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Master thesis

Biological reaction-diffusion models

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Prohlášení

Prohlašuji, že jsem diplomovou práci vypracoval samostatně a výhradně s použitím uvedených zdrojů.

V Plzni dne 11. května, 2016 ....

Martin Fencl

Poděkování

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Abstrakt

V této práci se zabýváme soustavou dvou parciálních diferenciálních reakčně-difuzních rovnic, se kterou souvisí tzv. Turingův efekt. Provedeme rešerši základní teorie týkající se tohoto systému a Turingova efektu a shrneme podmínky na existenci tohoto efektu. Dále se budeme věnovat systému s jednostranným zdrojovým členem v první rovnici a jeho vlivu na rozložení kritických a bifurkačních bodů tohoto systému v kladném kvadrantu $\mathbb{R}^2_+$ pro dva různé typy okrajových podmínek. V druhé části práce se zaměříme na numerické experimenty týkajících se konkrétního modelu s různými jednostrannými členy.

Klíčová slova: systém reakčně-difuzních rovnic, Turingův efekt, difuzí řízená nestabilita, jednostranný člen, spektrální analýza, vzorek, numerické experimenty

Abstract

We consider a system of two partial differential reaction-diffusion equations. The first goal is to present so called Turing effect and the appropriate theory. Then we focus on the system with an unilateral source term in the first equation of this system. We shall investigate an influence of this unilateral term on the displacement of critical and bifurcation points in positive quadrant $\mathbb{R}^2_+$. Eventually we use numerical methods to experiment with the concrete model with various unilateral terms.

Key words: system of reaction-diffusion equations, Turing effect, diffusion driven instability, unilateral term, spectral analysis, pattern, numerical experiments

Contents

Preface

The aim of this master thesis is to study a system of reaction-diffusion equations. This type of equations can describe an interaction of for example chemical substances or populations in some space. We consider a system

$$\begin{aligned} \frac{du}{dt} &= d_1 \Delta u + f(u,v), \\ \frac{dv}{dt} &= d_2 \Delta v + g(u,v), \quad \text{in } \Omega \times [0,+\infty) \end{aligned} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$. We suppose a Neumann boundary condition on the part Γ_N of the boundary and a Dirichlet boundary condition on the part Γ_D of the boundary. The diffusion parameters d_1, d_2 are positive and functions f, g describe a reaction of the substances u, v.

We study so called Turing effect presented by Alan Turing in the article [13] in 1952. If this effect occurs, spatially non-homogeneous solutions describing spatial patterns arise. In the first chapter, we summarize a basic theory concerning the system (1) and the Turing effect from [7] and [11].

The second chapter deals with a linearized system with a unilateral source term τu^-. We consider the mixed boundary conditions here, i.e. $\Gamma_D \neq \emptyset$. We are interested in the influence of the unilateral term on a distribution of critical and bifurcation points. The main results are inspired by the article [6]. The system considered in Chapter 3 is the same as in Chapter 2. However we will consider only pure Neumann conditions, i.e. $\Gamma_D = \emptyset$.

The last chapter is focused on numerical experiments with a concrete model and various unilateral terms. We continue the work of Vejchodský et al from the article [12]. We experiment with the unilateral term τu^- in the first equation, but larger part of this chapter is focused on the unilateral terms with saturation added to the second equation of the model. We are interested in qualitative properties of produced patterns and also a possibility to generate patterns for a large ratio $D = \frac{d_1}{d_2}$.

In the analytical part of this thesis, theory of partial differential equations and functional analysis is used. We put usually known statements and definitions into the appendix and recall on them when it is necessary. The theory of partial differential equations is taken from [2] and [4], while the theory of functional analysis is taken from [1], [3] and [9].

Notation

$\mathbb{N}_0; \mathbb{N}_{j_0}$	$\mathbb{N} \cup \{0\}$; set $\mathbb{N}$ starting with j_0,
$\mathbb{R}_+; \mathbb{R}^2_+$	$\{x \in \mathbb{R} : x > 0\}; \{[x, y] \in \mathbb{R}^2 : x > 0 \wedge y > 0\}$,
$\mathbb{R}^+_0$	$\{x \in \mathbb{R} : x \geq 0\}$,
$det(\mathbf{A})$	determinant of matrix $\mathbf{A}$,
$tr(\mathbf{A})$	trace of matrix $\mathbf{A}$,
$\mathbf{I}$	identity matrix,
$\partial\Omega$	boundary of domain Ω,
$\overline{\Omega}$	$\Omega \cup \partial\Omega$,
Δ	Laplace operator,
∇	gradient operator,
$\|\cdot\|_X$	norm on appropriate space X,
$\|F\|_*$	dual norm of linear continuous functional F,
$(\cdot, \cdot)$	inner product,
Tu	linear operator T on element u,
$T(u)$	nonlinear operator T on element u,
$F(\varphi)$	functional of variable φ,
$\to_X$	strong convergence on space X,
$\rightharpoonup_X$	weak convergence on space X,
$\sigma(T)$	spectrum of operator T,
$X \hookrightarrow Y$	continuous embedding of space X to space Y,
$X \hookrightarrow\hookrightarrow Y$	compact embedding of space X to space Y,
c_{emb}	embedding constant,
$\mathcal{C}(X, Y)$	space of compact operators between X and Y,
$Ker(T)$	kernel of operator T,

Chapter 1

Introduction

Let's consider a reaction-diffusion system

$$\begin{aligned} \frac{du}{dt} &= d_1 \Delta u + f(u,v), \\ \frac{dv}{dt} &= d_2 \Delta v + g(u,v), \quad \text{in } \Omega \times [0,+\infty) \end{aligned} \tag{1.1}$$

with mixed boundary conditions

$$\begin{aligned} u = \overline{u}, v = \overline{v} \quad &\text{on } \Gamma_D, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad &\text{on } \Gamma_N. \end{aligned} \tag{1.2}$$

We assume that the domain $\Omega \subset \mathbb{R}^N$ is bounded with Lipschitz boundary, $u = u(\mathbf{x}, t)$, $v = v(\mathbf{x}, t)$ where $\mathbf{x} = [x_1, x_2, \ldots, x_N] \in \mathbb{R}^N$ are space variables and t is the time variable. The functions f, g are smooth, d_1 and d_2 are positive diffusion parameters and n is a unit outward-pointing normal vector of the boundary $\partial\Omega$. We suppose that Γ_N, Γ_D are open disjoint subsets of $\partial\Omega$ and $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$. We will distinguish two cases, that is $\Gamma_D \neq \emptyset$ and $\Gamma_D = \emptyset$. Let's assume that the functions f, g satisfy

$$f(\overline{u}, \overline{v}) = g(\overline{u}, \overline{v}) = 0. \tag{1.3}$$

Apparently $[\overline{u}, \overline{v}]$ is the constant stationary solution of the problem (1.1), (1.2).

1.1 Turing effect and biological motivation

The effect we study in the most of this work was discovered by Alan M. Turing, the British mathematician, logician, cryptanalyst and theoretical biologist. He published his ideas in article called "The chemical basis of morphogenesis" in 1952 (see [13]). The main idea is that the stationary solution $[\overline{u}, \overline{v}]$ of the problem (1.1), (1.2) without diffusion ($d_1 = d_2 = 0$) is stable, but for some values of d_1, d_2 the stationary solution of the problem

(1.1), (1.2) with diffusion is unstable. Since we consider the zero Neumann condition, i.e. zero flow through the boundary, this effect can be caused only by the diffusion. For this reason this effect is sometimes called also "Turing diffusion driven instability". The loss of the stability of the constant stationary solution $[\overline{u}, \overline{v}]$ give rise to the stationary spatially non-homogeneous solutions. These solutions describe the spatial patterns, which has applications as for example patterns on the animal coat. For details see [11].

1.2 Conditions of diffusion driven instability

Since we will study the stability of the stationary solution $\overline{W} := [\overline{u}, \overline{v}]$, it is convenient to use a linearization. After we apply Taylor expansion on the neighbourhood of $\overline{W}$, we get

$$\begin{aligned} f(u,v) &= \nabla f(\overline{u},\overline{v})(u-\overline{u}, v-\overline{v})^T + n_1(u,v), \\ g(u,v) &= \nabla g(\overline{u},\overline{v})(u-\overline{u}, v-\overline{v})^T + n_2(u,v), \end{aligned} \tag{1.4}$$

where $n_{1,2}(u,v)$ are higher order terms of Taylor expansion. We set

$$\begin{aligned} b_{1,1} &= \frac{\partial f}{\partial u}(\overline{u},\overline{v}), \quad b_{1,2} = \frac{\partial f}{\partial v}(\overline{u},\overline{v}), \\ b_{2,1} &= \frac{\partial g}{\partial u}(\overline{u},\overline{v}), \quad b_{2,2} = \frac{\partial g}{\partial v}(\overline{u},\overline{v}), \end{aligned}$$

and $\mathbf{B} := (b_{i,j})_{i,j=1,2}$, which is Jacobi matrix of the mapping f, g.
Now we can write the system (1.1) as

$$\begin{aligned} \frac{du}{dt} &= d_1 \Delta u + b_{1,1}(u-\overline{u}) + b_{1,2}(v-\overline{v}) + n_1(u,v), \\ \frac{dv}{dt} &= d_2 \Delta v + b_{2,1}(u-\overline{u}) + b_{2,2}(v-\overline{v}) + n_2(u,v). \end{aligned} \tag{1.5}$$

It is convenient to substitute $w := u - \overline{u}$, $z = v - \overline{v}$, to introduce $\overline{n}_{1,2}(w,z) = n_{1,2}(w + \overline{u}, z + \overline{v})$ and solve the equivalent problem with stationary solution $[0, 0]$. The stability of the zero stationary solution is stability of $[\overline{u}, \overline{v}]$. The higher order terms $\overline{n}_{1,2}$ are only small perturbations in the neighbourhood of zero (stationary solution), i.e.

$$\overline{n}_{1,2}(u,v) = o(|u| + |v|) \text{ as } |u| + |v| \to 0. \tag{1.6}$$

In the further text we will use the model with stationary solution in zero, but we will write again $[u, v]$ instead of $[w, z]$:

$$\begin{aligned} \frac{du}{dt} &= d_1 \Delta u + b_{1,1} u + b_{1,2} v + \overline{n}_1(u,v), \\ \frac{dv}{dt} &= d_2 \Delta v + b_{2,1} u + b_{2,2} v + \overline{n}_2(u,v), \end{aligned} \tag{1.7}$$

with

$$\begin{aligned} &u = v = 0 \quad \text{on } \Gamma_D, \\ &\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \Gamma_N. \end{aligned} \tag{1.8}$$

1.2.1 Conditions of stability without diffusion

In the previous section we presented, that Turing effect assumes the stability of the system without diffusion. Let's take a brief look at our problem with $d_1 = d_2 = 0$. Such a system is actually a system of ordinary differential equations and we can use Theorem $B.1$. The eigenvalues $\lambda_{1,2}$ of $\mathbf{B}$ must be negative, so that the stationary solution would be stable.

Let's compute the eigenvalues of $\mathbf{B}$, which we get as solutions of the equation

$$det(\mathbf{B} - \lambda \mathbf{I}) = 0.$$

This equation leads to

$$\lambda^2 - \lambda(b_{1,1} + b_{2,2}) + b_{1,1}b_{2,2} - b_{1,2}b_{2,1} = 0,$$

and solutions

$$\lambda_1 = \frac{b_{1,1} + b_{2,2} + \sqrt{(b_{1,1} + b_{2,2})^2 - 4(b_{1,1}b_{2,2} - b_{1,2}b_{2,1})}}{2},$$
$$\lambda_2 = \frac{b_{1,1} + b_{2,2} - \sqrt{(b_{1,1} + b_{2,2})^2 - 4(b_{1,1}b_{2,2} - b_{1,2}b_{2,1})}}{2}.$$

If we look closer at λ_1, it is clear that $b_{1,1} + b_{2,2}$ must be negative. That will also ensure the negativity of λ_2. Furthermore the positivity of $b_{1,1}b_{2,2} - b_{1,2}b_{2,1}$ will give us the negativity of λ_1.

