ and $\varepsilon > 0$ such that for each $k \in \mathbb{N}$ there is $\mathbf{u}_k \in \mathbb{B}(\mathbf{u}, 1/k)$ such that

$$\partial_C\mathbf{h}(\mathbf{u}_k) \setminus \big(\partial_C\mathbf{h}(\mathbf{u}) + \mathbb{B}[\mathbf{0}, \varepsilon]\big) \ne \emptyset.$$

For each $k \in \mathbb{N}$, there is $\mathbf{A}_k \in \partial_B\mathbf{h}(\mathbf{u}_k)$ such that $\mathbf{A}_k \notin \partial_C\mathbf{h}(\mathbf{u}) + \mathbb{B}[\mathbf{0}, \varepsilon]$. Indeed, the set on the right-side is convex because it is the Minkowski sum of two convex sets. Hence, if $\partial_B\mathbf{h}(\mathbf{u}_k) \subset \partial_C\mathbf{h}(\mathbf{u}) + \mathbb{B}[\mathbf{0}, \varepsilon]$, then, as $\partial_C\mathbf{h}(\mathbf{u}_k)$ is the convex hull of $\partial_B\mathbf{h}(\mathbf{u}_k)$, one would obtain that

$$\partial_C\mathbf{h}(\mathbf{u}_k) \subset \partial_C\mathbf{h}(\mathbf{u}) + \mathbb{B}[\mathbf{0}, \varepsilon].$$

Since $(\mathbf{A}_k)_{k\in\mathbb{N}}$ is bounded by (v), passing to a sub-sequence, if necessary, we may assume that it converges to an $\mathbf{A} \in \mathbb{R}^{d\times m}$. Taking into account that $\partial_B \mathbf{h}$ has closed graph, one infers that $\mathbf{A} \in \partial_B \mathbf{h}(\mathbf{u}) \subset \partial_C \mathbf{h}(\mathbf{u})$. The choice of $\mathbf{A}_k$'s, implies that

$$0 = d\big(\mathbf{A}, \partial_C \mathbf{h}(\mathbf{u})\big) = \lim_{k\to+\infty} d\big(\mathbf{A}_k, \partial_C \mathbf{h}(\mathbf{u})\big) \geq \varepsilon > 0,$$

a contradiction. Therefore (vi) holds.

Lemma 3.2 (i) says that (vi) and (iii) imply (iv). □

Note that the sum rule in (ii) fails if both the functions are locally Lipschitz only. Indeed, is suffices to consider a function $h(u) := |u| + (-|u|) = 0$, $u \in \mathbb{R}$.

Example 4.6. A combination of Lemma 4.3 (i), Example 4.4, and Proposition 4.5 yields that, for the function φ in (4.4), one has

$$\partial_B \varphi(\mathbf{u}) = \begin{cases} \left\{ \frac{\mathbf{u}}{\|\mathbf{u}\|} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, & \text{if} \quad \mathbf{u} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}, \\ \mathbb{B}[(-1,-1)^T, 1] \setminus \mathbb{B}((-1,-1)^T, 1), & \text{if} \quad \mathbf{u} = \mathbf{0}; \end{cases}$$

and

$$\partial_C \varphi(\mathbf{u}) = \begin{cases} \left\{ \frac{\mathbf{u}}{\|\mathbf{u}\|} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, & \text{if} \quad \mathbf{u} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}, \\ \mathbb{B}[(-1,-1)^T, 1], & \text{if} \quad \mathbf{u} = \mathbf{0}. \end{cases}$$

We are going to use following assumptions:

(S1) $\mathbf{h} : \mathbb{R}^m \to \mathbb{R}^d$ is locally Lipschitz at $\bar{\mathbf{u}} \in \mathbb{R}^m$;

(S2) There is $\mathcal{H} : \mathbb{R}^m \rightrightarrows \mathbb{R}^{d\times m}$ such that

- (a) $\mathcal{H}(\bar{\mathbf{u}})$ is compact;
- (b) $\mathcal{H}$ is Pompeiu-Hausdorff outer semi-continuous at $\bar{\mathbf{u}}$, the interior point of $\operatorname{dom} \mathcal{H}$;
- (c)

$$\lim_{\mathbf{0}\neq\mathbf{v}\to\mathbf{0}} \frac{\sup_{\mathbf{A}\in\mathcal{H}(\bar{\mathbf{u}}+\mathbf{v})} \|\mathbf{h}(\bar{\mathbf{u}}+\mathbf{v}) - \mathbf{h}(\bar{\mathbf{u}}) - \mathbf{A}\mathbf{v}\|}{\|\mathbf{v}\|} = 0.$$

Example 4.7. For any $\bar{\mathbf{u}} \in \mathbb{R}^2$, the function φ in (4.4) satisfies (S2) for $\mathcal{H} := \partial_* \varphi$ with $* \in \{B, C\}$. In view of the previous consideration, it suffices to prove (S2) (c). If $\bar{\mathbf{u}}$ is non-zero, then φ is continuously differentiable at $\bar{\mathbf{u}}$. Which means that $\partial_B \varphi(\mathbf{u}) = \partial_C \varphi(\mathbf{u}) = \{\nabla \varphi(\mathbf{u})\}$ for any $\mathbf{u}$ in a vicinity of $\bar{\mathbf{u}}$. Then (S2) (c) holds trivially (see the steps in Example 2.14). Suppose that $\bar{\mathbf{u}} = \mathbf{0}$ and fix any non-zero $\mathbf{v} \in \mathbb{R}^2$. Then $\partial_C \varphi(\mathbf{v}) = \left\{ \frac{\mathbf{v}}{\|\mathbf{v}\|} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$. Thus

$$\sup_{\mathbf{a}\in\partial_C\varphi(\mathbf{v})} \|\varphi(\mathbf{v}) - \langle \mathbf{a}, \mathbf{v}\rangle\| = \left\| \|\mathbf{v}\| - \langle (1,1)^T, \mathbf{v}\rangle - \left(\frac{\langle \mathbf{v}, \mathbf{v}\rangle}{\|\mathbf{v}\|} - \langle (1,1)^T, \mathbf{v}\rangle \right) \right\| = 0.$$

Conditions (S1) and (S2) hold if $\mathbf{h}$ is *semi-smooth* at $\bar{\mathbf{u}}$, which means that, in addition, it is directionally differentiable in any direction. This additional assumption is not needed in the proof on the speed of convergence of the iterative scheme for solving (4.3) . One cannot avoid (S2) (c) in general. Let us point out, that this condition should not be taken as a definition of "differentiability" without imposing additional requests on $\mathcal{H}$. Xu [33] defined this class as functions having a point-based set-valued approximation, while in his recent book J.-P. Penot [28] used the name slantly differentiable functions. In [7], A. L. Dontchev named this kind of differentiability after B. Kummer, with the intention to give credit to the individual who introduced it. As it turns out, however, *every function acting between Banach spaces is Kummer/point-based/slantly differentiable.* This simple fact is explicitly shown in the proof of [28, Lemma 2.64], but perhaps well-known much earlier since a finite-dimensional version of it was given in [33] and credited there to a referee of that paper.

The class of semi-smooth functions includes, for example, smooth functions (see the proof of Example 2.14), convex functions, piece-wise smooth functions and tame functions [4]. Moreover, compositions and products of semi-smooth functions are again semi-smooth. In particular, the function in (4.5) is semi-smooth. Also, [13, Exercise 2D.9] says that a projection mapping on the set

$$K := \{\mathbf{u} \in \mathbb{R}^m : \ \psi_i(\mathbf{u}) \leq 0, \ i = 1, 2, \dots, d\},$$

with twice continuously differentiable convex functions $\psi_i : \mathbb{R}^m \to \mathbb{R}$, is piece-wise smooth on a neighborhood of $\bar{\mathbf{u}} \in K$ provided that the gradients of active constraints at this point are linearly independent. Therefore, in this case, the projection mapping is semi-smooth at $\bar{\mathbf{u}}$.

Algorithm 1. (non-smooth Newton's method)

STEP 1. *Choose a starting point* $\mathbf{u}_0 \in \mathbb{R}^m$ *and set* $k = 0$*;*
STEP 2. *Until a stopping criterion holds continue;*
STEP 3. *Given* $\mathbf{u}_k \in \mathbb{R}^m$ *compute an element* $\mathbf{A}_k \in \mathcal{H}(\mathbf{u}_k)$*;*
STEP 4. *Find* $\mathbf{u}_{k+1} \in \mathbb{R}^m$ *such that*

$$\mathbf{h}(\mathbf{u}_k) + \mathbf{A}_k(\mathbf{u}_{k+1} - \mathbf{u}_k) = \mathbf{0};$$

STEP 5. *Set* $k := k + 1$ *and go to* STEP 2.

Let us point out, that we are not going to discuss possible choices of stopping criteria. We are interested in showing the speed of the convergence only, therefore our stopping criterion will be simply $\mathbf{h}(\mathbf{u}_k) = \mathbf{0}$ which is a nonsense in any practical implementation of this algorithm on the computer. Recall that a sequence $(\mathbf{u}_k)_{k \in \mathbb{N}}$ with $\mathbf{u}_k \neq \bar{\mathbf{u}}$ is *q-super-linearly convergent* to $\bar{\mathbf{u}}$ when

$$\lim_{k \to +\infty} \frac{\|\mathbf{u}_{k+1} - \bar{\mathbf{u}}\|}{\|\mathbf{u}_k - \bar{\mathbf{u}}\|} = 0;$$

and *q-quadratically convergent* to $\bar{\mathbf{u}}$ when there exist $\gamma > 0$ and $k_0 \in \mathbb{N}$ such that

$$\|\mathbf{u}_{k+1} - \bar{\mathbf{u}}\| \leq \gamma \|\mathbf{u}_k - \bar{\mathbf{u}}\|^2 \quad \text{for all } k > k_0.$$

A detailed discussion on the topic can be found in [16]. The following statement goes back to L. Qi and J. Sun [19].

Theorem 4.8. *Given a solution* $\bar{\mathbf{u}} \in \mathbb{R}^m$ *to* (4.3) *such that (S1) – (S2) hold with* $d := m$*, assume that all matrices* $\mathbf{A} \in \mathcal{H}(\bar{\mathbf{u}})$ *are non-singular. Then there is a neighborhood* U *of* $\bar{\mathbf{u}}$ *such that, for any starting point* $\mathbf{u}_0 \in U$*, Algorithm 1 either terminates after a finite number of steps or generates a sequence which converges to* $\bar{\mathbf{u}}$ *q-super-linearly and this sequence is unique in* U.

Proof. First, we `claim` that there are positive δ and c such that $\mathbf{A}$ is non-singular and $\|\mathbf{A}^{-1}\| \leq c$ whenever $\mathbf{A} \in \mathcal{H}(\mathbf{u})$ and $\mathbf{u} \in \mathbb{B}(\bar{\mathbf{u}}, \delta)$. Suppose on the contrary that there is $(\mathbf{u}_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^m$ converging to $\bar{\mathbf{u}}$ along with $(\mathbf{A}_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^{m\times m}$, satisfying $\mathbf{A}_k \in \mathcal{H}(\mathbf{u}_k)$ for each $k \in \mathbb{N}$, such that either all $\mathbf{A}_k$'s are singular or $\|{\mathbf{A}_k}^{-1}\| \to +\infty$ as $k \to +\infty$. The conditions (a) and (b) in (S2) imply that passing to a sub-sequence, if necessary, one has that

$$\mathbf{A}_k \in \mathcal{H}(\mathbf{u}_k) \subset \mathcal{H}(\bar{\mathbf{u}}) + \mathbb{B}(\mathbf{0}, 1/k) \quad \text{for each} \quad k \in \mathbb{N}.$$

In particular, $(\mathbf{A}_k)_{k\in\mathbb{N}}$ is bounded. Extract a sub-sequence of it which converges to an $\mathbf{A} \in \mathbb{R}^{m\times m}$. By the assumption, this $\mathbf{A}$ has to be singular. The last inclusion reveals that

$$d(\mathbf{A}, \mathcal{H}(\bar{\mathbf{u}})) = \lim_{k\to+\infty} d(\mathbf{A}_k, \mathcal{H}(\bar{\mathbf{u}})) = 0.$$

Since $\mathcal{H}(\bar{\mathbf{u}})$ is closed, it contains $\mathbf{A}$ which is singular, a contradiction.

In view of (S2) (c), shrinking δ, one can suppose that

$$\sup_{\mathbf{A}\in\mathcal{H}(\mathbf{u})} \|\mathbf{h}(\mathbf{u}) - \mathbf{A}(\mathbf{u} - \bar{\mathbf{u}})\| \leq \frac{1}{2c}\|\mathbf{u} - \bar{\mathbf{u}}\| \quad \text{whenever} \quad \mathbf{u} \in \mathbb{B}(\bar{\mathbf{u}}, \delta). \tag{4.6}$$

Let $U := \mathbb{B}(\bar{\mathbf{u}}, \delta)$ and take any $\mathbf{u}_0 \in U$. Assume that the Algorithm 1 has already generated $\mathbf{u}_0$, $\mathbf{u}_1$, ..., $\mathbf{u}_k$ in U for some $k \in \mathbb{N}_0$ and has not stopped. Choose any $\mathbf{A}_k \in \mathcal{H}(\mathbf{u}_k)$. As $\mathbf{A}_k$ is non-singular, there is a unique $\mathbf{u}_{k+1} \in \mathbb{R}^m$ such that

$$\mathbf{h}(\mathbf{u}_k) + \mathbf{A}_k(\mathbf{u}_{k+1} - \mathbf{u}_k) = \mathbf{0}.$$

Then

$$\begin{aligned}
\|\mathbf{u}_{k+1} - \bar{\mathbf{u}}\| &= \|\mathbf{u}_k - {\mathbf{A}_k}^{-1}\mathbf{h}(\mathbf{u}_k) - \bar{\mathbf{u}}\| \leq \|{\mathbf{A}_k}^{-1}\| \, \|\mathbf{A}_k(\mathbf{u}_k - \bar{\mathbf{u}}) - \mathbf{h}(\mathbf{u}_k)\| \\
&\underset{(4.6)}{\leq} c\frac{1}{2c}\|\mathbf{u}_k - \bar{\mathbf{u}}\| = \frac{1}{2}\|\mathbf{u}_k - \bar{\mathbf{u}}\|.
\end{aligned}$$

This means that $\mathbf{u}_{k+1} \in U$. Hence the algorithm either stops after a finite number of steps or generates an infinite sequence $(\mathbf{u}_k)_{k\in\mathbb{N}}$ with elements in U which are uniquely determined by the previous iterate and all are different from $\bar{\mathbf{u}}$. The last chain of inequalities also implies that

$$\|\mathbf{u}_k - \bar{\mathbf{u}}\| \leq \frac{1}{2^k}\|\mathbf{u}_0 - \bar{\mathbf{u}}\| \quad \text{for each} \quad k \in \mathbb{N}.$$

Therefore $(\mathbf{u}_k)_{k\in\mathbb{N}}$ converges to $\bar{\mathbf{u}}$ (in a q-linear way). Using (S2) (c) together with the very definition of $\mathbf{u}_{k+1}$, we see that

$$\begin{aligned}
0 &\leq \limsup_{k\to+\infty} \frac{\|\mathbf{u}_{k+1} - \bar{\mathbf{u}}\|}{\|\mathbf{u}_k - \bar{\mathbf{u}}\|} \leq \limsup_{k\to+\infty} \frac{c\|\mathbf{A}_k(\mathbf{u}_k - \bar{\mathbf{u}}) - \mathbf{h}(\mathbf{u}_k)\|}{\|\mathbf{u}_k - \bar{\mathbf{u}}\|} \\
&\leq c \lim_{k\to+\infty} \frac{\sup_{\mathbf{A}\in\mathcal{H}(\mathbf{u}_k)} \|\mathbf{A}(\mathbf{u}_k - \bar{\mathbf{u}}) - \mathbf{h}(\mathbf{u}_k)\|}{\|\mathbf{u}_k - \bar{\mathbf{u}}\|} = 0.
\end{aligned}$$

Hence the convergence is, in fact, q-super-linear. $\square$

To perform the third step in the algorithm, one wants $\mathcal{H}$ to have large values. On the other hand, Theorem 4.8 says the smaller $\mathcal{H}(\bar{\mathbf{u}})$ is the better.

Remark 4.9. (i) In the proof of the previous result, the gist is that the set of non-singular matrices is open in $\mathbb{R}^{m\times m}$. Indeed, fix any non-singular $\mathbf{A} \in \mathbb{R}^{m\times m}$, equivalently, the matrix $\mathbf{A}$ with a non-zero determinant $\det \mathbf{A}$. The function $\mathbb{R}^{m\times m} \ni \mathbf{B} \mapsto \det \mathbf{B}$ is continuous. This can be easily seen either by the very definition or via induction on the dimension. As the statement clearly holds for $m = 1$, suppose that for $k \in \mathbb{N}$ we have that $\mathbb{R}^{k\times k} \ni \mathbf{B} \mapsto \det \mathbf{B}$ is continuous. Given $\mathbf{B} \in \mathbb{R}^{(k+1)\times(k+1)}$, $\det \mathbf{B}$ is a sum of $k + 1$ terms being scalar multiples of determinants of $k \times k$ matrices, hence it is continuous. We conclude that $\det \mathbf{B}$ is non-zero for any matrix $\mathbf{B}$ sufficiently close to $\mathbf{A}$;

(ii) If the condition (S2) (c) is replaced by a (stronger) request that there is $\gamma > 0$ such that

$$\limsup_{\mathbf{0}\neq\mathbf{v}\to\mathbf{0}} \frac{\sup_{\mathbf{A}\in\mathcal{H}(\bar{\mathbf{u}}+\mathbf{v})} \|\mathbf{h}(\bar{\mathbf{u}}+\mathbf{v}) - \mathbf{h}(\bar{\mathbf{u}}) - \mathbf{A}\mathbf{v}\|}{\|\mathbf{v}\|^2} < \gamma,$$

then the convergence is q-quadratic as can be seen immediately from the last chain of inequalities in the proof. This condition is satisfied for *strongly semi-smooth functions*, for example, if $\mathbf{h}$ is continuously differentiable at $\bar{\mathbf{u}}$ and its derivative is locally Lipschitz continuous at this point (modify the proof of Example 2.14 in an obvious way);

(iii) When $\mathbf{h} : \mathbb{R}^m \to \mathbb{R}^d$ with $d < m$, then the assumption that all matrices $\mathbf{A} \in \mathcal{H}(\bar{\mathbf{u}})$ have full rank guarantees the existence of a q-super-linearly convergent sequence lying in U. However, this sequence is not unique.

The proof of the above statement works even in general Banach spaces for generalized equations with a non-smooth single-valued part. The matrices $\mathbf{A}_k$ can be chosen close to $\mathcal{H}(\mathbf{u}_k)$ (not necessarily inside), and $\mathbf{u}_{k+1} \in \mathbb{R}^m$ does not need to be an exact solution of

$$\mathbf{h}(\mathbf{u}_k) + \mathbf{A}_k(\mathbf{u}_{k+1} - \mathbf{u}_k) = \mathbf{0}.$$

Inexact Newton methods for solving equations

$$\mathbf{h}(\mathbf{x}) = \mathbf{0},$$

where $\mathbf{h}$ is continuously differentiable, were introduced by Dembo, Eisenstat and Steihaug [6]. Specifically, they defined the following iteration: given $\mathbf{u}_k$ find $\mathbf{u}_{k+1}$ such that

$$\|\mathbf{h}(\mathbf{u}_k) + \nabla\mathbf{h}(\mathbf{u}_k)(\mathbf{u}_{k+1} - \mathbf{u}_k)\| \leq \eta_k\|\mathbf{h}(\mathbf{u}_k)\|, \tag{4.7}$$

that is, $\mathbf{u}_{k+1}$ is obtained by a Newton iteration "only approximately and in some *unspecified* manner," as Dembo et al. say in [6]. They proved among other results that if $\mathbf{h}$ is continuously differentiable in a neighborhood of $\bar{\mathbf{u}}$, a zero of $\mathbf{h}$, the Jacobian $\nabla\mathbf{h}(\bar{\mathbf{u}})$ is nonsingular, and the *forcing* sequence $\eta_k \searrow 0$, then any sequence $(\mathbf{u}_k)_{k\in\mathbb{N}}$ generated by (4.7) which is convergent to $\bar{\mathbf{u}}$ is convergent q-super-linearly.

Moreover, to cite Martin Vohralík: *Statements as Theorem 4.8 are nice from the theoretical point of view but totally useless in practice.* The reason is that the conditions guaranteeing convergence are imposed at the *unknown solution we are searching for.* All the above mentioned issues are discussed in the next chapter.

5. Iterative Methods for Generalized Equations

In this chapter, which is taken from [11] and [10], we study iterative methods of Newton type for solving a generalized equation in Banach spaces in the form

$$f(x) + F(x) \ni 0, \tag{5.1}$$

where $f : X \to Y$ is a function and $F : X \rightrightarrows Y$ is generally a set-valued mapping but may also be another function. To simplify some of the arguments used, we adopt the standing assumption that f is continuous on X and F has closed graph. Observe that Dembo-Eisenstat-Steihaug inexact Newton iteration (cf. (4.7)) can be also written as the inclusion

$$f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) \in \mathbb{B}[0, \eta_k \|f(x_k)\|]. \tag{5.2}$$

As in the previous chapter, we introduce a mapping $\mathcal{H} : X \rightrightarrows \mathcal{L}(X, Y)$ viewed as a *generalized set-valued derivative* of a (non-smooth) function f, and consider the following iteration: *given $x_k \in X$ choose any $A_k \in \mathcal{H}(x_k)$ and then find $x_{k+1} \in X$ to satisfy*

$$\big(f(x_k) + A_k(x_{k+1} - x_k) + F(x_{k+1})\big) \cap R_k(x_k) \neq \emptyset. \tag{5.3}$$

Our convergence results for the method (5.3) utilize three groups of assumptions. The first group concerns the non-smoothness of the function f. Namely, we associate to the function f and to the reference point $\bar{x} \in X$ a mapping $\mathcal{H} : X \rightrightarrows \mathcal{L}(X, Y)$ defined in a vicinity of $\bar{x}$, which will be required to satisfy one of the following conditions:

(A1) For every $\varepsilon > 0$ there exists a neighborhood U of $\bar{x}$ such that

$$\|f(x) - f(\bar{x}) - A(x - \bar{x})\| \leq \varepsilon \|x - \bar{x}\| \quad \text{whenever} \quad x \in U \text{ and } A \in \mathcal{H}(x).$$

(A2) There exist a positive β and a neighborhood U of $\bar{x}$ such that

$$\|f(x) - f(\bar{x}) - A(x - \bar{x})\| \leq \beta \|x - \bar{x}\|^2 \quad \text{whenever} \quad x \in U \text{ and } A \in \mathcal{H}(x).$$

(A3) For every $\varepsilon > 0$ there exists a neighborhood U of $\bar{x}$ such that for every $x, x' \in U$ there exists $A \in \mathcal{H}(\bar{x})$ satisfying

$$\|f(x) - f(x') - A(x - x')\| \leq \varepsilon \|x - x'\|.$$

Clearly, (A2) $\Rightarrow$ (A1). If f is Fréchet differentiable around $\bar{x}$, then $\mathcal{H}(x)$ can be identified with the derivative $Df(x)$; in this case both (A1) and (A3) hold when Df is continuous at $\bar{x}$ and (A2) holds when Df is Lipschitz continuous around $\bar{x}$. In finite dimensions, with $\mathcal{H}$ identified with Clarke's generalized Jacobian, condition (A1) holds if f is semi-smooth; condition (A2) is valid when f is strongly semi-smooth; while (A3) holds automatically (a proof of the last claim can be traced back to [14] if not earlier). Note that for $\mathcal{H}$ identified with Bouligand's limiting Jacobian, the same is true in case of both (A1) and (A2); while (A3) fails (see Example 2.17). In Banach spaces condition (A3) enters the definition of the strict prederivative in the sense of Ioffe [20], which is a set-valued generalization of the usual strict derivative. Other extensions of the notion of generalized Jacobian to infinite dimensions are given in [26], [27].