In conclusion, conditions

$$tr(\mathbf{B}) = b_{1,1} + b_{2,2} < 0, \tag{1.9}$$
$$det(\mathbf{B}) = b_{1,1}b_{2,2} - b_{1,2}b_{2,1} > 0, \tag{1.10}$$

will ensure the stability of the trivial solution of the system (1.7) without diffusion ($d_1 = d_2 = 0$).

1.2.2 Conditions of instability with diffusion

In this subsection, we shall focus on finding conditions, which will make sure that the stationary solution $\overline{W}$ of the system with diffusion is unstable. We can expect conditions not only on elements of the matrix $\mathbf{B}$, but also parameters d_1, d_2. Let's consider the system

$$\begin{aligned} d_1\Delta u + b_{1,1}u + b_{1,2}v &= \lambda u, \\ d_2\Delta v + b_{2,1}u + b_{2,2}v &= \lambda v. \end{aligned} \tag{1.11}$$

We will consider a Hilbert space

$$H^1_D(\Omega) := \{\phi \in W^{1,2}(\Omega) : \phi = 0 \text{ on } \Gamma_D \text{ "in the sense of traces (see } B.5\text{)" } \} \tag{1.12}$$

equipped with the inner product $(u, v) = \int_\Omega (\nabla u \nabla v + uv)\ d\Omega$. We will denote the norm induced by the inner product by $\|u\|_{W^{1,2}}$.

Remark 1.1. *By a solution of any problem we will always mean a weak solution introduced bellow.*

The signs of eigenvalues λ of the system (1.11) decide the stability of $\overline{W}$ as the following theorem states.

Theorem 1.1:
If there exists $\varepsilon > 0$ such that all eigenvalues λ of the problem (1.11), (1.8) fulfil $Re(\lambda) < -\varepsilon$, then the stationary solution $\overline{W}$ of the problem (1.11), (1.8) is stable in the norm of the space $H^1_D(\Omega)$. If there exists an eigenvalue such that $Re(\lambda) > 0$, then $\overline{W}$ is unstable.

Let a pair of functions $u, v \in H^1_D(\Omega)$ fulfils integral identity

$$\begin{aligned} &\int_\Omega d_1 \nabla u \nabla \varphi - (b_{1,1}u + b_{1,2}v - \lambda u)\varphi\ d\Omega = 0, \\ &\int_\Omega d_2 \nabla v \nabla \varphi - (b_{2,1}u + b_{2,2}v - \lambda v)\varphi\ d\Omega = 0, \end{aligned} \qquad \forall \varphi \in H^1_D(\Omega). \tag{1.13}$$

Then such a pair $u, v \in H^1_D(\Omega)$ is called a weak solution of the problem (1.11), (1.8). Before we proceed further, it would be good to check the finiteness of the integrals in (1.13). We will use properties of Lebesgue integral, Hölder's inequality $B.1$ and continuous embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$:

$$\begin{aligned} \int_\Omega \nabla u \nabla \varphi\ d\Omega &\leq \left| \int_\Omega \nabla u \nabla \varphi\ d\Omega \right| \leq \int_\Omega |\nabla u \nabla \varphi|\ d\Omega \leq \\ &\leq \left(\int_\Omega |\nabla u|^2\ d\Omega \right)^{\frac{1}{2}} \cdot \left(\int_\Omega |\nabla \varphi|^2\ d\Omega \right)^{\frac{1}{2}} \leq c_1 \|u\|_{W^{1,2}} \|\varphi\|_{W^{1,2}}, \\ \int_\Omega u\varphi\ d\Omega &\leq \left| \int_\Omega u\varphi\ d\Omega \right| \leq \int_\Omega |u\varphi|\ d\Omega \leq \\ &\leq \|u\|_{L^2} \|\varphi\|_{L^2} \leq c_2 \|u\|_{W^{1,2}} \|\varphi\|_{W^{1,2}}. \end{aligned}$$

The other integrals are only variations of these two.

Let's define an operator $A : H^1_D(\Omega) \mapsto H^1_D(\Omega)$ for any fixed $u \in H^1_D$ by

$$(Au, \varphi) = \int_\Omega u\varphi\ d\Omega, \quad \forall \varphi \in H^1_D(\Omega). \tag{1.14}$$

We show that this definition is correct thanks to Riesz representation theorem $B.2$. If for fixed u we set

$$T(\varphi) = \int_\Omega u\varphi\ d\Omega, \quad \forall \varphi \in H^1_D(\Omega),$$

then T is a linear continuous functional. Hence by Riesz representation theorem there exists exactly one $Au \in H^1_D(\Omega)$ such that

$$(Au, \varphi) = T(\varphi) = \int_\Omega u\varphi \, d\Omega, \quad \forall \varphi \in H^1_D(\Omega).$$

Moreover

$$\|Au\|_{W^{1,2}} = \|T\|_* = \sup_{\|\varphi\|_{W^{1,2}} \leq 1} |T(\varphi)| = \sup_{\|\varphi\|_{W^{1,2}} \leq 1} |(Au, \varphi)|.$$

Lemma 1.1:
Let the operator $A : H^1_D(\Omega) \mapsto H^1_D(\Omega)$ be defined by (1.14). Then A is a linear, continuous, symmetric and compact operator.

Proof.

$\forall u_1, u_2 \in H^1_D(\Omega) \; \forall \alpha \in \mathbb{R}$:

$$(A(\alpha u_1 + u_2), \varphi) = \int_\Omega (\alpha u_1 + u_2)\varphi \, d\Omega = \alpha \int_\Omega u_1\varphi \, d\Omega + \int_\Omega u_2\varphi \, d\Omega = \alpha(Au_1, \varphi) + (Au_2, \varphi).$$

Hence A is linear. Since a continuity is the same as a boundedness for the linear operators, we will show the boundedness:

$$\begin{aligned} \|Au\|_{W^{1,2}} &= \sup_{\|\varphi\|_{W^{1,2}} \leq 1} |(Au, \varphi)| = \sup_{\|\varphi\|_{W^{1,2}} \leq 1} \left| \int_\Omega u\varphi \, d\Omega \right| \leq \\ &\leq \sup_{\|\varphi\|_{W^{1,2}} \leq 1} \{ \|u\|_{L^2} \|\varphi\|_{L^2} \} \leq c\|u\|_{W^{1,2}}. \end{aligned}$$

We used properties of supremum, Hölder inequality and the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$. We have

$$(Au, \varphi) = \int_\Omega u\varphi \, d\Omega = (A\varphi, u) = (u, A\varphi), \quad \forall u, \varphi \in H^1_D(\Omega),$$

hence the operator A is symmetric.

In order to show the compactness, we will need a compact embedding $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ from Rellich-Kondrachov theorem $B.3$. Its necessary to show that the operator A maps every bounded sequence (u_n) to (Au_n), which has a convergent subsequence.

Let (u_n) be a bounded sequence. From Riesz representation theorem we have that every Hilbert space is reflexive, hence by Eberlain-Šmuljan theorem $B.4$ the sequence (u_n) has weakly convergent subsequence $u_{n_k} \rightharpoonup_{H^1_D} u$. By the compact embedding we have that $u_{n_k} \to_{L^2} u$. Using Hölder's inequality we get

$$\begin{aligned} \|Au_{n_k} - Au\|_{W^{1,2}} &= \sup_{\|\varphi\|_{W^{1,2}} \leq 1} |(Au_{n_k} - Au, \varphi)| = \sup_{\|\varphi\|_{W^{1,2}} \leq 1} |\int_\Omega (u_{n_k} - u)\varphi \, d\Omega| \leq \\ &\leq \sup_{\|\varphi\|_{W^{1,2}} \leq 1} \{ (\int_\Omega |u_{n_k} - u|^2 \, d\Omega)^{\frac{1}{2}} \cdot (\int_\Omega |\varphi|^2 \, d\Omega)^{\frac{1}{2}} \} \leq \\ &\leq c\|u_{n_k} - u\|_{L^2}. \end{aligned}$$

From this inequality we have that $Au_{n_k} \to_{H^1_D} Au$, i.e. (Au_{n_k}) is a strongly convergent subsequence of (Au_n). □

We equivalently rewrite the system (1.13) as

$$\left.\begin{aligned} \int_\Omega d_1(\nabla u \nabla \varphi + u\varphi) - (b_{1,1}u + b_{1,2}v + d_1 u - \lambda u)\varphi \, d\Omega = 0, \\ \int_\Omega d_2(\nabla v \nabla \varphi + v\varphi) - (b_{2,1}u + b_{2,2}v + d_2 v - \lambda v)\varphi \, d\Omega = 0, \end{aligned}\right. \quad \forall \varphi \in H^1_D(\Omega).$$

Further we can use the definition of the inner product and of the operator A to formulation

$$\begin{aligned} (d_1 u - (b_{1,1}Au + b_{1,2}Av + d_1 Au - \lambda Au), \varphi) = 0, \\ (d_2 v - (b_{2,1}Au + b_{2,2}Av + d_2 Av - \lambda Av), \varphi) = 0, \end{aligned} \quad \forall \varphi \in H^1_D(\Omega).$$

Since both inner products must be equal to zero for all $\varphi \in H^1_D(\Omega)$, we get

$$\begin{aligned} d_1 u - (d_1 + b_{1,1})Au - b_{1,2}Av + \lambda Au = 0, \\ d_2 v - b_{2,1}Au - (d_2 + b_{2,2})Av + \lambda Av = 0. \end{aligned} \tag{1.15}$$

We will rewrite this system of two operator equations by denoting

$$W = \begin{bmatrix} u \\ v \end{bmatrix}, \quad D(d) = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \quad B(d) = \begin{bmatrix} d_1 + b_{1,1} & b_{1,2} \\ b_{2,1} & d_2 + b_{2,2} \end{bmatrix}, \quad AW = \begin{bmatrix} Au \\ Av \end{bmatrix},$$

into the vector form

$$D(d)W - B(d)AW + \lambda AW = 0. \tag{1.16}$$

In order to get an eigenvalue λ, we need to express the eigenvalues of A depending on the eigenvalues of the Laplace operator.

Remark 1.2.
We consider an eigenvalue problem

$$\begin{aligned} -\Delta u &= \kappa u, \\ u &= 0 \ \text{on } \Gamma_D, \\ \frac{\partial u}{\partial n} &= 0 \ \text{on } \Gamma_N. \end{aligned} \tag{1.17}$$

A sequence of eigenvalues (κ_j) of (1.17) is infinite and non-decreasing. The eigenvalues are not necessarily simple and $\kappa_j \to \infty$ as $j \to \infty$.