The second set of our assumptions concerns the mappings R_k representing the inexactness in (5.3). First, we always assume that $0 \in R_k(\bar{x})$ for every $k \in \mathbb{N}_0$ and when the mapping R_k appears together with $\mathcal{H}$, the point $\bar{x}$ lies in the interior of $\operatorname{dom}\mathcal{H}\bigcap\left(\bigcap_{k\in\mathbb{N}_0}\operatorname{dom}R_k\right)$. Furthermore, we utilize some growth conditions for R_k that are implanted in the statements of the theorems presented.

The third set of conditions revolves around metric regularity properties of mappings (two of them have already been mentioned in the previous chapters). The following notions are (local) extensions to nonlinear and even set-valued mappings of three basic properties of linear mappings in linear algebra and analysis: surjectivity, invertiblity, and injectivity. Let us start with surjectivity. A mapping $F: X \rightrightarrows Y$ with $(\bar{x},\bar{y})\in\operatorname{gph}F$ is said to be *metrically regular* at $\bar{x}$ for $\bar{y}$ when there is a constant $\kappa>0$ together with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d\big(x,F^{-1}(y)\big)\le\kappa d(y,F(x))\quad\text{for all}\quad x\in U, y\in V. \tag{5.4}$$

A mapping $A\in\mathcal{L}(X,Y)$ is metrically regular at any point if and only if it is surjective; this is one of the statements of the Banach open mapping principle. The infimum over all $\kappa>0$ such that (5.4) holds for some neighborhoods U and V is the *regularity modulus* of F at $\bar{x}$ for $\bar{y}$ denoted by $\operatorname{reg}(F;\bar{x}\,|\,\bar{y})$. We use the convention that a mapping F is metrically regular at $\bar{x}$ for $\bar{y}$ if and only if $\operatorname{reg}(F;\bar{x}\,|\,\bar{y})<+\infty$. If a mapping $F: X\rightrightarrows Y$ is metrically regular at $\bar{x}$ for $\bar{y}$ and moreover its inverse F^{-1} has a *single-valued graphical localization* around $\bar{y}$ for $\bar{x}$, meaning that there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that the mapping $V\ni y\mapsto F^{-1}(y)\cap U$ is single-valued, then F is said to be *strongly metrically regular* at $\bar{x}$ for $\bar{y}$. Equivalently, a mapping F is strongly metrically regular at $\bar{x}$ for $\bar{y}$ if and only if its inverse F^{-1} has a single-valued graphical localization around $\bar{y}$ for $\bar{x}$ which is Lipschitz continuous around $\bar{y}$ with Lipschitz modulus at $\bar{y}$ equal $\operatorname{reg}(F;\bar{x}\,|\,\bar{y})$. Clearly, this is an extension of invertibility because a mapping $A\in\mathcal{L}(X,Y)$ is strongly metrically regular at any point if and only if it is invertible. Finally, a mapping $F: X\rightrightarrows Y$ is said to be *strongly metrically sub-regular* at $\bar{x}$ for $\bar{y}$ when $(\bar{x},\bar{y})\in\operatorname{gph}F$ and there is a constant $\kappa>0$ together with a neighborhood U of $\bar{x}$ such that

$$\|x-\bar{x}\|\le\kappa d(\bar{y},F(x))\quad\text{for all}\quad x\in U. \tag{5.5}$$

The infimum over all $\kappa>0$ such that (5.5) holds for some neighborhood U is the *sub-regularity modulus* of F at $\bar{x}$ for $\bar{y}$ denoted by $\operatorname{subreg}(F;\bar{x}\,|\,\bar{y})$. For $A\in\mathcal{L}(X,Y)$ we have $\operatorname{subreg}A<+\infty$ if and only if A is injective.

5.1. **Local Convergence.** In the proofs of Theorems 5.2 and 5.3 we utilize the following result given in [13, Theorem 5G.3].

Theorem 5.1. *Consider a mapping $F: X\rightrightarrows Y$ with closed graph and a point $(\bar{x},\bar{y})\in\operatorname{gph}F$ at which F is metrically regular, that is, there exist positive constants a, b, and κ such that (5.4) holds with $U=\mathbb{B}[\bar{x},a]$ and $V=\mathbb{B}[\bar{y},b]$. Let $\nu>0$ be such that $\kappa\nu<1$ and let $\kappa'>\kappa/(1-\kappa\nu)$. Then for every positive α and β such that*

$$\alpha\le a/2,\quad 2\nu\alpha+2\beta\le b\quad\text{and}\quad 2\kappa'\beta\le\alpha$$

and for every function $g: X\to Y$ satisfying

$$\|g(\bar{x})\|\le\beta\quad\text{and}\quad\|g(x)-g(x')\|\le\nu\|x-x'\|\quad\text{for every}\ \ x,x'\in\mathbb{B}[\bar{x},2\alpha], \tag{5.6}$$

the mapping $g + F$ has the following property: for every $y, y' \in \mathbb{B}[\bar{y}, \beta]$ and every $x \in (g + F)^{-1}(y) \cap \mathbb{B}[\bar{x}, \alpha]$ there exists $x' \in (g + F)^{-1}(y')$ such that

$$\|x - x'\| \leq \kappa' \|y - y'\|.$$

In addition, if the mapping F is strongly metrically regular at $\bar{x}$ for $\bar{y}$; that is, the mapping $y \mapsto F^{-1}(y) \cap \mathbb{B}[\bar{x}, a]$ is single-valued and Lipschitz continuous on $\mathbb{B}[\bar{y}, b]$ with a Lipschitz constant κ, then for ν, κ', α and β as above and any function g satisfying (5.6)*, the mapping $y \mapsto (g + F)^{-1}(y) \cap \mathbb{B}[\bar{x}, \alpha]$ is a Lipschitz continuous function on $\mathbb{B}[\bar{y}, \beta]$ with a Lipschitz constant κ'.*

Given a set $\mathcal{A}$ in $\mathcal{L}(X, Y)$, the measure of non-compactness $\chi(\mathcal{A})$ of $\mathcal{A}$ is defined as

$$\chi(\mathcal{A}) = \inf \left\{ r > 0 \,\middle|\, \mathcal{A} \subset \bigcup \left\{ \mathbb{B}(A, r) \,\middle|\, A \in \mathcal{B} \right\},\ \mathcal{B} \subset \mathcal{A} \text{ finite} \right\}.$$

Our first result shows linear convergence of the method (5.3).

Theorem 5.2. *Consider the inexact Newton-type method* (5.3) *applied to the generalized equation* (5.1) *with a mapping $\mathcal{H} : X \rightrightarrows \mathcal{L}(X, Y)$ which is outer semicontinuous at $\bar{x}$, a solution of the generalized equation* (5.1)*, and satisfies condition (A1). Define*

$$G_A : x \mapsto f(\bar{x}) + A(x - \bar{x}) + F(x) \quad \text{for } A \in \mathcal{H}(\bar{x}) \tag{5.7}$$

and assume that

$$\mathfrak{c} := \chi(\mathcal{H}(\bar{x})) \quad \text{and} \quad \mathfrak{m} := \sup_{A \in \mathcal{H}(\bar{x})} \operatorname{reg}(G_A; \bar{x} \,|\, 0) \tag{5.8}$$

are finite constants that satisfy

$$\mathfrak{m}\mathfrak{c} < 1. \tag{5.9}$$

Furthermore, suppose that the sequence $(R_k)_{k \in \mathbb{N}_0}$ satisfies

$$\limsup_{\bar{x} \neq x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \sup_{k \in \mathbb{N}_0} \sup_{u \in R_k(x)} \|u\| < 1/\mathfrak{m} - \mathfrak{c}. \tag{5.10}$$

Then there exist $t \in (0, 1)$ and $r > 0$ such that for every $x \in X$ with $0 < \|x - \bar{x}\| \leq r$, every $A \in \mathcal{H}(x)$, every $k \in \mathbb{N}_0$, and every $u_k \in R_k(x)$ there exists x', which depends on the choice of x, A, k and u_k, such that

$$f(x) + A(x' - x) + F(x') \ni u_k, \tag{5.11}$$

and

$$\|x' - \bar{x}\| \leq t \|x - \bar{x}\|. \tag{5.12}$$

Consequently, for any starting point $x_0 \in \mathbb{B}[\bar{x}, r]$ there exists a sequence $(x_k)_{k \in \mathbb{N}}$ generated by (5.3) *which is q-linearly convergent to $\bar{x}$.*

Proof. In the first part of the proof we show for the mapping G_A defined in (5.7) that there exist positive δ, b and Θ such that for every $A \in \mathcal{H}(\mathbb{B}[\bar{x}, \delta])$ and for every $y \in \mathbb{B}[0, b]$ there exists $x \in G_A^{-1}(y)$ satisfying

$$\|x - \bar{x}\| \leq \Theta \|y\|. \tag{5.13}$$

On the basis of (5.10), pick any $\gamma > 0$ such that

$$\limsup_{\bar{x} \neq x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \sup_{k \in \mathbb{N}_0} \sup_{u \in R_k(x)} \|u\| < \gamma < 1/\mathfrak{m} - \mathfrak{c}. \tag{5.14}$$

Utilizing (5.9) and (5.14), one can find $\mu > \mathfrak{c}$, $\kappa > \mathfrak{m}$, $\varepsilon > 0$ and $t \in (0, 1)$ satisfying

$$\mu\kappa < 1, \quad \mathfrak{c} + 2\varepsilon < \mu \text{ and } \kappa(\varepsilon + \gamma) < t(1 - \kappa\mu). \tag{5.15}$$

From the first inequality in (5.14), there exists $\delta > 0$ such that

$$\|v\| < \gamma\|x - \bar{x}\| \quad \text{whenever} \quad x \in \mathbb{B}[\bar{x}, \delta] \setminus \{\bar{x}\},\ k \in \mathbb{N}_0, \text{ and } v \in R_k(x). \tag{5.16}$$

Make $\delta > 0$ smaller if necessary to obtain

$$\mathbb{B}[\bar{x}, \delta] \subset \operatorname{dom} \mathcal{H} \cap (\cap_{k \in \mathbb{N}_0} \operatorname{dom} R_k),$$

and also

$$\mathcal{H}(x) \subset \mathcal{H}(\bar{x}) + \mathbb{B}[0, \varepsilon] \quad \text{for each} \quad x \in \mathbb{B}[\bar{x}, \delta]. \tag{5.17}$$

By the definition of measure of non-compactness, there is a finite set $\mathcal{A}_F \subset \mathcal{H}(\bar{x})$ such that

$$\mathcal{H}(\bar{x}) \subset \mathcal{A}_F + \mathbb{B}[0, \chi(\mathcal{H}(\bar{x})) + \varepsilon].$$

Hence, from (5.17), for any $x \in \mathbb{B}[\bar{x}, \delta]$ we get

$$\mathcal{H}(x) \subset \mathcal{A}_F + \mathbb{B}[0, \chi(\mathcal{H}(\bar{x})) + \varepsilon] + \mathbb{B}[0, \varepsilon] = \mathcal{A}_F + \mathbb{B}[0, \mathfrak{c} + 2\varepsilon],$$

that is, from the second inequality in (5.15),

$$\mathcal{H}(x) \subset \mathcal{A}_F + \mathbb{B}[0, \mu] \quad \text{for every} \quad x \in \mathbb{B}[\bar{x}, \delta]. \tag{5.18}$$

Choose Θ to satisfy

$$\mathfrak{m}/(1 - \mu\mathfrak{m}) < \Theta < \kappa/(1 - \mu\kappa)$$

and then choose $\tau \in (\mathfrak{m}, \kappa)$ with $\tau/(1-\mu\tau) < \Theta$. Pick any $\tilde{A} \in \mathcal{A}_F$, any $A' \in \mathbb{B}[0, \mu]$. Then there exist $\alpha_{\tilde{A}} > 0$ and $\beta_{\tilde{A}} > 0$ such that $G_{\tilde{A}}$ is metrically regular at $\bar{x}$ for 0 with the constant τ and neighborhoods $\mathbb{B}[\bar{x}, \alpha_{\tilde{A}}]$ and $\mathbb{B}[0, \beta_{\tilde{A}}]$. Let $g(x) := A'(x - \bar{x})$, $x \in X$; then $G_{\tilde{A}+A'} = G_{\tilde{A}} + g$. Observe that g is single-valued, Lipschitz continuous with Lipschitz constant μ such that $\mu\tau < 1$, and $g(\bar{x}) = 0$. We can apply Theorem 5.1 with $F = G_{\tilde{A}}$, $\kappa = \tau$, $\nu = \mu$, $y' = y$, $y = \bar{y} = 0$, and $x = \bar{x}$, obtaining that there is $\beta'_{\tilde{A}} > 0$ (independent of A') such that for each $y \in \mathbb{B}[0, \beta'_{\tilde{A}}]$ there is $x \in \left(G_{\tilde{A}+A'}\right)^{-1}(y)$ such that $\|x - \bar{x}\| \le \Theta\|y\|$. Summarizing, given $\tilde{A} \in \mathcal{A}_F$, there exists $\beta'_{\tilde{A}} > 0$ such that for each $A' \in \mathbb{B}[0, \mu]$ and each $y \in \mathbb{B}[0, \beta'_{\tilde{A}}]$ there is $x \in \left(G_{\tilde{A}+A'}\right)^{-1}(y)$ satisfying $\|x - \bar{x}\| \le \Theta\|y\|$. Let $b = \min_{\tilde{A} \in \mathcal{A}_F} \beta'_{\tilde{A}}$. Taking into account (5.18) one has $\mathcal{H}(\mathbb{B}[\bar{x}, \delta]) \subset \mathcal{A}_F + \mathbb{B}[0, \mu]$, hence we obtain that for every $A \in \mathcal{H}(\mathbb{B}[\bar{x}, \delta])$ and for every $y \in \mathbb{B}[0, b]$ there is $x \in G_A^{-1}(y)$ satisfying (5.13).

Coming to the second part of the proof, first we make the constant δ smaller if necessary so that (A1) is satisfied with the already chosen ε and $U = \mathbb{B}[\bar{x}, \delta]$, that is,

$$\sup_{A \in \mathcal{H}(x)} \|f(x) - f(\bar{x}) - A(x - \bar{x})\| \le \varepsilon\|x - \bar{x}\| \quad \text{for every } x \in \mathbb{B}[\bar{x}, \delta]. \tag{5.19}$$

Fix r such that

$$0 < r < \min\{b/(\varepsilon + \gamma), \delta\}. \tag{5.20}$$

Fix $x \in X$ satisfying $0 < \|x - \bar{x}\| \le r$. Choose any $A \in \mathcal{H}(x)$, any $k \in \mathbb{N}_0$, and any $u_k \in R_k(x)$; then from (5.16) and (5.20) u_k satisfies $\|u_k\| < \gamma\|x - \bar{x}\|$. Denote

$$y_k := f(x) - f(\bar{x}) - A(x - \bar{x}) - u_k. \tag{5.21}$$

If $y_k = 0$ then $x' := \bar{x}$ satisfies (5.11) because $-f(\bar{x}) \in F(\bar{x})$ and (5.12) holds trivially. Assume that $y_k \neq 0$. Using (5.19), and (5.20), we get

$$\|y_k\| \le \|f(x) - f(\bar{x}) - A(x - \bar{x})\| + \|u_k\| < (\varepsilon + \gamma)\|x - \bar{x}\| < b.$$

Applying (5.13) with $y = -y_k$ and taking into account the last inequality in (5.15) and that $\Theta < \kappa/(1 - \mu\kappa)$, we obtain that there exists $x' \in G_A^{-1}(-y_k)$ such that

$$\begin{aligned} \|x' - \bar{x}\| &\le \Theta\|y_k\| < (\varepsilon + \gamma)\Theta\|x - \bar{x}\| \\ &< \frac{t(1 - \mu\kappa)}{\kappa}\frac{\kappa}{1 - \mu\kappa}\|x - \bar{x}\| = t\|x - \bar{x}\|. \end{aligned}$$

Hence $\|x' - \bar{x}\| < r$ because $t \in (0, 1)$. Furthermore,

$$-f(x) + f(\bar{x}) + A(x - \bar{x}) + u_k \in G_A(x') = f(\bar{x}) + A(x' - \bar{x}) + F(x').$$

Thus, x' satisfies (5.11) and (5.12).

To finish the proof, consider the iteration (5.3) and choose any $k \in \mathbb{N}_0$, any $x_k \in \mathbb{B}[\bar{x}, r]$ and any $A_k \in \mathcal{H}(x_k)$. If $x_k \neq \bar{x}$, applying (5.11) and (5.12) just proved with $x = x_k$, we obtain that for any $u_k \in R_k(x_k)$ there exists $x_{k+1} := x' \in \mathbb{B}[\bar{x}, r]$ such that

$$f(x_k) + A_k(x_{k+1} - x_k) + F(x_{k+1}) \ni u_k \tag{5.22}$$

and

$$\|x_{k+1} - \bar{x}\| \le t\|x_k - \bar{x}\|. \tag{5.23}$$

The inclusion (5.22) yields that x_{k+1} satisfies (5.3). If $x_k = \bar{x}$, then $x_{k+1} := \bar{x}$ verifies (5.3) because $0 \in R_k(\bar{x})$. It remains to choose any $x_0 \in \mathbb{B}[\bar{x}, r]$ to obtain in this way an infinite sequence $(x_k)_{k \in \mathbb{N}}$ with $x_k \in \mathbb{B}[\bar{x}, r]$ generated by (5.3) and satisfying (5.23) for all $k \in \mathbb{N}_0$. Since $t \in (0, 1)$, $(x_k)_{k \in \mathbb{N}}$ converges q-linearly to $\bar{x}$. □

The next theorem shows that under stronger conditions every convergent sequence, in particular those whose existence is claimed in Theorem 5.2, is actually convergent q-super-linearly, or q-quadratically, depending on the assumptions for the mappings R_k.

Theorem 5.3. *Consider the inexact Newton-type method* (5.3) *applied to the generalized equation* (5.1) *and suppose that the assumptions of Theorem 5.2 are satisfied. In addition, suppose that for every $A \in \mathcal{H}(\bar{x})$ the mapping G_A defined in* (5.7) *is strongly metrically regular at $\bar{x}$ for* 0. *Then every sequence $(x_k)_{k \in \mathbb{N}}$ generated by* (5.3) *which is convergent to $\bar{x}$ is in fact q-linearly convergent.*

Assume that the sequence $(R_k)_{k \in \mathbb{N}_0}$ satisfies

$$\lim_{\bar{x} \neq x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \sup_{k \in \mathbb{N}_0} \sup_{u \in R_k(x)} \|u\| = 0. \tag{5.24}$$

Then every sequence $(x_k)_{k\in\mathbb{N}}$ generated by (5.3) *which is convergent to $\bar{x}$ is in fact q-super-linearly convergent.*

Finally, suppose that the mapping $\mathcal{H} : X \rightrightarrows \mathcal{L}(X, Y)$ satisfies condition (A2) and the sequence $(R_k)_{k\in\mathbb{N}_0}$ satisfies

$$\limsup_{\bar{x}\neq x\to\bar{x}} \frac{1}{\|x-\bar{x}\|^2} \sup_{k\in\mathbb{N}_0} \sup_{u\in R_k(x)} \|u\| < +\infty. \tag{5.25}$$

Then every sequence $(x_k)_{k\in\mathbb{N}}$ generated by (5.3) *which is convergent to $\bar{x}$ is in fact q-quadratically convergent.*

Proof. Consider a sequence $(x_k)_{k\in\mathbb{N}}$ generated by (5.3) which converges to $\bar{x}$. Then there are sequences $(A_k)_{k\in\mathbb{N}_0}$ and $(u_k)_{k\in\mathbb{N}_0}$, with $A_k \in \mathcal{H}(x_k)$ and $u_k \in R_k(x_k)$ for each $k \in \mathbb{N}_0$ such that (5.11) holds. In parallel to the proof of (5.13) and based on the strong regularity part of Theorem 5.1 we obtain that there are positive a, b, δ, and Θ such that for each $A \in \mathcal{H}(\mathbb{B}[\bar{x}, \delta])$ the mapping $\mathbb{B}[0, b] \ni y \mapsto \sigma_A := G_A^{-1}(y) \cap \mathbb{B}[\bar{x}, a]$ is a Lipschitz continuous function on $\mathbb{B}[0, b]$ with a Lipschitz constant Θ.

For each $k \in \mathbb{N}_0$ define y_k by (5.21). We will show that for sufficiently large k we have

$$\|x_{k+1} - \bar{x}\| \leq \Theta\|y_k\|. \tag{5.26}$$

Fix $r \in (0, \min\{b/(\varepsilon + \gamma), \delta, a\})$. Since $x_k \to \bar{x}$ as $k \to +\infty$, we have $x_k \in \mathbb{B}[\bar{x}, r]$ for all sufficiently large k. Fix any such an index k. As in the proof of Theorem 5.2 we get that $\|y_k\| < b$. Noting that $x_{k+1} \in \mathbb{B}[\bar{x}, r] \subset \mathbb{B}[\bar{x}, a]$, the single-valuedness of σ_{A_k} on $\mathbb{B}[0, b]$ implies that $x_{k+1} = \sigma_{A_k}(-y_k)$. Taking into account that $\bar{x} = \sigma_{A_k}(0)$ we get (5.26).