An integral identity

$$\int_\Omega \nabla u \nabla \varphi - \kappa u \varphi \, d\Omega = 0 \quad \forall \varphi \in H^1_D(\Omega) \tag{1.18}$$

is a weak formulation of (1.17) *and* $u \in H^1_D(\Omega)$ *satisfying* (1.18) *is the weak solution of* (1.17)*.*
i.) Assume the case $\Gamma_D \neq \emptyset$***:***
The first eigenvalue κ_1 *is simple. The eigenfunction* e_1 *corresponding to the first eigenvalue* κ_1 *doesn't change the sign on the domain* Ω *and we can choose it to be positive. Actually, the eigenfunction* e_1 *is the only eigenfunction that doesn't change the sign on* Ω *(see* [5]*).*
ii.) Assume the case $\Gamma_D = \emptyset$***:***
The first eigenvalue $\kappa_0 = 0$ *is simple and the corresponding eigenfunction* e_0 *is constant. All other eigenfunctions change the sign on the domain* Ω*.*

By the definition (1.14) *of the operator* A *and of the inner product we get*

$$\begin{aligned} (u - (1+\kappa)Au, \varphi) &= 0, \quad \forall \varphi \in H^1_D(\Omega), \\ Au &= \frac{1}{1+\kappa} u. \end{aligned} \tag{1.19}$$

Hence $\mu_j := \frac{1}{1+\kappa_j}$ *are the eigenvalues of the operator* A*, where* κ_j *are the eigenvalues of the problem* (1.17)*. The corresponding eigenfunctions* e *of the operator* A *coincide with the eigenfunctions of* (1.17) *in both cases.*

We can choose an complete orthonormal system composed of eigenfunctions of the Laplacian and denote it e_j*, where* $j = 0, 1, \ldots$ *if* $\Gamma_D = \emptyset$ *and* $j = 1, \ldots$ *if* $\Gamma_D \neq \emptyset$*.*

We can express the solution W in a form of series, which is convergent in the space $H^1_D \times H^1_D$. The index j_0 in the further text is $j_0 = 0$ if $\Gamma_D = \emptyset$ and $j_0 = 1$ if $\Gamma_D \neq \emptyset$. We get

$$W = \sum_{j=j_0}^{+\infty} F_j e_j. \tag{1.20}$$

where $F_j = [f^1_j, f^2_j]^T$ are Fourier coefficients and $u = \sum_{j=j_0}^{+\infty} f^1_j e_j, v = \sum_{j=j_0}^{+\infty} f^2_j e_j$. Hence

$$AW = A\left(\sum_{j=j_0}^{+\infty} F_j e_j\right) = A\left(\lim_{n\to\infty} \sum_{j=j_0}^{n} F_j e_j\right).$$

Since the operator A is continuous, we can switch the limit and the operator, and thanks to the linearity we can switch A and the finite sum:

$$AW = A\left(\lim_{n\to\infty} \sum_{j=j_0}^{n} F_j e_j\right) = \lim_{n\to\infty} A\left(\sum_{j=j_0}^{n} F_j e_j\right) = \sum_{j=j_0}^{\infty} F_j A e_j. \tag{1.21}$$

We are able to use expressions (1.21) and (1.20) in the equation (1.16) to get

$$\sum_{j=j_0}^{\infty} \left(D(d) - B(d)A + \lambda A\right) F_j e_j = 0.$$

We can use the second line of (**??**) in the previous equation and then multiply it by $(1+\kappa_j)$. Hence we get

$$\sum_{j=j_0}^{\infty} \left(D(d)(1+\kappa_j) - B(d) + \lambda \mathbf{I}\right) F_j e_j = 0.$$

As we stated above, e_j is the orthonormal system in H^1_D, which means e_j are linearly independent and only way to fulfil the equality is that

$$\left(D(d)(1+\kappa_j) - B(d) + \lambda \mathbf{I}\right) F_j = 0, \quad \forall j \in \mathbb{N}_{j_0}.$$

Moreover $D(d) - B(d) = -\mathbf{B}$, hence

$$\left(D(d)\kappa_j - \mathbf{B} + \lambda \mathbf{I}\right) F_j = 0, \quad \forall j \in \mathbb{N}_{j_0}. \tag{1.22}$$

The problem (1.22) is a system of two linear algebraic equations (for every j) where $F_j = [f_j^1, f_j^2]^T$. The number λ from (1.11) is an eigenvalue if and only if there exist some j such that (1.22) has a non-trivial solution F_j. Hence must be

$$det(D(d)\kappa_j - \mathbf{B} + \lambda \mathbf{I}) = 0$$

that means

$$\lambda^2 + \lambda \left[\kappa_j(d_1+d_2) - (b_{1,1}+b_{2,2})\right] + (\kappa_j d_1 - b_{1,1})(\kappa_j d_2 - b_{2,2}) - b_{1,2}b_{2,1} = 0. \tag{1.23}$$

At first let's denote

$$H_d(\kappa_j) := (\kappa_j d_1 - b_{1,1})(\kappa_j d_2 - b_{2,2}) - b_{1,2}b_{2,1} \tag{1.24}$$

for further use. The number λ is the eigenvalue of problem (1.11) if and only if for some j it is a solution of (1.23), i.e. $\lambda = \lambda_1^j$ or $\lambda = \lambda_2^j$. From the quadratic equation we get

$$\lambda_{1,2}^j = \frac{b_{1,1}+b_{2,2} - \kappa_j(d_1+d_2) \pm \sqrt{(\kappa_j(d_1+d_2) - b_{1,1} - b_{2,2})^2 - 4H_d(\kappa_j)}}{2}. \tag{1.25}$$

Theorem 1.1 states that at least one eigenvalue of (1.11) must satisfy $Re(\lambda) > 0$ to ensure the instability of the stationary solution. Since we assume (1.9), $d_1, d_2 > 0$ and the eigenvalues of the Laplacian κ_j are non-negative, the eigenvalue λ_2^j is always negative. Hence we need to find conditions such that $Re(\lambda_1^j) > 0$.

Let's look closer on influence of $H_d(\kappa_j)$ on λ_1^j. If $H_d(\kappa_j) > 0$ then $Re(\lambda_1^j) < 0$. From the definition of $H_d(\kappa_j)$ we can see that for $\kappa_0 = 0$ (in case $\Gamma_D = \emptyset$) is $H_d(\kappa_j) > 0$. So we will only work with $\kappa_j > 0$. Hence we need to find out under which conditions we get $H_d(\kappa_j) < 0$. After multiplying brackets we get

$$H_d(\kappa_j) = b_{1,1}b_{2,2} + \kappa_j^2 d_1 d_2 - \kappa_j(b_{1,1}d_2 + b_{2,2}d_1) - b_{1,2}b_{2,1} = det(\mathbf{B}) + \kappa_j^2 d_1 d_2 - \kappa_j(b_{1,1}d_2 + b_{2,2}d_1).$$

We suppose that condition (1.10) holds, $d_1, d_2 > 0$ and $\kappa_j > 0$. Hence we need guaranteeing

$$b_{1,1}d_2 + b_{2,2}d_1 > 0, \tag{1.26}$$

which is the first condition (necessary condition). We can rewrite $H_d(\kappa_j) < 0$ as

$$det(\mathbf{B}) < \kappa_j(b_{1,1}d_2 + b_{2,2}d_1) - \kappa_j^2 d_1 d_2.$$

Since $d_1, d_2 > 0$, the function with respect to κ on the right-hand side of this inequality is a parabola with vertex direction up. It would be problematic to find out what function values of this parabola are bigger then $det(\mathbf{B})$. However with opposite inequality (i.e. $H_d(\kappa_j) > 0$), it is pretty easy. We can compute coordinates of the vertex by searching maximum of the parabola or from the usual formula. Either way coordinates of the vertex are $[\frac{b_{1,1}d_2+b_{2,2}d_1}{2d_1d_2}, \frac{(b_{1,1}d_2+b_{2,2}d_1)^2}{4d_1d_2}]$, and for $det(\mathbf{B}) > \frac{(b_{1,1}d_2+b_{2,2}d_1)^2}{4d_1d_2}$ we have $H_d(\kappa_j) > 0$. Therefore

$$det(\mathbf{B}) < \frac{(b_{1,1}d_2 + b_{2,2}d_1)^2}{4d_1d_2} \tag{1.27}$$

implies $H_d(\kappa_j) < 0$ and this is the second condition.

Let's sum up under which conditions Turing effect occurs:

$$\begin{aligned} tr(\mathbf{B}) < 0, \quad det(\mathbf{B}) &>0, \quad b_{1,1}d_2 + b_{2,2}d_1 > 0, \\ det(\mathbf{B}) &< \frac{(b_{1,1}d_2 + b_{2,2}d_1)^2}{4d_1d_2}. \end{aligned} \tag{1.28}$$

These conditions are quite complicated, so we will present some important corollaries. From (1.9) and (1.26) we can see that $b_{1,1}$ and $b_{2,2}$ must be of a different sign and also that $d_1 < d_2$ (important for experimental chapter). If we add condition (1.10), we have that product $-b_{1,2}b_{2,1}$ must be positive, which implies that $b_{1,2}$ and $b_{2,1}$ have also a different sign.

The matrix $\mathbf{B}$ then must be one of type

$$\begin{bmatrix} + & - \\ + & - \end{bmatrix}, \begin{bmatrix} + & + \\ - & - \end{bmatrix}, \begin{bmatrix} - & + \\ - & + \end{bmatrix}, \begin{bmatrix} - & - \\ + & + \end{bmatrix}. \tag{1.29}$$

If we swap u and v in the system, then it is clear that the first type is the same as the third and the second is the same as the fourth, so we can assume only first two. The system with both these types has an interesting chemical interpretation. The functions $u(\mathbf{x}, t)$ and $v(\mathbf{x}, t)$ represent difference of concentrations from $\overline{u}$ and $\overline{v}$ in time t and point $\mathbf{x} \in \mathbb{R}^N$. Let's assume

$$b_{1,1} > 0, \; b_{1,2} < 0, \; b_{2,1} > 0, \; b_{2,2} < 0, \tag{1.30}$$

i.e. the matrix $\mathbf{B}$ is of the first type. Then the system (1.7) describing the chemical reaction of u and v in the neighbourhood of the stationary solution is called "activator-inhibitor". From (1.30) we can see that if $u > 0$ then u is increasing production of both concentration of u and v and if $v > 0$ then v is decreasing production of concentration of u and v. Substance u is then activator of reaction and v inhibitor of reaction. Let's assume the second option

$$b_{1,1} > 0, \; b_{1,2} > 0, \; b_{2,1} < 0, \; b_{2,2} < 0, \tag{1.31}$$

i.e. the matrix of the second type. This type of the system is called "positive feedback" or "substrate depletion". In this case the growth of both u a v close to the stationary solution is increasing production of u and decreasing production of v.

1.2.3 Regions of stability and instability

In the previous subsection we found out, that the stability of the stationary solution (of the system with diffusion (1.7)) is mostly based on $H_d(\kappa_j)$. Now we will use it to split a positive quadrant of points $[d_1, d_2] \in \mathbb{R}_+^2$ on regions of stability and instability.

We already know that for $H_d(\kappa_j) > 0$ is $Re(\lambda_1^j) < 0$ and for $H_d(\kappa_j) < 0$ is $Re(\lambda_1^j) > 0$. Then it is easy to see that the sign of $Re(\lambda_1^j)$ is changing in when $H_d(\kappa_j) = 0$. This relation is equivalent with

$$d_2 = \frac{1}{\kappa_j}\left(\frac{b_{1,2}b_{2,1}}{d_1\kappa_j - b_{1,1}} + b_{2,2}\right), \tag{1.32}$$

under the assumption $d_1 \neq \frac{b_{1,1}}{\kappa_j}$. We can define the set of points $[d_1, d_2]$ such that $H_d(\kappa_j) = 0$:

$$C_j := \{[d_1, d_2] \in \mathbb{R}_+^2 : \ d_2 = \frac{1}{\kappa_j}\left(\frac{b_{1,2}b_{2,1}}{d_1\kappa_j - b_{1,1}} + b_{2,2}\right)\}. \tag{1.33}$$

The set C_j for $j \in \mathbb{N}$ is a hyperbola (or its part in $\mathbb{R}_+^2$) with asymptotes $d_1 = \frac{b_{1,1}}{\kappa_j}$ and $d_2 = \frac{b_{2,2}}{\kappa_j}$ (see Figure 1.1). The $j-th$ hyperbola corresponds to the eigenvalue κ_j of the Laplacian. In the case that $\Gamma_D = \emptyset$ the hyperbola corresponding to $\kappa_0 = 0$ is $C_0 = \emptyset$. If all eigenvalues of the Laplacian are simple, i.e. $\kappa_j < \kappa_{j+1}$ for all $j \in \mathbb{N}$ then $C_j \neq C_{j+1}$ for all j. If an eigenvalue has multiplicity n, then $\kappa_j < \kappa_{j+1} = \ldots = \kappa_{j+n} < \kappa_{j+n+1}$ and $C_j \neq C_{j+1} = \ldots = C_{j+n} \neq C_{j+n+1}$.

Remark 1.3. *If $[d_1, d_2]$ is close to some hyperbola C_j, there is $H_d(\kappa_j) \approx 0$ and the eigenvalues $\lambda_{1,2}^j$ are real.*

Remark 1.4. *Denote $d := [d_1, d_2]$. For any $j \in \mathbb{N}$, if d is on the left of C_j then $Re(\lambda_1^j) > 0$, if d is on the right of C_j then $Re(\lambda_1^j) < 0$ and if $d \in C_j$ then $\lambda_1^j = 0$.*

Let C_E be an envelope of all hyperbolas C_j, $j \in \mathbb{N}$, i.e.

$$C_E := \{d = [d_1, d_2] \in \mathbb{R}_+^2 : \ d_1 = \max_{m \in \mathbb{R}_+}\{m : [m, d_2] \in \bigcup_{j=1}^{\infty} C_j\}\}.$$

Hence we can define regions D_S and D_U:

$$D_S := \{d := [d_1, d_2] \in \mathbb{R}_+^2 : \ d \text{ lies on the right-hand side of } C_E\},$$
$$D_U := \{d := [d_1, d_2] \in \mathbb{R}_+^2 : \ d \text{ lies on the left-hand side of } C_E\}.$$

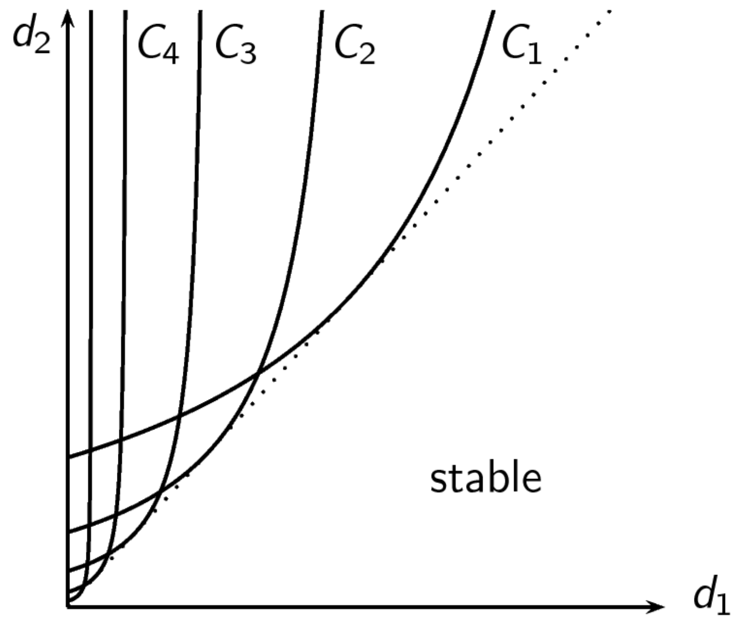


Figure 1.1: Illustration of the hyperbolas C_j

Proposition 1.1:
Let (1.9), (1.10), (1.26) *be fulfilled. If* $d \in D_S$ *then there exists* $\varepsilon > 0$ *such that* $Re(\lambda) < -\varepsilon$ *for all eigenvalues* λ *of* (1.11) *with* (1.8). *If* $d \in D_U$ *then there exists eigenvalue* λ *of* (1.11) *with* (1.8) *such that* $Re(\lambda) > 0$. *Consequently for* $d \in D_S$, $\overline{W}$ *is stable in the norm of* $H^1_D \times H^1_D$ *and for* $d \in D_U$, $\overline{W}$ *is unstable in the norm of* $H^1_D \times H^1_D$.

Proof.
Let's suppose the opposite, i.e. for some $d \in D_S$ for any $\varepsilon > 0$ there exists λ such that $0 > Re(\lambda) > -\varepsilon$. Then there exists a sequence of indices n_j such that $Re(\lambda_{n_j}) \to 0$ and thus from (1.25) $H_d(\kappa_{n_j}) \to 0$. However as $n_j \to \infty$, $\kappa_{n_j} \to \infty$ and therefore $H_d(\kappa_{n_j}) \to \infty$, which is a contradiction. Hence existence of $\varepsilon > 0$ is proved.

If $d \in D_U$, then d is on the left of C_E, i.e. there exists at least one j_0 such that d is on the left of the hyperbola C_{j_0}. Hence by Remark 1.4 we have $Re(\lambda_1^{j_0}) > 0$.

By Theorem 1.1, $\overline{W}$ is stable in the norm of H^1_D for $d \in D_S$ and unstable for $d \in D_U$. □

The previous proposition allows us to call D_S the region of stability and D_U the region of instability.

Chapter 2

Problem with a unilateral term and mixed boundary conditions

In this chapter we will study an influence of adding a unilateral source term to the first equation of the system (1.1) on the distribution of critical and bifurcation points (see definitions 2.2,2.3) in regions of stability and instability

2.1 Formulation

Let's consider the stationary problem

$$\begin{aligned} d_1\Delta u + b_{1,1}u + b_{1,2}v + \tau u^- = 0, \\ d_2\Delta v + b_{2,1}u + b_{2,2}v = 0, \end{aligned} \quad \text{on } \Omega \tag{2.1}$$

where d_1, d_2 are positive parameters and $(b_{i,j})_{i,j=1,2}$ are as in the previous chapter. The domain $\Omega \subset \mathbb{R}^N$ is bounded with Lipschitz boundary. We will suppose that Γ_D, Γ_N are open disjoint subsets of $\partial\Omega$, $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, $\Gamma_D \neq \emptyset$ and the boundary conditions

$$\begin{aligned} u = v = 0 \text{ on } \Gamma_D, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_N \end{aligned} \tag{2.2}$$

where n is a unit outward-pointing normal vector of the boundary $\partial\Omega$. The function u^- is defined as

$$u^- = \begin{cases} -u, & u < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

The parameter τ is positive and we will consider it fixed in this chapter. The term τu^- is not smooth in zero and it can be represented as a switch, which activates production of the substance u in case that u falls under zero.

Let's assume that the following conditions hold:

$$b_{1,1} > 0, b_{1,2} < 0, b_{2,1} > 0, b_{2,2} < 0, tr(\mathbf{B}) < 0, det(\mathbf{B}) > 0. \tag{2.4}$$

Under the assumption (2.4), the stability of the trivial solution of (2.1) depends only on the values of d_1, d_2.

We will consider the function space $H^1_D(\Omega)$ from the previous chapter, this time equipped with a simpler inner product $(u,v) = \int_\Omega \nabla u \nabla v \; d\Omega$. The corresponding norm $\|u\|_{H^1_D} = \sqrt{\int_\Omega (\nabla u)^2 \; d\Omega}$ is equivalent to the norm $\|\cdot\|_{W^{1,2}}$ from the previous chapter.

Weak solutions of the problem (2.1) are functions $u, v \in H^1_D(\Omega)$ such that

$$\begin{aligned} \int_\Omega d_1 \nabla u \nabla \varphi \; d\Omega - \left(\int_\Omega b_{1,1} u \varphi \; d\Omega + \int_\Omega b_{1,2} v \varphi \; d\Omega + \int_\Omega \tau u^- \varphi \; d\Omega \right) &= 0, \\ \int_\Omega d_2 \nabla v \nabla \varphi \; d\Omega - \left(\int_\Omega b_{2,1} u \varphi \; d\Omega + \int_\Omega b_{2,2} v \varphi \; d\Omega \right) &= 0, \end{aligned} \quad \forall \varphi \in H^1_D(\Omega). \tag{2.5}$$

In the previous chapter we have proven that integrals in (2.5) (excluding one with the unilateral term) are finite. Let's show the finiteness of the integral with the unilateral term:

$$\begin{aligned} \int_\Omega \tau u^- \varphi \; d\Omega &\le \tau \left| \int_\Omega u^- \varphi \; d\Omega \right| \le \tau \int_\Omega |u^- \varphi| \; d\Omega \le \\ &\le \tau \left(\int_\Omega |u^-|^2 \; d\Omega \right)^{\frac{1}{2}} \cdot \left(\int_\Omega |\varphi|^2 \; d\Omega \right)^{\frac{1}{2}} \le \tau C \|u\|_{H^1_D} \|\varphi\|_{H^1_D}. \end{aligned}$$

We used Hölder's inequality, continuous embedding $W^{1,2} \hookrightarrow L^2$ and a property $|u^-| \le |u|$, which holds almost everywhere on Ω, from the definition of the unilateral term (2.3).

We will assume the operator A defined in (1.14). Let's remind that A is linear, continuous, symmetric and compact. Further let's define operator $\beta : H^1_D \mapsto H^1_D$ for any fixed $u \in H^1_D$ by

$$(\beta(u), \varphi) = - \int_\Omega u^- \varphi \; d\Omega \quad \forall \varphi \in H^1_D. \tag{2.6}$$

Lemma 2.1:
Let the operator $\beta : H^1_D(\Omega) \mapsto H^1_D(\Omega)$ be defined by (2.6). Then β is positively homogeneous (see A.1), bounded and satisfies

$$\forall (u_n) \subset H^1_D : u_n \rightharpoonup u \in H^1_D \implies \beta(u_n) \to \beta(u). \tag{2.7}$$

Proof.
Let $t \in \mathbb{R}^+_0$. Since $t \ge 0$ we have

$$(\beta(tu), \varphi) = - \int_\Omega (tu)^- \varphi \; d\Omega = -t \int_\Omega u^- \varphi \; d\Omega = (t\beta(u), \varphi) \quad \forall u, \varphi \in H^1_D.$$

Hence β is positively homogeneous.

The boundedness of β can be shown similarly as we proved the finiteness of the corresponding integral in the weak formulation:

$$\|\beta(u)\| = \sup_{\|\varphi\|_{H^1_D}\le 1} |(\beta(u),\varphi)| = \sup_{\|\varphi\|_{H^1_D}\le 1} \left|\int_\Omega u^-\varphi \, d\Omega\right| \le \sup_{\|\varphi\|_{H^1_D}\le 1} \int_\Omega |u^-|\cdot|\varphi| \, d\Omega \le$$
$$\le \sup_{\|\varphi\|_{H^1_D}\le 1} \int_\Omega |u|\cdot|\varphi| \, d\Omega \le \sup_{\|\varphi\|_{H^1_D}\le 1} \{\|u\|_{L^2}\cdot\|\varphi\|_{L^2}\} \le c_{emb} \sup_{\|\varphi\|_{H^1_D}\le 1} \left\{\|u\|_{H^1_D}\cdot\|\varphi\|_{H^1_D}\right\} \le$$
$$\le C\|u\|_{H^1_D}.$$

Now let's have a sequence $(u_n) \subset H^1_D$ such that $u_n \rightharpoonup u \in H^1_D$. Then by compact embedding $W^{1,2} \hookrightarrow\hookrightarrow L^2$, we get $u_n \to_{L^2} u$.