Using exactly the same steps as in the proof of Theorem 5.2, one shows that (5.26) and (5.10) imply (5.23) which yields q-linear convergence.

Instead of (5.10), suppose that a stronger condition (5.24) holds. To show q-super-linear convergence, let $\nu > 0$. From the fact that $x_k \to \bar{x}$ and from (A1), for sufficiently large k, we have that

$$\|f(x_k) - f(\bar{x}) - A_k(x_k - \bar{x})\| \leq \nu/(2\Theta)\|x_k - \bar{x}\|$$

and, from (5.24), also that

$$\|u_k\| \leq \nu/(2\Theta)\|x_k - \bar{x}\|.$$

From the last two inequalities and (5.26), for all sufficiently large k such that $x_k \neq \bar{x}$ we obtain

$$\begin{aligned} \frac{\|x_{k+1} - \bar{x}\|}{\|x_k - \bar{x}\|} &\leq \frac{\Theta\|y_k\|}{\|x_k - \bar{x}\|} \leq \frac{\Theta\|f(x_k) - f(\bar{x}) - A_k(x_k - \bar{x})\|}{\|x_k - \bar{x}\|} + \frac{\Theta\|u_k\|}{\|x_k - \bar{x}\|} \\ &\leq \nu/2 + \nu/2 = \nu. \end{aligned}$$

Since ν can be arbitrarily small, this yields q-super-linear convergence of $(x_k)_{k\in\mathbb{N}}$.

For the quadratic convergence claim, condition (5.25) yields that there exists $\gamma > 0$ such that $\|u_k\| \leq \gamma\|x_k - \bar{x}\|^2$ for any sufficiently large $k \in \mathbb{N}_0$. By repeating the argument of the proof of the q-super-linear convergence by using (A2) and (5.25) instead of (A1) and (5.24), we get

$$\|x_{k+1} - \bar{x}\| \leq \Theta\|y_k\| \leq \Theta\|f(x_k) - f(\bar{x}) - A_k(x_k - \bar{x})\| + \Theta\|u_k\| \leq \Theta(\beta + \gamma)\|x_k - \bar{x}\|^2.$$

This yields q-quadratic convergence of $(x_k)_{k\in\mathbb{N}}$. $\square$

If $\mathcal{H}$ were compact valued, which is the case when $\mathcal{H}$ is taken to be Clarke's generalized Jacobian in finite dimensions, then $\mathfrak{c}$ is just zero and (5.9) is always satisfied when all mappings G_A with $A \in \mathcal{H}(\bar{x})$ are metrically regular at $\bar{x}$ for 0. If we deal with an equation in finite dimensions solved via Algorithm 1 (which means that $F \equiv 0$ and $R_k \equiv 0$ for each $k \in \mathbb{N}_0$) we get an extension of Theorem 4.8 (see also Remark 4.9).

5.2. **Dennis–Moré Theorems.** Dennis-Moré theorem [5] characterizes q-super-linear convergence of quasi-Newton methods of the form

$$f(x_k) + B_k(x_{k+1} - x_k) = 0, \quad k = 0, 1, \dots, \quad x_0 \text{ given,} \tag{5.27}$$

for finding a zero of a smooth function f, where B_k is a sequence of quasi-Newton updates constructed in certain way, which will not be discussed here. Throughout, for a sequence $(x_k)_{k\in\mathbb{N}}$ and a point $\bar{x}$, denote $s_k = x_{k+1} - x_k$ and $e_k = x_k - \bar{x}$. We start with a statement of the Dennis-Moré theorem for a smooth function f acting in Banach spaces.

Theorem 5.4. *Suppose that $f : X \to Y$ is strictly Fréchet differentiable at $\bar{x}$ and the derivative $Df(\bar{x})$ is invertible, meaning that $\|Df(\bar{x})^{-1}\| < +\infty$. Let $(B_k)_{k\in\mathbb{N}_0}$ be a sequence in $\mathcal{L}(X,Y)$, let $E_k = B_k - Df(\bar{x})$, and let the sequence $(x_k)_{k\in\mathbb{N}_0}$ be generated by* (5.27) *and converge to $\bar{x}$. Then $x_k \to \bar{x}$ q-super-linearly and $f(\bar{x}) = 0$ if and only if*

$$\lim_{k\to+\infty} \frac{\|E_k s_k\|}{\|s_k\|} = 0. \tag{5.28}$$

In this section we focus on inexact nonsmooth quasi-Newton methods for (5.1), of the form

$$\big(f(x_k) + B_k(x_{k+1} - x_k) + F(x_{k+1})\big) \cap R_k(x_k) \neq \emptyset, \tag{5.29}$$

where $B_k \in \mathcal{L}(X,Y)$ now represents a quasi-Newton update.

In the following theorem we use an immediate consequence of condition (A3): If the mapping $\mathcal{H} : X \rightrightarrows \mathcal{L}(X,Y)$ satisfies condition (A3) at $\bar{x}$ and $x_k \to \bar{x}$, $x_{k+1} \neq x_k$ for all k, then there exists a sequence $(A_k)_{k\in\mathbb{N}_0}$ of mappings such that $A_k \in \mathcal{H}(\bar{x})$ for each $k \in \mathbb{N}_0$ and

$$\lim_{k\to+\infty} \frac{\|f(x_{k+1}) - f(x_k) - A_k s_k\|}{\|s_k\|} = 0. \tag{5.30}$$

The first result in this section follows.

Theorem 5.5. *Let $\bar{x} \in X$ be such that the function f and the mapping $\mathcal{H}$ satisfy condition (A3) at $\bar{x}$, the sequence $(R_k)_{k\in\mathbb{N}_0}$ satisfies condition* (5.24), *and let the set $\mathcal{H}(\bar{x})$ be bounded. Consider a sequence $(x_k)_{k\in\mathbb{N}_0}$ generated by the method* (5.29), *for a sequence $(B_k)_{k\in\mathbb{N}_0}$ in $\mathcal{L}(X,Y)$, which converges to $\bar{x}$ and such that $x_k \neq \bar{x}$ for all $k \in \mathbb{N}_0$. Let $(A_k)_{k\in\mathbb{N}_0}$ be a sequence of mappings in $\mathcal{H}(\bar{x})$ satisfying* (5.30), *and let $E_k = B_k - A_k$.*

(i) *If* $x_k \to \bar{x}$ *q-super-linearly, then*

$$\lim_{k\to+\infty} \frac{d(0, f(\bar{x}) + E_k s_k + F(x_{k+1}))}{\|s_k\|} = 0. \tag{5.31}$$

(ii) *If*

$$\lim_{k\to+\infty} \frac{\|E_k s_k\|}{\|s_k\|} = 0, \tag{5.32}$$

then $\bar{x}$ *is a solution of the generalized equation* (5.1)*. If, in addition, the mapping* $f + F$ *is strongly metrically sub-regular at* $\bar{x}$ *for* 0 *then* $x_k \to \bar{x}$ *q-super-linearly.*

Proof. First, observe that, by (A3), there is $\delta > 0$ such that for any $x, y \in \mathbb{B}[\bar{x}, \delta]$ there exists $A \in \mathcal{H}(\bar{x})$ satisfying

$$\|f(y) - f(x) - A(y - x)\| \le \|y - x\|.$$

Let $\mu > 0$ be such that $\mathcal{H}(\bar{x}) \subset \mathbb{B}[0, \mu]$. Fix any $x, y \in \mathbb{B}[\bar{x}, \delta]$. Then

$$\|f(y) - f(x)\| \le \|f(y) - f(x) - A(y - x)\| + \|A(y - x)\| \le (1 + \mu)\|y - x\|,$$

which gives us Lipschitz continuity of f on $\mathbb{B}[\bar{x}, \delta]$ with Lipschitz constant $1 + \mu$.

Consider a sequence $x_k \to \bar{x}$ with $\|e_k\| \neq 0$ for all $k \in \mathbb{N}_0$ generated by (5.29) for sequences of mappings $(B_k)_{k\in\mathbb{N}_0}$ and $(R_k)_{k\in\mathbb{N}_0}$. For each $k \in \mathbb{N}_0$, set

$$\gamma_k = \frac{1}{\|e_k\|} \sup_{u\in R_k(x_k)} \|u\|.$$

By (5.24), we have that $\gamma_k \to 0$ as $k \to +\infty$. From iteration (5.29), there exists $u_k \in R_k(x_k)$ such that

$$f(x_k) + B_k s_k + F(x_{k+1}) \ni u_k \text{ and } \|u_k\| \le \gamma_k \|e_k\| \quad \text{for all} \quad k \in \mathbb{N}_0. \tag{5.33}$$

Let $x_k \to \bar{x}$ q-super-linearly and let $\varepsilon > 0$. In [5, Lemma 2.1] it is shown that

$$\frac{\|s_k\|}{\|e_k\|} \to 1 \text{ as } k \to +\infty. \tag{5.34}$$

Indeed,

$$\left|\frac{\|s_k\|}{\|e_k\|} - 1\right| = \frac{|\|s_k\| - \| - e_k\||}{\|e_k\|} \le \frac{\|s_k + e_k\|}{\|e_k\|} = \frac{\|e_{k+1}\|}{\|e_k\|} \to 0 \text{ as } k \to +\infty.$$

Therefore

$$\frac{\|e_{k+1}\|}{\|s_k\|} = \frac{\|e_{k+1}\|}{\|e_k\|}\frac{\|e_k\|}{\|s_k\|} \to 0 \text{ as } k \to +\infty.$$

Then for k sufficiently large we get

$$\|e_{k+1}\| \le \varepsilon\|s_k\|, \qquad \|e_k\| \le 2\|s_k\| \quad \text{and} \quad \gamma_k < \varepsilon. \tag{5.35}$$

Hence, from the inequality in (5.33) and the last two inequalities in (5.35),

$$\|u_k\| \le 2\gamma_k\|s_k\| \le 2\varepsilon\|s_k\|. \tag{5.36}$$

Adding and subtracting to the inclusion in (5.33) we have

$$f(\bar{x}) - f(x_{k+1}) + f(x_{k+1}) - f(x_k) - A_k s_k + u_k \in f(\bar{x}) + E_k s_k + F(x_{k+1}). \tag{5.37}$$

Then, from the first inequality in (5.35), for all sufficiently large k we get

$$\|f(\bar{x}) - f(x_{k+1})\| \leq (1+\mu)\|e_{k+1}\| \leq (1+\mu)\varepsilon\|s_k\|. \tag{5.38}$$

Further, from (5.30), for large k,

$$\|f(x_{k+1}) - f(x_k) - A_k s_k\| \leq \varepsilon\|s_k\|. \tag{5.39}$$

Using (5.36), (5.38), and (5.39), we obtain

$$\begin{aligned}
&\|f(\bar{x}) - f(x_{k+1}) + f(x_{k+1}) - f(x_k) - A_k s_k + u_k\| \\
&\quad \leq \|u_k\| + \|f(\bar{x}) - f(x_{k+1})\| + \|f(x_{k+1}) - f(x_k) - A_k s_k\| \\
&\quad \leq 2\varepsilon\|s_k\| + (1+\mu)\varepsilon\|s_k\| + \varepsilon\|s_k\|.
\end{aligned}$$

Taking into account (5.37), this yields

$$d(0, f(\bar{x}) + E_k s_k + F(x_{k+1})) \leq (4+\mu)\varepsilon\|s_k\|.$$

Since ε can be arbitrarily small, we obtain (5.31) and (i) is proved.

To prove (ii), let $(A_k)_{k\in\mathbb{N}_0}$ be a sequence of mappings in $\mathcal{H}(\bar{x})$ satisfying (5.30) and suppose that (5.32) holds. From (5.33), there exists a sequence $(y_k)_{k\in\mathbb{N}_0}$ such that for each $k \in \mathbb{N}_0$ we have

$$u_k = f(x_k) + B_k s_k + y_k, \quad y_k \in F(x_{k+1}), \quad \text{and} \quad u_k \in R_k(x_k).$$

Then, from the inequality in (5.33),

$$\|u_k\| \leq \gamma_k\|e_k\| \to 0 \quad \text{as} \quad k \to +\infty,$$

and, taking into account that the sequence $(A_k)_{k\in\mathbb{N}_0}$ is bounded, we get

$$\|B_k s_k\| \leq \|E_k s_k\| + \|A_k s_k\| \to 0 \quad \text{as} \quad k \to +\infty.$$

Therefore $y_k \to -f(\bar{x})$. Since the graph of F is closed, we obtain $-f(\bar{x}) \in F(\bar{x})$; that is, $\bar{x}$ is a solution of (5.1).

Now, suppose that the mapping $f + F$ is strongly metrically sub-regular at the solution $\bar{x}$ for 0. From the strong sub-regularity, there exists a constant $\kappa > 0$ such that, for large k,

$$\|e_{k+1}\| \leq \kappa d(0, f(x_{k+1}) + F(x_{k+1})). \tag{5.40}$$

From (5.33) for all $k \in \mathbb{N}_0$ we have

$$u_k - f(x_k) - A_k s_k - E_k s_k + f(x_{k+1}) \in f(x_{k+1}) + F(x_{k+1}). \tag{5.41}$$

Hence, from (5.40),

$$\begin{aligned}
\|e_{k+1}\| &\leq \kappa\|u_k - f(x_k) - A_k s_k - E_k s_k + f(x_{k+1})\| \\
&\leq \kappa\|u_k\| + \kappa\|f(x_{k+1}) - f(x_k) - A_k s_k\| + \kappa\|E_k s_k\|.
\end{aligned} \tag{5.42}$$

Let $\varepsilon \in (0, 1/(2\kappa))$. From (5.32) we get

$$\|E_k s_k\| \leq \varepsilon\|s_k\| \quad \text{for all} \quad k \quad \text{sufficiently large}. \tag{5.43}$$

Using (5.39), (5.43), the last inequality in (5.35), and (5.42), we obtain

$$\|e_{k+1}\| \leq \kappa\gamma_k\|e_k\| + 2\kappa\varepsilon\|s_k\| \leq \kappa\varepsilon\|e_k\| + 2\kappa\varepsilon\|e_{k+1}\| + 2\kappa\varepsilon\|e_k\|.$$

Hence,

$$\frac{\|e_{k+1}\|}{\|e_k\|} \leq \frac{3\kappa\varepsilon}{1-2\kappa\varepsilon}.$$

Since ε can be arbitrarily small we get q-super-linear convergence of $(x_k)_{k\in\mathbb{N}_0}$. $\square$

When $F \equiv 0$ we have $f(\bar{x}) = 0$ and then, taking $R_k \equiv 0$ for each $k \in \mathbb{N}_0$, we come to Theorem 5.4. Theorem 5.5 is a generalization of [7, Theorem 3] for both nonsmooth functions and inexact quasi-Newton methods.

Now, we will show that if the function f and the mapping $\mathcal{H}$ satisfy condition (A1), $\mathcal{H}$ is outer semi-continuous at $\bar{x}$ and $\mathcal{H}(\bar{x})$ is a bounded set, then the particular element A_k of $\mathcal{H}(\bar{x})$ in Theorem 5.5 which satisfies (5.30) can be replaced by *any* $A_k \in \mathcal{H}(x_k)$ in the necessity part involving (5.31) and those $A_k \in \mathcal{H}(x_k)$ in the sufficiency part involving (5.32) that are approximated by B_k in the same way as the derivative $Df(\bar{x})$ is approximated in the classical Dennis-Moré Theorem 5.4.

Theorem 5.6. *Let $\bar{x} \in X$ be such that the mapping $\mathcal{H}$ is outer semi-continuous at $\bar{x}$ and satisfies condition (A1) for f at $\bar{x}$, that the sequence $(R_k)_{k\in\mathbb{N}_0}$ satisfies condition* (5.24) *and also that $\mathcal{H}(\bar{x})$ is a bounded set. Consider a sequence $(x_k)_{k\in\mathbb{N}_0}$ generated by the method* (5.29), *for a sequence $(B_k)_{k\in\mathbb{N}_0}$ in $\mathcal{L}(X,Y)$, which converges to $\bar{x}$ and such that $x_k \neq \bar{x}$ for all $k \in \mathbb{N}_0$.*

(i) *Suppose that $x_k \to \bar{x}$ q-super-linearly. Then, for every sequence $(A_k)_{k\in\mathbb{N}_0}$ of mappings such that $A_k \in \mathcal{H}(x_k)$ for all sufficiently large $k \in \mathbb{N}$, condition* (5.31) *holds with $E_k = B_k - A_k$.*

(ii) *If there exists a sequence $(A_k)_{k\in\mathbb{N}_0}$ such that $A_k \in \mathcal{H}(x_k)$ for all sufficiently large $k \in \mathbb{N}$ and that* (5.32) *is satisfied for $E_k = B_k - A_k$, then $\bar{x}$ is a solution of* (5.1). *If, in addition, for every $A \in \mathcal{H}(\bar{x})$ the mapping G_A defined in* (5.7) *is strongly metrically sub-regular at $\bar{x}$ for* 0 *and*

$$\mathfrak{c} := \chi(\mathcal{H}(\bar{x})) \quad \text{and} \quad \mathfrak{m} := \sup_{A\in\mathcal{H}(\bar{x})} \operatorname{subreg}(G_A; \bar{x}\,|\,0) \tag{5.44}$$

are finite constants satisfying

$$\mathfrak{m}\mathfrak{c} < 1, \tag{5.45}$$

then $x_k \to \bar{x}$ q-super-linearly.

Proof. Let $x_k \to \bar{x}$ q-super-linearly and let $\varepsilon > 0$. Choose a sequence $(A_k)_{k\in\mathbb{N}_0}$ of mappings $A_k \in \mathcal{H}(x_k)$ for all $k \in \mathbb{N}$ sufficiently large. Repeat the proof of Theorem 5.5 starting from the second paragraph until formula (5.37) where we write instead

$$f(\bar{x}) - f(x_k) - A_k s_k + u_k \in f(\bar{x}) + E_k s_k + F(x_{k+1}). \tag{5.46}$$

From the assumed outer semi-continuity of $\mathcal{H}$ and the boundedness of $\mathcal{H}(\bar{x})$, there exists a constant λ such that

$$\|A_k\| \leq \lambda \quad \text{for all} \quad k \quad \text{large enough.} \tag{5.47}$$

For k sufficiently large, condition (A1) yields

$$\|f(x_k) - f(\bar{x}) - A_k e_k\| \leq \varepsilon\|e_k\|. \tag{5.48}$$

Then, from (5.36), (5.35), (5.47) and (5.48), for such k we obtain

$$\begin{aligned}\|f(\bar{x}) - f(x_k) - A_k s_k + u_k\| &\le \|u_k\| + \|f(x_k) - f(\bar{x}) - A_k e_k\| + \|A_k\|\|e_{k+1}\| \\ &\le 2\varepsilon\|s_k\| + \varepsilon\|e_k\| + \lambda\|e_{k+1}\| \le (\lambda+4)\varepsilon\|s_k\|.\end{aligned}$$

The inclusion (5.46) then implies

$$d(0, f(\bar{x}) + E_k s_k + F(x_{k+1})) \le (\lambda+4)\varepsilon\|s_k\|.$$

Since ε can be arbitrarily small, we obtain (5.31) and hence (i) is proved.

For the second part of the statement, consider a sequence $(x_k)_{k\in\mathbb{N}_0}$ which converges to $\bar{x}$ and is generated by (5.29) for a sequence $(B_k)_{k\in\mathbb{N}_0}$ in $\mathcal{L}(X,Y)$ and a sequence $(R_k)_{k\in\mathbb{N}_0}$ satisfying (5.24). For each $k \in \mathbb{N}_0$, find $u_k \in R_k(x_k)$ verifying (5.33). Observe that (5.32) implies that $\bar{x}$ is a solution of (5.1) as in Theorem 5.5.

We show next that there exist positive a and Θ such that

$$\text{(5.49)} \quad \|x - \bar{x}\| \le \Theta d(0, G_A(x)) \quad \text{whenever} \quad x \in \mathbb{B}[\bar{x}, a] \quad \text{and} \quad A \in \mathcal{H}(\mathbb{B}[\bar{x}, a]).$$

Use (5.45) to find $\mu > \mathfrak{c}$, $\kappa > \mathfrak{m}$ and $\varepsilon > 0$ satisfying

$$\text{(5.50)} \qquad \mu\kappa < 1 \quad \text{and} \quad \mathfrak{c} + 2\varepsilon < \mu.$$

Let $\Theta := \kappa/(1-\mu\kappa) > 0$. There exists $\delta > 0$ such that

$$\text{(5.51)} \qquad \mathcal{H}(u) \subset \mathcal{H}(\bar{x}) + \mathbb{B}[0,\varepsilon] \quad \text{for each } u \in \mathbb{B}[\bar{x}, \delta].$$

By the definition of measure of non-compactness, there is a finite set $\mathcal{A}_F \subset \mathcal{H}(\bar{x})$ such that

$$\mathcal{H}(\bar{x}) \subset \mathcal{A}_F + \mathbb{B}[0, \chi(\mathcal{H}(\bar{x})) + \varepsilon].$$

Hence, from (5.51), for any $u \in \mathbb{B}[\bar{x}, \delta]$ we get

$$\mathcal{H}(u) \subset \mathcal{A}_F + \mathbb{B}[0, \chi(\mathcal{H}(\bar{x})) + \varepsilon] + \mathbb{B}[0,\varepsilon] = \mathcal{A}_F + \mathbb{B}[0, \mathfrak{c} + 2\varepsilon],$$

that is, from the second inequality in (5.50),

$$\text{(5.52)} \qquad \mathcal{H}(\mathbb{B}[\bar{x}, \delta]) \subset \mathcal{A}_F + \mathbb{B}[0, \mu].$$

Pick any $\tilde{A} \in \mathcal{A}_F$, any $A' \in \mathbb{B}[0,\mu]$. Then there exists $\alpha_{\tilde{A}} > 0$ such that

$$\|x - \bar{x}\| \le \kappa d(0, G_{\tilde{A}}(x)) \quad \text{whenever} \quad x \in \mathbb{B}[\bar{x}, \alpha_{\tilde{A}}].$$

Fix any $x \in \mathbb{B}[\bar{x}, \alpha_{\tilde{A}}]$. As $G_{\tilde{A}+A'} = G_{\tilde{A}} + A'(x - \bar{x})$, one gets

$$\begin{aligned}\|x - \bar{x}\| &\le \kappa d(0, G_{\tilde{A}}(x)) = \kappa d\big(0, G_{\tilde{A}+A'}(x) - A'(x-\bar{x})\big) \\ &= \kappa d\big(A'(x-\bar{x}), G_{\tilde{A}+A'}(x)\big) \le \kappa\|A'(x-\bar{x})\| + \kappa d\big(0, G_{\tilde{A}+A'}(x)\big) \\ &\le \kappa\mu\|x - \bar{x}\| + \kappa d\big(0, G_{\tilde{A}+A'}(x)\big).\end{aligned}$$

Summarizing, given $\tilde{A} \in \mathcal{A}_F$, there exists $\alpha_{\tilde{A}} > 0$ such that for each $A' \in \mathbb{B}[0,\mu]$ we have $\|x - \bar{x}\| \le \Theta d\big(0, G_{\tilde{A}+A'}(x)\big)$ whenever $x \in \mathbb{B}[\bar{x}, \alpha_{\tilde{A}}]$.
Let $a = \min\big\{\delta, \min_{\tilde{A}\in\mathcal{A}_F} \alpha_{\tilde{A}}\big\}$. Taking into account (5.52) one has $\mathcal{H}(\mathbb{B}[\bar{x}, a]) \subset \mathcal{A}_F + \mathbb{B}[0,\mu]$, hence we obtain (5.49).