Let's show that

$$|u_n^- - u^-| \le |u_n - u| \text{ holds almost everywhere on } \Omega :$$

$$u_n(x), u(x) \ge 0 \implies |u_n^-(x) - u^-(x)| = 0,$$
$$u_n(x), u(x) < 0 \implies |u_n^-(x) - u^-(x)| = |-u_n(x) - (-u(x))| = |u_n(x) - u(x)|,$$
$$u_n(x) < 0 < u(x) \implies |u_n^-(x) - u^-(x)| = |u_n(x)| \le |u_n(x) - u(x)|.$$

Hence

$$\|\beta(u_n) - \beta(u)\|_{H^1_D} = \sup_{\|\varphi\|_{H^1_D}\le 1} |(\beta(u_n) - \beta(u),\varphi| \le \sup_{\|\varphi\|_{H^1_D}\le 1} \int_\Omega |u_n^- - u^-|\cdot|\varphi| \, d\Omega \le$$
$$\le C\|u_n - u\|_{L^2}.$$

Then by the strong convergence $u_n \to_{L^2} u$ we get $\beta(u_n) \to_{H^1_D} \beta(u)$. □

If we rewrite the system (2.5) in form of operator equations, we get

$$\begin{aligned} d_1 u - b_{1,1}Au - b_{1,2}Av + \tau\beta(u) &= 0, \\ d_2 v - b_{2,1}Au - b_{2,2}Av &= 0. \end{aligned} \tag{2.8}$$

2.2 Reduction to one operator equation

The system of the operator equations (2.8) can be reduced to one operator equation by expressing v from the second equation and inserting to the first equation. **We will consider an arbitrary fixed d_2 in the further text**. Then we can rewrite the second equation of (2.8) as

$$(d_2 I - b_{2,2}A)v = b_{2,1}Au.$$

The operator on the left-hand side of the equation is linear. Since $b_{2,2} < 0$, we have that $\frac{d_2}{b_{2,2}} < 0$. Hence $\frac{d_2}{b_{2,2}}$ cannot be an eigenvalue of the operator A. In that case according to Inverse theorem (see $B.6$), the operator $(d_2 I - b_{2,2}A)$ is invertible and the inverse is also linear. Hence we can write

$$v = (d_2 I - b_{2,2}A)^{-1} b_{2,1}Au$$

and after inserting to the first equation we get

$$d_1u - b_{1,1}Au - b_{1,2}A(d_2I - b_{2,2}A)^{-1}b_{2,1}Au + \tau\beta(u) = 0. \tag{2.9}$$

Let's define a new operator $S : H^1_D \mapsto H^1_D$ as

$$S := b_{1,1}A + b_{1,2}A(d_2I - b_{2,2}A)^{-1}b_{2,1}A. \tag{2.10}$$

Then we can write the equation (2.9) as

$$d_1u - Su + \tau\beta(u) = 0. \tag{2.11}$$

Observation 2.1:
The system of the operator equations

$$\begin{aligned} &d_1u - Su + \tau\beta(u) = 0,\\ &v = (d_2I - b_{2,2}A)^{-1}b_{2,1}Au \end{aligned} \tag{2.12}$$

is equivalent with the system (2.8) *and likewise the system of the operator equations*

$$\begin{aligned} &d_1u - Su = 0,\\ &v = (d_2I - b_{2,2}A)^{-1}b_{2,1}Au, \end{aligned} \tag{2.13}$$

is equivalent with the system

$$\begin{aligned} &d_1u - b_{1,1}Au - b_{1,2}Av = 0,\\ &d_2v - b_{2,1}Au - b_{2,2}Av = 0. \end{aligned} \tag{2.14}$$

Lemma 2.2:
Let the operator $S : H^1_D \mapsto H^1_D$ be defined by (2.10)*. Then S is linear, continuous, symmetric and compact operator.*

Proof.
Since S is a composition of only linear operators, S is also linear. Then we can prove the boundedness instead of the continuity. The operator A is bounded and product or sum of two bounded operators is also bounded. Hence we need to show that $(d_2I - b_{2,2}A)^{-1}$ is bounded. We will use Bounded inverse theorem $B.7$. We already know that $(d_2I - b_{2,2}A)$ is linear, bounded and injective (under assumption $\frac{d_2}{b_{2,2}} \neq \frac{1}{\kappa_j}$). Then it is necessary to show that the operator is surjective, i.e.

$$\forall w \in H^1_D \; \exists z \in H^1_D : (d_2I - b_{2,2}A)z = w. \tag{2.15}$$

We will use Fredholm alternative $B.8$. The operator A is compact, symmetric and consequently self-adjoint (the symmetry and self-adjointness is the same on a Hilbert space). Moreover $\frac{d_2}{b_{2,2}} \neq 0$ is not the eigenvalue of A. Then by Fredholm alternative (2.15) is true and by Bounded inverse theorem we get that $(d_2I - b_{2,2}A)^{-1}$ is bounded, i.e. the operator

S is also bounded.

We know that A is compact and $(d_2I - b_{2,2}A)^{-1}$ is bounded. Then their product is compact by Lemma B.2 and $b_{1,2}A(d_2I - b_{2,2}A)^{-1}b_{2,1}A$ is also compact. Since sum of two compact operators is compact as well, the operator S is compact.

Since the operator A and identity I are symmetric, it is obvious that $(d_2I - b_{2,2}A)$ is also symmetric, i.e.

$$((d_2I - b_{2,2}A)u, \varphi) = (u, (d_2I - b_{2,2}A)\varphi).$$

Let's set $z := (d_2I - b_{2,2}A)u$ and $w := (d_2I - b_{2,2}A)\varphi$. Then $u = (d_2I - b_{2,2}A)^{-1}z$ and $\varphi = (d_2I - b_{2,2}A)^{-1}w$, which holds for all $z, w \in H^1_D$, because we have already shown $(d_2I - b_{2,2}A)$ is bijective. Hence we get

$$(z, (d_2I - b_{2,2}A)^{-1}w) = ((d_2I - b_{2,2}A)^{-1}z, w), \quad \forall z, w \in H^1_D$$

so we have the symmetry of this inverse. To show the symmetry of the composition, we will use lemma about self-adjointness of the product B.3. Let's prove that $b_{2,1}A$ and $(d_2I - b_{2,2}A)^{-1}$ commute:

$$\begin{aligned}
Au &= Au, \qquad \forall u \in H^1_D \\
b_{2,1}Au &= b_{2,1}Au, \\
b_{2,1}Au &= b_{2,1}A(d_2I - b_{2,2}A)(d_2I - b_{2,2}A)^{-1}u.
\end{aligned}$$

Since A commute with the identity and itself trivially, we can write

$$\begin{aligned}
b_{2,1}Au &= (d_2I - b_{2,2}A)b_{2,1}A(d_2I - b_{2,2}A)^{-1}u, \\
(d_2I - b_{2,2}A)^{-1}b_{2,1}Au &= b_{2,1}A(d_2I - b_{2,2}A)^{-1}u.
\end{aligned}$$

Hence $(d_2I - b_{2,2}A)^{-1}b_{2,1}A$ is a symmetric operator. Again $b_{1,2}A$ and $b_{2,1}A$ commutes trivially and then $b_{1,2}A(d_2I - b_{2,2}A)^{-1}b_{2,1}A$ is symmetric too. It is true, that sum of two general symmetric operators T, R is symmetric operator:

$$((T + R)u, \varphi) = (Tu, \varphi) + (Ru, \varphi) = (u, T\varphi) + (u, R\varphi) = (u, (T + R)\varphi).$$

Hence we get that S is a symmetric operator. □

Remark 2.1. *Since we use a different inner product in this chapter, the operator A defined using this simpler inner product is slightly different. The operator A shares its eigenfunctions e_j with the Laplacian and the form of the eigenvalues of the operator A is $\mu_j = \frac{1}{\kappa_j}, j = 1, 2, \ldots$. The sequences κ_j and e_j possess the same properties as in the case $\Gamma_D \neq \emptyset$ of Remark* 1.2.

The following lemma will explain connections between the eigenvalues of S and of the Laplace operator.

Lemma 2.3:
The sequence e_j is by Remark 1.2 the sequence of the eigenfunctions of the Laplacian corresponding to the eigenvalues κ_j. The eigenvalues of the operator S are $d_1^j = \frac{1}{\kappa_j}(\frac{b_{1,2}b_{2,1}}{d_2\kappa_j - b_{2,2}} + b_{1,1})$ and the corresponding eigenfunctions e_j are the same as the eigenfunctions of the Laplacian corresponding to κ_j.

Proof.
From the definition of S, we have

$$Se_j = (b_{1,1}A + b_{1,2}A(d_2I - b_{2,2}A)^{-1}b_{2,1}A)e_j = b_{1,1}Ae_j + b_{1,2}A(d_2I - b_{2,2}A)^{-1}b_{2,1}Ae_j.$$

Since the operator A share its eigenfunctions with the Laplacian (Remark 2.1) and corresponding eigenvalues of A are $\frac{1}{\kappa_j}$, we get

$$Se_j = \frac{b_{1,1}}{\kappa_j}e_j + b_{1,2}A(d_2I - b_{2,2}A)^{-1}\frac{b_{2,1}}{\kappa_j}e_j.$$

Now we need to do one auxiliary process:

$$(d_2I - b_{2,2}A)^{-1}e_j = \lambda e_j \Leftrightarrow \frac{1}{\lambda}e_j = d_2e_j - b_{2,2}Ae_j =$$
$$= d_2e_j - \frac{b_{2,2}}{\kappa_j}e_j = \frac{d_2\kappa_j - b_{2,2}}{\kappa_j}e_j \Leftrightarrow \lambda = \frac{\kappa_j}{d_2\kappa_j - b_{2,2}}.$$

Then we get

$$Se_j = \frac{b_{1,1}}{\kappa_j}e_j + \frac{b_{1,2}b_{2,1}}{\kappa_j}\frac{\kappa_j}{d_2\kappa_j - b_{2,2}}Ae_j =$$
$$= \frac{b_{1,1}}{\kappa_j}e_j + \frac{b_{1,2}b_{2,1}}{d_2\kappa_j - b_{2,2}}\frac{1}{\kappa_j}e_j = \frac{1}{\kappa_j}(\frac{b_{1,2}b_{2,1}}{d_2\kappa_j - b_{2,2}} + b_{1,1})e_j.$$

In the conclusion e_j are the eigenfunctions of S corresponding to the eigenvalues
$d_1^j = \frac{1}{\kappa_j}(\frac{b_{1,2}b_{2,1}}{d_2\kappa_j - b_{2,2}} + b_{1,1})$. □

Observation 2.2:
From the form of the eigenvalues d_1^j of S and the equation $H_d(\kappa_j) = 0$ from the previous chapter, we can see that for fixed d_2, the points $[d_1^j, d_2]$ lies always on the hyperbolas C_j in the positive quadrant of the plane $[d_1, d_2]$. Since the sequence κ_j of the eigenvalues of the Laplacian is non-decreasing and $\kappa_j \to \infty$ as $j \to \infty$, we get $d_1^j \to 0$ as $j \to \infty$. The maximal eigenvalue d_1^{MAX} of S is such that $[d_1^{MAX}, d_2]$ lies always on the envelope C_E.

We will often use maximal eigenvalues of operators. The following remark will clarify relation of a maximal eigenvalue and a maximizing function.

Remark 2.2. *Let H be a Hilbert space and $T : H \mapsto H$ be a linear, symmetric, compact operator. Then the maximal eigenvalue of T can be expressed as*

$$\lambda_{MAX} = \max_{u \in H, u \neq o} \frac{(Tu, u)}{\|u\|_H^2}.$$

The function v is an eigenfunction corresponding to λ_{MAX} if and only if

$$\lambda_{MAX} = \max_{u\in H, u\neq o} \frac{(Tu,u)}{\|u\|_H^2} = \frac{(Tv,v)}{\|v\|_H^2}.$$

2.3 Critical and bifurcation points of the system with a unilateral term

We will define and deal with critical and bifurcation points in this section. Critical points will be usually related to the system (2.14) or (2.8), which we consider most often. Bifurcation points are related to stationary problems

$$\begin{aligned} d_1\Delta u + b_{1,1}u + b_{1,2}v + \overline{n}_1(u,v) = 0, \\ d_2\Delta v + b_{2,1}u + b_{2,2}v + \overline{n}_2(u,v) = 0, \end{aligned} \quad \text{on } \Omega \tag{2.16}$$

and

$$\begin{aligned} d_1\Delta u + b_{1,1}u + b_{1,2}v + \tau u^- + \overline{n}_1(u,v) = 0, \\ d_2\Delta v + b_{2,1}u + b_{2,2}v + \overline{n}_2(u,v) = 0, \end{aligned} \quad \text{on } \Omega \tag{2.17}$$

with boundary conditions (2.2). The non-linearities $\overline{n}_{1,2}$ are as in the system (1.7) from Chapter 1.