Observe that in (5.49) we do not assume that $A \in \mathcal{H}(x)$. Fix any $\varepsilon \in (0, 1/\Theta)$. Let $(\gamma_k)_{k\in\mathbb{N}_0}$ be defined as in the proof of Theorem 3.2. Since $\gamma_k \to 0$ and $x_k \to \bar{x}$ for $k \to +\infty$, there is $k_0 \in \mathbb{N}$ such that

$$\text{(5.53)} \qquad \gamma_k < \varepsilon \quad \text{and} \quad x_k \in \mathbb{B}[\bar{x}, a] \quad \text{whenever} \quad k > k_0.$$

Taking into account (A1) and (5.32), we also have

$$\|f(x_k) - f(\bar{x}) - A_k e_k\| \le \varepsilon \|e_k\| \text{ and } \|E_k s_k\| \le \varepsilon \|s_k\| \text{ whenever } k > k_0. \tag{5.54}$$

Then (5.33) and (5.53) imply, for $k > k_0$, that $\|u_k\| \le \varepsilon \|e_k\|$ as well as that

$$u_k - f(x_k) + A_k e_k + f(\bar{x}) - E_k s_k \in f(\bar{x}) + A_k e_{k+1} + F(x_{k+1}). \tag{5.55}$$

Therefore, for $k > k_0$, one can estimate

$$\begin{aligned}
\|e_{k+1}\| &\underset{(5.49)}{\le} \Theta d(0, f(\bar{x}) + A_k e_{k+1} + F(x_{k+1})) \\
&\underset{(5.55)}{\le} \Theta \|u_k - f(x_k) + A_k e_k + f(\bar{x}) - E_k s_k\| \\
&\le \Theta \|u_k\| + \Theta \|f(x_k) - A_k e_k - f(\bar{x})\| + \Theta \|E_k s_k\| \\
&\underset{(5.54)}{\le} \Theta \varepsilon \|e_k\| + \Theta \varepsilon \|e_k\| + \Theta \varepsilon \|s_k\| \\
&\le 2\Theta \varepsilon \|e_k\| + \Theta \varepsilon (\|e_{k+1}\| + \|e_k\|) = 3\Theta \varepsilon \|e_k\| + \Theta \varepsilon \|e_{k+1}\|.
\end{aligned}$$

That is

$$\frac{\|e_{k+1}\|}{\|e_k\|} \le \frac{3\Theta\varepsilon}{1 - \Theta\varepsilon} \quad \text{whenever} \quad k > k_0.$$

Since ε can be arbitrarily small, $(x_k)_{k \in \mathbb{N}_0}$ converges q-super-linearly. □

To put Theorem 5.6 in the perspective of basic results for equations, let a function $\mathbf{h} : \mathbb{R}^m \to \mathbb{R}^m$ and a point $\mathbf{u}_0 \in \mathbb{R}^m$ be given. Consider the inexact quasi-Newton method (cf. (4.7)): given $\mathbf{u}_k \in \mathbb{R}^m$ find $\mathbf{u}_{k+1} \in \mathbb{R}^m$ such that

$$\|\mathbf{h}(\mathbf{u}_k) + \mathbf{B}_k(\mathbf{u}_{k+1} - \mathbf{u}_k)\| \le \eta_k \|\mathbf{h}(\mathbf{u}_k)\| \tag{5.56}$$

for a sequence of matrices $\mathbf{B}_k \in \mathbb{R}^{m \times m}$ and for a forcing sequence $\eta_k \searrow 0$.

Corollary 5.7. *Consider a function $\mathbf{h} : \mathbb{R}^m \to \mathbb{R}^m$ which is semi-smooth at $\bar{\mathbf{u}} \in \mathbb{R}^m$. Let $* \in \{B, C\}$ and suppose that all matrices $\mathbf{A} \in \partial_* \mathbf{h}(\bar{\mathbf{u}})$ are non-singular. Consider a sequence $(\mathbf{u}_k)_{k \in \mathbb{N}}$ generated by (5.56) which is convergent to $\bar{\mathbf{u}}$. Then $\mathbf{u}_k \to \bar{\mathbf{u}}$ q-super-linearly and $\mathbf{h}(\bar{\mathbf{u}}) = \mathbf{0}$ if and only if there exists a sequence $(\mathbf{A}_k)_{k \in \mathbb{N}_0}$, with $\mathbf{A}_k \in \partial_* \mathbf{h}(\mathbf{u}_k)$ for all sufficiently large $k \in \mathbb{N}$, such that*

$$\lim_{k \to +\infty} \frac{\|(\mathbf{B}_k - \mathbf{A}_k)(\mathbf{u}_{k+1} - \mathbf{u}_k)\|}{\|\mathbf{u}_{k+1} - \mathbf{u}_k\|} = 0.$$

5.3. **Kantorovich-type Theorems.** L. V. Kantorovich [21] was the first to obtain convergence of the method on assumptions involving the point where iterations begin. Specifically, Kantorovich considered the Newton's method for solving the equation $f(x) = 0$ and proved convergence by imposing conditions on the derivative $Df(x_0)$ of the function f and the residual $\|f(x_0)\|$ at the starting point x_0. These conditions can be actually checked, in contrast to the conventional approach to assume that the derivative $Df(\bar{x})$ at a (unknown) root $\bar{x}$ of the equation is invertible and then claim that if the iteration starts close enough to $\bar{x}$ then it generates a convergent to $\bar{x}$ sequence. For this reason Kantorovich's theorem is usually called a semi-local convergence theorem whereas conventional convergence theorems are described as local theorems.

Theorem 5.8 (Kantorovich)**.** *Let X and Y be Banach spaces. Consider a function $f : X \to Y$, a point $x_0 \in X$ and a real $a > 0$, and suppose that f is continuously Fréchet differentiable in an open neighborhood of the ball $\mathbb{B}[x_0, a]$ and its derivative Df is Lipschitz continuous in $\mathbb{B}[x_0, a]$ with a constant $L > 0$. Assume that there exist positive reals κ and η such that*

$$\|Df(x_0)^{-1}\| \leq \kappa \quad \textit{and} \quad \|Df(x_0)^{-1}f(x_0)\| < \eta.$$

If $\alpha := \kappa L \eta a < \frac{1}{2}$ and $a \geq a_0 := \frac{1-\sqrt{1-2\alpha}}{\kappa L}$, then there exists a unique sequence $(x_k)_{k\in\mathbb{N}}$ satisfying the iteration

$$f(x_k) + Df(x_k)(x_{k+1} - x_k) = 0, \quad k = 0, 1, \ldots, \tag{5.57}$$

with a starting point x_0; this sequence converges to a unique zero $\bar{x}$ of f in $\mathbb{B}[x_0, a_0]$ and the convergence rate is r-quadratic:

$$\|x_k - \bar{x}\| \leq \frac{\eta}{\alpha}(2\alpha)^{2^k}, \quad k = 0, 1, \ldots.$$

In a related development, Kantorovich showed in [22, Chapter 18] that, under the same assumptions as in Theorem 5.8, to achieve linear convergence to a solution there is no need to calculate during iterations the derivative $Df(x_k)$ at the current point x_k— it is enough to use at each iteration the value of the derivative $Df(x_0)$ at the starting point, that is,

$$f(x_k) + Df(x_0)(x_{k+1} - x_k) = 0, \quad k = 0, 1, \ldots. \tag{5.58}$$

He called this method the *modified Newton process*. This method is also known as the *chord method* in the literature. The work of Kantorovich has been extended in a number of ways by, in particular, utilizing various extensions of the majorization technique. We focus on a version of Kantorovich's theorem due to R. G. Bartle [3], which has been largely forgotten if not ignored in the literature. A version of Bartle's theorem, without referring to [3], was given recently in [8, Theorem 5].

Specifically, Bartle [3] considered a function f acting between Banach spaces X and Y and the equation $f(x) = 0$ which is solved by the iteration

$$f(x_k) + Df(z_k)(x_{k+1} - x_k) = 0, \quad k = 0, 1, \ldots, \tag{5.59}$$

where Df is the Fréchet derivative mapping of f and z_k are, to quote [3], "arbitrarily selected points ... sufficiently close to the solution desired." For $z_k = x_k$ one obtains the usual Newton's method, and for $z_k = x_0$ the chord method, but z_k may be chosen in other ways. For example as x_0 for the first s iterations and then the derivative could be calculated again every s iterations, obtaining in this way a *hybrid* version of the method. If computing the derivatives, in particular in the case they are obtained numerically, involves time consuming procedures, it is quite plausible to expect that for large scale problems the chord method or a hybrid version of it would possibly be faster than the usual method. We present here the following somewhat modified statement of Bartle's theorem which fits our purposes:

Theorem 5.9 (Bartle [3])**.** *Assume that the function $f : X \to Y$ is continuously Fréchet differentiable in an open set O. Let $x_0 \in O$ and let there exist positive reals* *a and κ such that for any three points $x_1, x_2, x_3 \in \mathbb{B}[x_0, a] \subset O$ we have*

$$\|Df(x_1)^{-1}\| < \kappa \text{ and } \|f(x_1) - f(x_2) - Df(x_3)(x_1 - x_2)\| \le \frac{1}{2\kappa}\|x_1 - x_2\|, \tag{5.60}$$

and also

$$\|f(x_0)\| < \frac{a}{2\kappa}. \tag{5.61}$$

Then for every sequence $(z_k)_{n\in\mathbb{N}_0}$ in $\mathbb{B}[x_0, a]$ there exists a unique sequence $(x_k)_{k\in\mathbb{N}}$ satisfying the iteration (5.59) with initial point x_0; this sequence converges to a root $\bar{x}$ of f which is unique in $\mathbb{B}[x_0, a]$ and the convergence rate is r-linear:

$$\|x_k - \bar{x}\| \le 2^{-k}a, \quad k = 0, 1, \ldots.$$

For a non-smooth function we get the following result.