The weak formulations of (2.16) with (2.2) and (2.17) with (2.2) are

$$\begin{aligned} \int_\Omega d_1\nabla u\nabla\varphi - (b_{1,1}u\varphi + b_{1,2}v\varphi + \overline{n}_1(u,v)\varphi)\ d\Omega = \ & 0 \\ \int_\Omega d_2\nabla v\nabla\varphi - (b_{2,1}u\varphi + b_{2,2}v\varphi + \overline{n}_2(u,v)\varphi)\ d\Omega = \ & 0 \end{aligned} \quad \forall\varphi\in H_D^1(\Omega) \tag{2.18}$$

and

$$\begin{aligned} \int_\Omega d_1\nabla u\nabla\varphi - \big(b_{1,1}u\varphi + b_{1,2}v\varphi + \tau u^-\varphi + \overline{n}_1(u,v)\varphi\big)\ d\Omega = \ & 0, \\ \int_\Omega d_2\nabla v\nabla\varphi - (b_{2,1}u\varphi + b_{2,2}v\varphi + \overline{n}_2(u,v)\varphi)\ d\Omega = \ & 0, \end{aligned} \quad \forall\varphi\in H_D^1(\Omega), \tag{2.19}$$

respectively. A pair of functions $u, v \in H_D^1(\Omega)$ satisfying (2.18) or (2.19) is called the weak solution of the problem (2.18) with (2.2) or (2.19) with (2.2), respectively.

Let's show that integrals exist. The non-linear functions $\overline{n}_1, \overline{n}_2$ are differentiable in u, v so the Caratheodory conditions (see $A.3$) are satisfied. Then we can define by Definition $A.4$ Nemytskii operator

$$\mathcal{N}_i(u,v)(x) = \overline{n}_i(u(x), v(x)) \quad i = 1,2.$$

By the Sobolev embedding theorem $B.10$ for $p = 2$ the Sobolev space $W^{1,2}$ is continuously embedded to L^{p^*}, for $N > 2$ and to L^r, $r \geq 1$ for $N = 2$. The number $p^* = \frac{2N}{N-2}$ is a

Sobolev conjugate (see $A.2$) of $p = 2$. Moreover L^{p^*} is continuously embedded to L^q for $q \leq p^*$. Under the previous assumptions, the space H^1_D is continuously embedded to L^q, for $q \in [1, p^*]$ if $N > 2$ and for $q \geq 1$ if $N = 2$. Hence we get that every $u, v, \varphi \in H^1_D$ is also in L^q.

We need to assume some constraints on the functions $\overline{n}_1, \overline{n}_2$. Let there exist constants $C_1, C_2 \in \mathbb{R}$ such that the functions $\overline{n}_1, \overline{n}_2 : \mathbb{R}^2 \mapsto \mathbb{R}$ satisfy

$$\begin{aligned} |\overline{n}_1(\chi, \xi)| &\leq C_1(1 + |\chi|^{q-1} + |\xi|^{q-1}) \quad \forall \chi, \xi \in \mathbb{R}, \\ |\overline{n}_2(\chi, \xi)| &\leq C_2(1 + |\chi|^{q-1} + |\xi|^{q-1}) \quad \forall \chi, \xi \in \mathbb{R}, \end{aligned} \tag{2.20}$$

for some $q > 2$ if $N = 2$ or $2 < q < \frac{2N}{N-2}$ if $N > 2$.

Let q^* be a dual index to q. We have $\frac{q}{q^*} = q - 1$. The exponent $q - 1$ from (2.20) is then equal to $\frac{q}{q^*}$. By Nemytskii theorem $B.9$ the Nemytskii operators $\mathcal{N}_{1,2}$ are well defined and continuous from the space $L^q \times L^q$ to the space L^{q^*}.

Now we can show that integrals with the functions $\overline{n}_{1,2}$ are finite:

$$\begin{aligned} \int_\Omega \overline{n}_i(u, v)\varphi \, d\Omega = \int_\Omega \mathcal{N}_i(u, v)\varphi \, d\Omega \leq \left(\int_\Omega |\mathcal{N}_i(u, v)|^{q^*} \, d\Omega \right)^{\frac{1}{q^*}} \cdot \left(\int_\Omega |\varphi|^q \, d\Omega \right)^{\frac{1}{q}} \leq \\ \leq c_{emb} \left(\int_\Omega |\mathcal{N}_i(u, v)|^{q^*} \, d\Omega \right)^{\frac{1}{q^*}} \|\varphi\|_{H^1_D} \quad \forall \varphi \in H^1_D, i = 1, 2. \end{aligned}$$

We used Hölder's inequality and continuous embedding $W^{1,2} \hookrightarrow L^q$.

We can define for any fixed $u, v \in H^1_D$ two new non-linear operators $N_1, N_2 : L^q \times L^q \mapsto L^{q^*}$ as

$$(N_i(u, v), \varphi) = \int_\Omega \overline{n}_i(u, v)\varphi \, d\Omega \quad \text{for } i = 1, 2, \forall \varphi \in H^1_D. \tag{2.21}$$

Now we can write the systems (2.18),(2.19) as systems of operator equations:

$$\begin{aligned} d_1 u - b_{1,1}Au - b_{1,2}Av - N_1(u, v) &= 0, \\ d_2 u - b_{2,1}Au - b_{2,2}Av - N_2(u, v) &= 0, \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} d_1 u - b_{1,1}Au - b_{1,2}Av - N_1(u, v) + \tau\beta(u) &= 0, \\ d_2 u - b_{2,1}Au - b_{2,2}Av - N_2(u, v) &= 0. \end{aligned} \tag{2.23}$$

According to appendix A.1 of [6] and (1.6) we have

$$\lim_{\|u\|_{H^1_D} + \|v\|_{H^1_D} \to 0} \frac{N_i(u, v)}{\|u\|_{H^1_D} + \|v\|_{H^1_D}} = 0, \quad i = 1, 2. \tag{2.24}$$

In this section we will intensively study eigenvalues of operators. Since the operator β, that we have encountered, is non-linear, we cannot use usual spectral theory.

Definition 2.1.
Let H be a Hilbert space and $T : H \mapsto H$ a positively homogeneous operator. Then $\lambda \in \mathbb{R}$ is an eigenvalue of T if there exists a non-zero function $v \in H$ such that

$$T(v) = \lambda v.$$

Let's define the concept of critical and bifurcation point and say something about their connection:

Definition 2.2 (Critical point)**.**
A parameter $d = [d_1, d_2] \in \mathbb{R}^2_+$ is a critical point of (2.14) or (2.8) if there exists a non-trivial solution of (2.14) or (2.8) for such a d, respectively.

Definition 2.3 (Bifurcation point)**.**
A parameter $d^0 = [d^0_1, d^0_2] \in \mathbb{R}^2_+$ is a bifurcation point of (2.22) or (2.23) if in any neighbourhood of $[d^0, 0, 0] \in \mathbb{R}^2_+ \times H^1_D \times H^1_D$ there exists $[d, W] = [d, u, v]$, $\|W\| \neq 0$ satisfying (2.22) or (2.23), respectively.

Lemma 2.4:
Every bifurcation point $[d_1, d_2]$ *of* (2.23) *is also a critical point of* (2.8).

Proof.
Let $d^0 = [d_1, d_2]$ be a bifurcation point of (2.23). Then there exists a sequence $d^n = [d^n_1, d^n_2]$ such that $d^n \to d^0$ and $W_n = [u_n, v_n] \to 0$ with $\|W_n\| \neq 0$ and d^n, W_n satisfy (2.23). Due to Eberlain-Šmuljan theorem, we can assume $\frac{W_n}{\|W_n\|} \rightharpoonup W = [w, z]$. Let's divide the system (2.23) by $\|W_n\|$. We get

$$\begin{aligned} d^n_1 \frac{u_n}{\|W_n\|} - b_{1,1} A \frac{u_n}{\|W_n\|} - b_{1,2} A \frac{v_n}{\|W_n\|} + \tau\beta\left(\frac{u_n}{\|W_n\|}\right) - \frac{N_1(u_n, v_n)}{\|W_n\|} &= 0, \\ d^n_2 \frac{v_n}{\|W_n\|} - b_{2,1} A \frac{u_n}{\|W_n\|} - b_{2,2} A \frac{v_n}{\|W_n\|} - \frac{N_2(u_n, v_n)}{\|W_n\|} &= 0, \end{aligned} \tag{2.25}$$

due to linearity of A and positive homogeneity of β. Due to (2.24), we have $\frac{N_{1,2}(u_n,v_n)}{\|W_n\|} \to 0$ as $n \to \infty$. Since we have $\frac{u_n}{\|W_n\|} \rightharpoonup w$ and $\frac{v_n}{\|W_n\|} \rightharpoonup z$, using compactness of A and (2.7), we get $A\frac{u_n}{\|W_n\|} \to Aw$ and $\beta\left(\frac{u_n}{\|W_n\|}\right) \to \beta(w)$ (analogously for v_n and z). Hence from the system (2.25), we have $\frac{u_n}{\|W_n\|} \to w$, $\frac{v_n}{\|W_n\|} \to z$ and

$$\begin{aligned} d^0_1 w - b_{1,1} Aw - b_{1,2} Az + \tau\beta(w) &= 0, \\ d^0_2 z - b_{2,1} Aw - b_{2,2} Az &= 0. \end{aligned}$$

The convergence of $\frac{W_n}{\|W_n\|}$ is strong, hence $\|W\| = 1$. The point d^0 is a critical point of the system (2.8). □

Remark 2.3. *Since the problem* (2.14) *is a special case of* (2.8) *for* $\tau = 0$ *(the same holds for* (2.22) *and* (2.23)*), Lemma* 2.4 *holds for the problem* (2.14) *and* (2.22) *too.*

Lemma 2.5:
The point $[d_1, d_2] \in \mathbb{R}^2_+$ *is a critical point of system* (2.14) *or* (2.8) *if and only if* d_1 *is an eigenvalue of the operator* S *or* $S - \tau\beta$*, respectively.*

Proof.
The proof of implication $\implies$:
If the point $[d_1, d_2] \in \mathbb{R}^2_+$ is a critical point of the system (2.14) or (2.8), then there exists a non-trivial solution $[u, v]$ of (2.14) or (2.8), respectively. By Observation 2.1 these systems are equivalent with systems (2.13) or (2.12), respectively. Hence there exists a non-trivial solution $u \in H^1_D$ of the operator equations $d_1 u = Su$ or $d_1 u = Su - \tau\beta(u)$ and d_1 is an eigenvalue of the operator S or $S - \tau\beta$, respectively.
The proof of implication $\impliedby$:
The number d_1 is an eigenvalue of S or $S - \tau\beta(u)$ if and only if there exists a non-trivial solution $u \in H^1_D$ of $d_1 u = Su$ or $d_1 u = Su - \tau\beta(u)$, respectively. By Observation 2.1 there exists a non-trivial solution of (2.14) or (2.8) for $d = [d_1, d_2]$, respectively. Hence by the definition 2.2, d is a critical point of the system (2.14) or (2.8), respectively. □

Corollary 2.1:
The region of stability D_S *doesn't contain any critical point of* (2.14) *or bifurcation point of* (2.22)*.*

Proof.
By Observation 2.2, the maximal eigenvalue d_1^{MAX} of S is such that the point $[d_1^{MAX}, d_2]$ lies on the envelope C_E. Combined with Lemma 2.5 and Remark 2.3 there are no critical points of (2.14) in D_S. Then by Lemma 2.4 there are also no bifurcation points of (2.22) in D_S. □

If we want to study the critical points of the system (2.12), we need to have some information about eigenvalues of the operator $S - \tau\beta$. Following result from the article *A variational approach to critical points of reaction-diffusion systems with jumping nonlinearities* by Kučera and Navrátil (see [8]) will help us acquire the maximal eigenvalue of the operator $S - \tau\beta$.