Theorem 5.10. *Let $f : X \to Y$ be a continuous function in a vicinity of $x_0 \in X$ and let numbers $a > 0$, $\kappa \ge 0$, $\delta \ge 0$ be such that*

$$\kappa\delta < 1 \quad \text{and} \quad \|f(x_0)\| < (1 - \kappa\delta)\frac{a}{\kappa}. \tag{5.62}$$

Consider a sequence $(A_k)_{k\in\mathbb{N}_0}$ in $\mathcal{L}(X, Y)$ such that for every $k \in \mathbb{N}_0$ we have

(5.63)

$$\|A_k^{-1}\| \le \kappa \text{ and } \|f(x) - f(x') - A_k(x - x')\| \le \delta\|x - x'\| \text{ for every } x, x' \in \mathbb{B}[x_0, a].$$

Then there exists a unique sequence $(x_k)_{k\in\mathbb{N}}$ satisfying

$$f(x_k) + A_k(x_{k+1} - x_k) = 0, \quad k = 0, 1, \ldots, \tag{5.64}$$

with initial point x_0. This sequence remains in $\mathbb{B}(x_0, a)$ and converges to a root $\bar{x} \in \mathbb{B}(x_0, a)$ of f which is unique in $\mathbb{B}[x_0, a]$; moreover, the convergence rate is r-linear: for each $\alpha \in (\kappa\delta, 1)$ we have

$$\|x_k - \bar{x}\| < \alpha^k a.$$

Proof. Without any loss of generality assume that $\alpha \in (\kappa\delta, 1)$ is such that

$$\|f(x_0)\| < (1 - \alpha)\frac{a}{\kappa}.$$

We will show, by induction, that there is a sequence $(x_k)_{k\in\mathbb{N}}$ with elements in $\mathbb{B}[x_0, a]$ satisfying (5.64) with the starting point x_0 such that

$$\|x_{j+1} - x_j\| \le \alpha^j\kappa\|f(x_0)\| < a\alpha^j(1 - \alpha), \quad j = 0, 1, \ldots. \tag{5.65}$$

Let $k := 0$. Since A_0 is invertible, there is a unique $x_1 \in X$ such that $A_0(x_1 - x_0) = -f(x_0)$. Therefore,

$$\|x_1 - x_0\| = \|A_0^{-1}A_0(x_1 - x_0)\| = \|A_0^{-1}f(x_0)\| \le \kappa\|f(x_0)\| < a(1 - \alpha).$$

Hence $x_1 \in \mathbb{B}(x_0, a)$. Suppose that, for some $k \in \mathbb{N}$, we have already found points $x_0, x_1, \ldots, x_k \in \mathbb{B}(x_0, a)$ satisfying (5.65) for each $j = 0, 1, \ldots, k - 1$. Since A_k is invertible, there is a unique $x_{k+1} \in X$ such that $A_k(x_{k+1} - x_k) = -f(x_k)$. Then (5.65) with $j := k - 1$ implies

$$\begin{aligned}
\|x_{k+1} - x_k\| &= \|A_k^{-1}A_k(x_{k+1} - x_k)\| = \|A_k^{-1}f(x_k)\| \le \kappa\|f(x_k)\| \\
&= \kappa\|f(x_k) - f(x_{k-1}) - A_{k-1}(x_k - x_{k-1})\| \\
&\le \kappa\delta\|x_k - x_{k-1}\| \le \alpha^k\kappa\|f(x_0)\| < a\alpha^k(1 - \alpha).
\end{aligned}$$

From (5.65), we have

$$\|x_{k+1}-x_0\| \le \sum_{j=0}^{k}\|x_{j+1}-x_j\| \le \sum_{j=0}^{k}\alpha^j\kappa\|f(x_0)\| < a\sum_{j=0}^{\infty}\alpha^j(1-\alpha) = a,$$

that is, $x_{k+1}\in\mathbb{B}(x_0,a)$. The induction step is complete.

For any natural k and p we have

(5.66)

$$\|x_{k+p+1}-x_k\| \le \sum_{j=k}^{k+p}\|x_{j+1}-x_j\| \le \sum_{j=k}^{k+p}\alpha^j\kappa\|f(x_0)\| < \frac{\alpha^k}{1-\alpha}\kappa\|f(x_0)\| < a\alpha^k.$$

Hence $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence; let it converge to $\bar{x}\in X$. Passing to the limit with $p\to+\infty$ in (5.66) we obtain

$$\|\bar{x}-x_k\| \le \frac{\alpha^k}{1-\alpha}\kappa\|f(x_0)\| < a\alpha^k \quad \text{for each} \quad k\in\mathbb{N}_0.$$

In particular, $\bar{x}\in\mathbb{B}(x_0,a)$. Using (5.64) and (5.63), we get

$$\begin{aligned} 0\le\|f(\bar{x})\| &= \lim_{k\to+\infty}\|f(x_k)\| = \lim_{k\to+\infty}\|f(x_k)-f(x_{k-1})-A_{k-1}(x_k-x_{k-1})\| \\ &\le \lim_{k\to+\infty}\delta\|x_k-x_{k-1}\| = 0. \end{aligned}$$

Hence, $f(\bar{x})=0$. Suppose that there is $\bar{y}\in\mathbb{B}[x_0,a]$ with $\bar{y}\ne\bar{x}$ and $f(\bar{y})=0$. Then

$$\begin{aligned} \|\bar{y}-\bar{x}\| &\le \kappa\|A_0(\bar{y}-\bar{x})\| = \kappa\|f(\bar{y})-f(\bar{x})-A_0(\bar{y}-\bar{x})\| \\ &\le \kappa\delta\|\bar{y}-\bar{x}\| < \alpha\|\bar{y}-\bar{x}\| < \|\bar{y}-\bar{x}\|, \end{aligned}$$

which is a contradiction. Hence $\bar{x}$ is a unique root of f in $\mathbb{B}[x_0,a]$. □

Extensions of Theorem 5.8 and Theorem 5.10 for a generalized equation together with numerical experiments can be found in [10], where the following model of an iterative procedure for solving (5.1) is considered. Given $k\in\mathbb{N}_0$, the current and prior iterates x_n $(n\le k)$ generate a “feasible" element $A_k\in\mathcal{L}(X,Y)$ and then choose the next iterate $x_{k+1}\in X$ according to the Newton-type iteration:

$$f(x_k)+A_k(x_{k+1}-x_k)+F(x_{k+1})\ni 0.$$

As in the previous two subsections, the invertibility of linear mappings appearing in iteration (5.64) is replaced by the (strong) metric regularity of mappings

$$f(x_0)+A_k(\cdot-x_0)+F, \quad k\in\mathbb{N}_0,$$

at x_0 for y_0 with a constant $\kappa>0$ and neighborhoods $\mathbb{B}[x_0,a]$ and $\mathbb{B}[y_0,b]$. Here a point $x_0\in X$ is the starting point of the iteration and $y_0\in f(x_0)+F(x_0)$, which plays the role of the initial residual, is supposed to have a sufficiently small norm. Moreover, different rates of r-convergence are considered.

References

[1] Aubin, J.-P.; Cellina, A. *Differential inclusions. Set-valued maps and viability theory.* Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin, 1984.

[2] Aubin, J.-P.; Frankowska, H. *Set-valued analysis.* Systems&Control: Foundations&Applications. Birkhäuser, Boston, 1990.

[3] Bartle, R. G. *Newton's method in Banach spaces.* Proc. Amer. Math. Soc. 6 (1955) 827–831.

[4] Bolte, J.; Daniilidis, A.; Lewis, A. *Tame functions are semismooth.* Math. Program. 117 (2009) 5–19.

[5] Dennis, J. E. Jr.; Moré, J. J. *A characterization of superlinear convergence and its application to quasi-Newton methods.* Math. Comp. 28 (1974) 549–560.

[6] Dembo, R. S.; Eisenstat, S. C.; Steihaug, T. *Inexact Newton methods.* SIAM J. Numer. Anal. 19 (1982) 400–408.

[7] Dontchev, A. L. *Generalizations of the Dennis-Moré theorem.* SIAM J. Optim. 22 (2012) 821–830.

[8] Ciarlet, P. G.; Mardare, C. *On the Newton-Kantorovich theorem.* Anal. Appl. (Singap.) 10 (2012) 249–269.

[9] Cibulka, R.; Dontchev, A. L. *A Nonsmooth Robinson's Inverse Function Theorem in Banach Spaces.* Math. Program. 156 (2016) 257–270.

[10] Cibulka, R.; Dontchev, A. L.; Preininger, J.; Roubal, T.; Veliov, V. *Kantorovich-type Theorems for Generalized Equations.* (2015), submitted for publication.

[11] Cibulka, R.; Geoffroy, M.; Dontchev, A. L. *Inexact Newton Methods and Dennis–Moré Theorems for Nonsmooth Generalized Equations.* SIAM J. Control Optim. 53 (2015) 1003–1019.

[12] Clarke, F. H. *On the inverse function theorem.* Pacific J. Math. 64 (1976) 97–102.

[13] Dontchev, A. L.; Rockafellar, R. T. *Implicit functions and solution mappings. A view from variational analysis.* Springer Monographs in Mathematics. Springer, Dordrecht, 2009.

[14] Fabian, M. *Concerning interior mapping theorem.* Comment. Math. Univ. Carolin. 20 (1979) 345–356.

[15] Facchinei, F.; Pang, J.-S. *Finite-dimensional variational inequalities and complementarity problems.* Vol. I. Springer Series in Operations Research. Springer-Verlag, New York, 2003.

[16] Facchinei, F.; Pang, J.-S. *Finite-dimensional variational inequalities and complementarity problems.* Vol. II. Springer Series in Operations Research. Springer-Verlag, New York, 2003.

[17] Filippov, A. F. *Differential equations with discontinuous righthand sides.* Translated from the Russian. Mathematics and its Applications (Soviet Series). Kluwer Academic Publishers Group, Dordrecht, 1988.

[18] Izmailov, A. F. *Strongly regular nonsmooth generalized equations.* Math. Program. 147 (2014) 581–590.

[19] Qi, L.; Sun, J. *A nonsmooth version of Newton's method.* Math. Program. 58 (1993) 353–367.

[20] Ioffe, A. D. *Nonsmooth analysis: differential calculus of nondifferentiable mappings.* Trans. Amer. Math. Soc. 266 (1981) 1–56.

[21] Kantorovich, L. V. *On Newton's method for functional equations.* (Russian), Doklady Akad. Nauk SSSR (N.S.) 59 (1948) 1237–1240.

[22] Kantorovich, L. V.; Akilov, G. P. *Functional analysis.* (Russian), 2nd Edition, revised, Nauka, Moscow, 1977.

[23] Klatte, D.; Kummer, B. *Nonsmooth equations in optimization. Regularity, calculus, methods and applications.* Kluwer, Dordrecht, 2002.

[24] Pang, J.-S.; Stewart, D. E. *Differential variational inequalities.* Math. Program. 113 (2008) 345–424.

[25] Pang, J.-S.; Stewart, D. E. *Solution dependence on initial conditions in differential variational inequalities.* Math. Program. 116 (2009) 429–460.

[26] Páles, Zs.; Zeidan, V. *Generalized Jacobian for functions with infinite dimensional range and domain.* Set-Valued Anal. 15 (2007) 331–375.

[27] Páles, Zs.; Zeidan, V. *Infinite dimensional generalized Jacobian: properties and calculus rules.* J. Math. Anal. Appl. 344 (2008) 55–75.

[28] Penot, J.- P. *Calculus without derivatives.* Springer, Dordrecht, 2013.

[29] Robinson, S. M. *Strongly regular generalized equations.* Math. Oper. Res. 5 (1980) 43–62.
[30] Rockafellar, R. T. *Convex analysis.* Princeton Mathematical Series, No. 28 Princeton University Press, Princeton, 1970.
[31] Simeon, B. *Differential-Algebraic Equations (DAEs) Differential-algebraische Gleichungen.* http://www.mathematik.uni-kl.de/fileadmin/das/simeon/notesdaes.pdf
[32] Stewart, D. E. *Dynamics with inequalities. Impacts and hard constraints.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011.
[33] Xu, H. *Set-valued approximations and Newton's method.* Math. Program. 84 (1999) 401–420.

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