Theorem 2.3.1 (Navrátil):
Let H *be a Hilbert space,* $T : H \mapsto H$*,* $L : H \mapsto H$ *linear, continuous, symmetric, compact operators and* $R : H \mapsto H$ *a bounded, positively homogeneous operator such that* $\forall (u_n) \subset H : u_n \rightharpoonup u \implies R(u_n) \to R(u)$*. Suppose* $\sigma(L) \subset [0, 1]$ *and let there exist*

$$\lambda_0 := \max_{\substack{u \in H \\ u \notin Ker(I-L)}}, \frac{(Tu, u) - (R(u), u)}{((I - L)u, u)} \in \mathbb{R}_+. \tag{2.26}$$

Let there exist a function $F : H \times \mathbb{R} \mapsto \mathbb{R}$ *such that*

$$(R(u_0 + th), u_0 + th) - (R(u_0), u_0)) \leq 2t(R(u_0), h) + F(h, t), \quad \forall t \in \mathbb{R},\ \forall h \in H, \tag{2.27}$$

$$\lim_{t\to 0} \frac{F(h,t)}{t} = 0 \text{ for any fixed } h \in H. \tag{2.28}$$

Then λ_0 is the largest eigenvalue of the problem $\lambda(I-L)u - Tu + R(u) = 0$.

Remark 2.4 (Navratil)**.** *In the special case $L \equiv 0$, the maximal eigenvalue of $T - R$ is*

$$\lambda_0 := \max_{\substack{u\in H\\ u\neq o}} \frac{(Tu,u)-(R(u),u)}{\|u\|_H^2}.$$

Remark 2.5. *We will always denote d_1^{MAX} the maximal eigenvalue of the operator S and $d_1^{MAX,\beta}$ the maximal eigenvalue of the operator $S - \tau\beta$.*

As a consequence of Theorem 2.3.1 and Remark 2.4 we have this corollary, which ties to our case:

Corollary 2.2:
The maximal eigenvalue of the operator $S - \tau\beta$ is

$$d_1^{MAX,\beta} := \max_{\substack{u\in H_D^1\\ u\neq o}} \frac{(Su,u)-\tau(\beta(u),u)}{\|u\|_{H_D^1}^2}. \tag{2.29}$$

Proof.
We will use Theorem 2.3.1 with $L \equiv 0$, $T \equiv S$ and $R \equiv \beta$. We have already proved properties of S and β in Lemma 2.2 and Lemma 2.1. Let's prove that there exists the maximum in (2.29). Let

$$M := \sup_{\substack{u\in H_D^1\\ u\neq o}} \frac{(Su,u)-\tau(\beta(u),u)}{\|u\|_{H_D^1}^2}.$$

We can choose a sequence $(u_n) \subset H_D^1$ and $\|u_n\|_{H_D^1} = 1$ such that

$$\lim_{n\to\infty} \frac{(Su_n,u_n)-\tau(\beta(u_n),u_n)}{\|u_n\|_{H_D^1}^2} = \lim_{n\to\infty} (Su_n,u_n)-\tau(\beta(u_n),u_n) = M. \tag{2.30}$$

Due to Eberlain-Šmuljan theorem we can automatically assume $u_n \rightharpoonup u_0 \in H_D^1$. Since S is compact and β satisfies (2.7), we get $Su_n \to Su_0$ and $\beta(u_n) \to \beta(u_0)$. Hence

$$(Su_n,u_n)-\tau(\beta(u_n),u_n) \to (Su_0,u_0)-\tau(\beta(u_0),u_0). \tag{2.31}$$

Hence the maximum in (2.29) exists and it is attained in u_0.

We need to check conditions (2.27),(2.28). Let's $u_0, h \in H_D^1$. We define three subsets of Ω:

$$\Omega_{th} := \{\mathbf{x} \in \Omega : \ |u_0(\mathbf{x})| \le |th(\mathbf{x})|\},$$
$$\Omega_{th}^+ := \{\mathbf{x} \in \Omega : \ |th(\mathbf{x})| \le u_0(\mathbf{x})\},$$
$$\Omega_{th}^- := \{\mathbf{x} \in \Omega : \ -|th(\mathbf{x})| \ge u_0(\mathbf{x})\}.$$

This distribution of Ω implies following properties of the functions u_0 and $u_0 + th$:

$$\forall \mathbf{x} \in \Omega_{th}^+ : u_0^-(\mathbf{x}) = 0,\ \forall \mathbf{x} \in \Omega_{th}^- : u_0^-(\mathbf{x}) = -u_0(\mathbf{x}), \quad (2.32)$$

$$\forall \mathbf{x} \in \Omega_{th}^+ : (u_0(\mathbf{x}) + th(\mathbf{x}))^- = 0,\ \forall \mathbf{x} \in \Omega_{th}^- : (u_0(\mathbf{x}) + th(\mathbf{x}))^- = -(u_0(\mathbf{x}) + th(\mathbf{x})), \quad (2.33)$$

$$\forall \mathbf{x} \in \Omega_{th} : [(u_0(\mathbf{x}) + th(\mathbf{x}))^-]^2 = v_0^2(\mathbf{x}) + 2u_0(\mathbf{x})th(\mathbf{x}) + t^2h^2(\mathbf{x}) \leq 4t^2h^2(\mathbf{x}). \quad (2.34)$$

Let's find estimates of inner products in (2.27) using properties (2.32), (2.33), (2.34).

$$\begin{aligned}
&(\beta(u_0 + th), u_0 + th) = -\int_\Omega (u_0 + th)^-(u_0 + th)\ d\Omega = \\
&= -\int_{\Omega_{th}} (u_0 + th)^-(u_0 + th)\ d\Omega_{th} - \int_{\Omega_{th}^+} (u_0 + th)^-(u_0 + th)\ d\Omega_{th}^+ \\
&- \int_{\Omega_{th}^-} (u_0 + th)^-(u_0 + th)\ d\Omega_{th}^- = \int_{\Omega_{th}^-} (u_0 + th)^2\ d\Omega_{th}^- + \int_{\Omega_{th}} [(u_0 + th)^-]^2\ d\Omega_{th} \leq \\
&\leq \int_{\Omega_{th}^-} (u_0 + th)^2\ d\Omega_{th}^- + 4t^2 \int_{\Omega_{th}} h^2\ d\Omega_{th} = \int_{\Omega_{th}^-} (u_0)^2 + 2u_0th + t^2h^2\ d\Omega_{th}^- + 4t^2 \int_{\Omega_{th}} h^2\ d\Omega_{th}
\end{aligned}$$

The second term in the previous estimate can be estimated like this:

$$\begin{aligned}
&\int_{\Omega_{th}^-} 2u_0th\ d\Omega_{th}^- = -2t \int_{\Omega_{th}^-} u_0^-h\ d\Omega_{th}^- - 2t \int_{\Omega_{th}^+} u_0^-h\ d\Omega_{th}^+ - 2t \int_{\Omega_{th}} u_0^-h\ d\Omega_{th} + 2t \int_{\Omega_{th}} u_0^-h\ d\Omega_{th} = \\
&= -2t \int_\Omega u_0^-h\ d\Omega + 2t \int_{\Omega_{th}} u_0^-h\ d\Omega_{th} = 2t(\beta(u_0, h)) + 2t \int_{\Omega_{th}} u_0^-h\ d\Omega_{th} \leq \\
&\leq 2t(\beta(u_0, h)) + 2t^2 \int_{\Omega_{th}} h^2\ d\Omega_{th}.
\end{aligned}$$

Hence we get the estimate of the first inner product in (2.27):

$$(\beta(u_0+th), u_0+th) \leq \int_{\Omega_{th}^-} (u_0)^2\ d\Omega_{th}^- + 2t(\beta(u_0), h) + \int_{\Omega_{th}^-} t^2h^2\ d\Omega_{th}^- + 6t^2 \int_{\Omega_{th}} h^2\ d\Omega_{th}. \quad (2.35)$$

The second inner product can be estimated as follows

$$\begin{aligned}
&(\beta(u_0), u_0) = -\int_\Omega u_0^-u_0\ d\Omega = \int_\Omega (u_0^-)^2\ d\Omega = \\
&= \int_{\Omega_{th}} (u_0^-)^2\ d\Omega_{th} + \int_{\Omega_{th}^+} (u_0^-)^2\ d\Omega_{th}^+ + \int_{\Omega_{th}^-} (u_0^-)^2\ d\Omega_{th}^- \geq \int_{\Omega_{th}^-} (u_0)^2\ d\Omega_{th}^-
\end{aligned} \quad (2.36)$$

If we combine (2.35) and (2.36), we get

$$\begin{aligned}
(\beta(u_0 + th), u_0 + th) - (\beta(u_0), u_0) &\leq 2t(\beta(u_0), h) + \int_{\Omega_{th}^-} t^2h^2\ d\Omega_{th}^- + 6t^2 \int_{\Omega_{th}} h^2\ d\Omega_{th} = \\
&= 2t(\beta(u_0), h) + F(h, t).
\end{aligned}$$

The condition (2.27) is then satisfied. Let's check condition (2.28):

$$\lim_{t\to 0}\frac{F(h,t)}{t} = \lim_{t\to 0}\frac{\int_{\Omega_{th}^-} t^2h^2\ d\Omega_{th}^- + 6t^2\int_{\Omega_{th}} h^2\ d\Omega_{th}}{t} =$$
$$= \lim_{t\to 0} t\left(\int_{\Omega_{th}^-} h^2\ d\Omega_{th}^- + 6\int_{\Omega_{th}} h^2\ d\Omega_{th}\right) = 0.$$

Then by Navratil's theorem (2.3.1) the maximal eigenvalue of $S-\tau\beta$ is d_1^{MAX} in (2.29). □

We will formulate the following auxiliary lemma, which will give us additional information and we will use it in further proofs.

Lemma 2.6:
The operator $\beta : H_D^1 \mapsto H_D^1$ from (2.6) satisfies

$$(\beta(\varphi),\varphi) \geq 0 \qquad \forall\varphi \in H_D^1. \tag{2.37}$$

Moreover, $(\beta(\varphi),\varphi) > 0$ if and only if the function φ change the sign or it is negative on the domain Ω.

Proof.
Let Ω_+ be the part of the domain where $\varphi \geq 0$ and Ω_- where $\varphi < 0$. These two sets are disjoint and $\Omega_+ \cup \Omega_- = \Omega$. Hence

$$-\int_\Omega \varphi\varphi^- d\Omega = -\int_{\Omega_+} \varphi\varphi^- d\Omega_+ - \int_{\Omega_-} \varphi\varphi^- d\Omega_-.$$

Since $\varphi^- = 0$ on Ω_+, the first integral is equal to zero. On Ω_- there is $\varphi^- = -\varphi$, thus the second integral is

$$\int_{\Omega_-} \varphi^2\, d\Omega_- > 0.$$

Hence $(\beta(\varphi),\varphi) \geq 0$.

We have $(\beta(\varphi),\varphi) > 0$ if and only if $\Omega_- \neq \emptyset$, i.e. either φ change the sign on Ω or it is negative a.e. on Ω. Also there is $(\beta(\varphi),\varphi) = 0$ if and only if $\Omega_- = \emptyset$, i.e. φ is non-negative a.e. on Ω. □

Now we are able to compare the maximal eigenvalue of the operator S and that of the operator $S - \tau\beta$.

Let's note that the following two theorems hold even for general $S : H \mapsto H$ and $\beta : H \mapsto H$, not just those we have defined before.

Theorem 2.1:
Let H be a Hilbert space. Let operators $S : H \mapsto H$ and $\beta : H \mapsto H$ satisfy assumptions of Theorem 2.3.1 for $T \equiv S$, $R \equiv \beta$ and $L \equiv 0$. Let's suppose that the operator β satisfies condition (2.37). Then $d_1^{MAX,\beta} \leq d_1^{MAX}$.

Proof.
Due to Remark 2.2 the maximal eigenvalue of the operator S is

$$d_1^{MAX} = \max_{\substack{u\in H\\ u\neq o}} \frac{(Su,u)}{\|u\|_H^2}.$$

By Theorem 2.3.1 we get that the maximal eigenvalue of the operator $S - \tau\beta$ is

$$d_1^{MAX,\beta} = \max_{\substack{u\in H\\ u\neq o}} \frac{(Su,u) - \tau(\beta(u),u)}{\|u\|_H^2}.$$

By condition (2.37) we have $\tau(\beta(u),u) \geq 0$ and therefore

$$\max_{\substack{u\in H\\ u\neq o}} \frac{(Su,u) - \tau(\beta(u),u)}{\|u\|_H^2} \leq \max_{\substack{u\in H\\ u\neq o}} \frac{(Su,u)}{\|u\|_H^2}$$

i.e. $d_1^{MAX,\beta} \leq d_1^{MAX}$. □

Corollary 2.3:
The region of stability D_S doesn't contain any critical point of (2.8) *or bifurcation point of* (2.23).

Proof.
By Observation 2.2 the point $[d_1^{MAX}, d_2]$ lies on the envelope C_E. By Theorem 2.1 all eigenvalues d_1^β of $S - \tau\beta$ are less or equal to the maximal eigenvalue of S. Hence all eigenvalues of $S - \tau\beta$ are such that $[d_1^\beta, d_2]$ lies in $\overline{D_U}$ and by Lemma 2.5 and Figure 1.1 there are no critical points of (2.8) in D_S. By Lemma 2.4 there are also no bifurcation points of (2.23) in D_S. □

Remark 2.6. *Some of the eigenvalues κ_j of the Laplacian can be multiple. If $\kappa_j = \kappa_{j+1} = \ldots = \kappa_{j+k}$ and the corresponding eigenfunctions are $e_j, \ldots, e_{j+k}$, then also every linear combination of $e_j, \ldots, e_{j+k}$ is an eigenfunction of the Laplacian (and consequently of the operator S). For this reason if we will speak about all eigenfunctions of S, we will denote them e instead of e_j.*

Theorem 2.2:
Let H be a Hilbert space. Let operators $S : H \mapsto H$ and $\beta : H \mapsto H$ satisfy assumptions of Theorem 2.3.1 *for $T \equiv S$, $R \equiv \beta$ and $L \equiv 0$. Let's suppose that the operator β satisfies condition* (2.37). *Let all eigenfunctions e of S corresponding to the maximal eigenvalue d_1^{MAX} of S satisfy $(\beta(e), e) > 0$. Then $d_1^{MAX,\beta} < d_1^{MAX}$.*

Proof.
By Theorem 2.3.1 we get

$$d_1^{MAX,\beta} = \max_{\substack{u\in H\\ u\neq o}} \frac{(Su,u) - \tau(\beta(u),u)}{\|u\|_H^2} = \frac{(Su_0,u_0) - \tau(\beta(u_0),u_0)}{\|u_0\|_H^2},$$

with some $u_0 \in H$. Let e be as in the assumptions. By Remark 2.2 we get

$$\frac{(Se,e)}{\|e\|_H^2} = \max_{\substack{u\in H\\ u\neq o}} \frac{(Su,u)}{\|u\|_H^2} = d_1^{MAX}.$$

Since e is the maximizer of $\frac{(Su,u)}{\|u\|_H^2}$ we have

$$\frac{(Su_0,u_0)}{\|u_0\|_H^2} \leq \frac{(Se,e)}{\|e\|_H^2}. \tag{2.38}$$

If u_0 is such that $\tau(\beta(u_0),u_0) = 0$ then it cannot be $u_0 = e$, because $(\beta(e),e) > 0$. Hence we have strict inequality in (2.38) and consequently we get $d_1^{MAX,\beta} < d_1^{MAX}$.
If u_0 is such that $\tau(\beta(u_0),u_0) > 0$, then we have

$$\frac{(Su_0,u_0) - \tau(\beta(u_0),u_0)}{\|u_0\|_H^2} < \frac{(Su_0,u_0)}{\|u_0\|_H^2} \leq \frac{(Se,e)}{\|e\|_H^2}$$

and consequently we get $d_1^{MAX,\beta} < d_1^{MAX}$. □

The last theorem gave us the comparison in case that the eigenfunctions change the sign. We already know that the eigenfunction e_1 of the Laplacian doesn't change the sign.

Lemma 2.7:
Let d_2^I be the second coordinate of $C_1 \cap C_2$ (the intersection point of the first and the second hyperbola). If $d_2 \geq d_2^I$, the point $[d_1^{MAX}, d_2] \in C_1$ is a critical point of both (2.8) *and* (2.14).

Proof.
Let d_2 be as in the assumptions. If there is $[d_1,d_2] \in C_1$, then we have $d_1 = d_1^{MAX}$ by Observation 2.2. The hyperbola C_1 corresponds to the eigenvalue κ_1 of the Laplacian. The corresponding eigenfunction e_1 can be chosen positive on the domain Ω. It means that $\beta(e_1) = 0$. Since the Laplacian and the operator S share eigenfunctions, due to Lemma 2.3 the eigenvalue corresponding to the eigenfunction e_1 is $d_1^{MAX} = \frac{1}{\kappa_1}(\frac{b_{1,2}b_{2,1}}{d_2\kappa_1 - b_{2,2}} + b_{1,1})$. Hence the following identity holds

$$Se_1 - \tau\beta(e_1) = d_1^{MAX} e_1,$$

i.e. $d_1^{MAX} = d_1^{MAX,\beta}$.

Hence by Lemma 2.5 the point $[d_1^{MAX}, d_2]$ is a critical point of the system (2.8) and (2.14). □

Following Theorem 2.2 we can formulate next Theorem about critical points of the system with the unilateral term.

Theorem 2.3:
Let d_2 be such that $[d_1^{MAX}, d_2] \in C_E \setminus C_1$. Then all critical points $[d_1,d_2]$ of the problem (2.8) *fulfil $d_1 \leq d_1^{MAX,\beta} < d_1^{MAX}$.*

Proof.
The assumption of Theorem 2.3.1 are due to Lemma 2.2, Lemma 2.1 and proof of Corollary 2.2 satisfied. The operator β satisfies condition (2.37) from Lemma 2.6. The eigenfunctions of the Laplacian excluding the first one change the sign. Hence by Lemma 2.6 we have $(\beta(e), e) > 0$ for these eigenfunctions.

Therefore we can use Theorem 2.2 and get that $d_1^{MAX,\beta} < d_1^{MAX}$. Let d_2 be as in the assumptions. If $[d_1, d_2]$ is a critical point of the problem (2.8), then by Lemma 2.5 is d_1 an eigenvalue of $S - \tau\beta$. Hence we have $d_1 \leq d_1^{MAX,\beta}$ and consequently there is

$$d_1 \leq d_1^{MAX,\beta} < d_1^{MAX}.$$

□

In the rest of this chapter we will consider variable d_2. We will define two sets in the positive quadrant of the plane $[d_1, d_2]$.

Definition 2.4.
Let $r, R, \varepsilon \in \mathbb{R}_+$ and $r < R$. We define
$C_r^R := \{d = [d_1, d_2] \in C_E : d_2 \in [r, R]\}$,
$C_r^R(\varepsilon) := \{d = [d_1, d_2] \in C_E \cup D_U : d_2 \in [r, R] \wedge dist(d, C_E) < \varepsilon\}$.

Remark 2.7. *The set C_r^R is a compact part of the envelope C_E.*

Now we can formulate the following theorem, which is the main result of this chapter.

Theorem 2.4:
For every part C_r^R of the envelope such that $0 < r < R < d_2^I$ (from Lemma 2.7), there exists $\varepsilon > 0$ such that there are no critical points of (2.8) in $C_r^R(\varepsilon)$ (ε depending on τ).

Proof.
Let's suppose the opposite. We can choose a sequence $d^n = [d_1^n, d_2^n] \in D_U$ and $W_n = [u_n, v_n]$ such that $d^n \in C_r^R(\varepsilon)$, $d^n \to d^0 \in C_r^R$, $\|W_n\| \neq 0$ and d^n, W_n satisfy (2.8). By Eberlain-Šmuljan theorem we can automatically assume $\frac{W_n}{\|W_n\|} \rightharpoonup W = [w, z]$. Let's divide system (2.8) by $\|W_n\|$ to get

$$\begin{aligned} d_1^n \frac{u_n}{\|W_n\|} - b_{1,1} A \frac{u_n}{\|W_n\|} - b_{1,2} A \frac{v_n}{\|W_n\|} + \tau\beta\left(\frac{u_n}{\|W_n\|}\right) &= 0, \\ d_2^n \frac{v_n}{\|W_n\|} - b_{2,1} A \frac{u_n}{\|W_n\|} - b_{2,2} A \frac{v_n}{\|W_n\|} &= 0. \end{aligned} \tag{2.39}$$

It is possible due to linearity of A and positive homogeneity of β. Since $\frac{u_n}{\|W_n\|} \rightharpoonup w$ and $\frac{v_n}{\|W_n\|} \rightharpoonup z$, using compactness of A and that β satisfies (2.7), we get $A\frac{u_n}{\|W_n\|} \to Aw$ and $\beta\left(\frac{u_n}{\|W_n\|}\right) \to \beta(w)$ (analogously for v_n and z). Hence from the system (2.39) we have $\frac{u_n}{\|W_n\|} \to w$, $\frac{v_n}{\|W_n\|} \to z$ and

$$\begin{aligned} d_1^0 w - b_{1,1} Aw - b_{1,2} Az - \tau\beta(z) &= 0, \\ d_2^0 z - b_{2,1} Aw - b_{2,2} Az &= 0. \end{aligned}$$

Hence the point $d^0 \in C_r^R$ is a critical point of the system (2.8), which contradicts Theorem 2.3. $\square$

Corollary 2.4:
For every part C_r^R of the envelope where $0 < r < R < d_2^I$ (from Lemma 2.7), there exists $\varepsilon > 0$ such that there are no bifurcation points of (2.23) in $C_r^R(\varepsilon)$ (ε depending on τ).

Proof.
By the previous Theorem 2.4 there are no critical points of (2.8) in $C_r^R(\varepsilon)$. Hence by Lemma 2.4 there are also no bifurcation points of (2.23) in $C_r^R(\varepsilon)$. $\square$

Chapter 3

Problem with a unilateral term and pure Neumann boundary conditions

Let's consider the stationary system (2.1) from the previous chapter. We will suppose $\Gamma_D = \emptyset$, i.e. pure Neumann boundary conditions presented in (2.2). Also assume that conditions (2.4) are satisfied.

Since $\Gamma_D = \emptyset$ the space $H^1_D(\Omega)$ is the same as the function space $W^{1,2}(\Omega)$. We will consider the inner product $(u,v) = \int_\Omega (\nabla u \nabla v + uv)\, d\Omega$ from Chapter 1. We call the pair of functions $u, v \in W^{1,2}$ satisfying

$$\begin{aligned} \int_\Omega d_1 \nabla u \nabla \varphi \, d\Omega - \left(\int_\Omega b_{1,1} u\varphi \, d\Omega + \int_\Omega b_{1,2} v\varphi \, d\Omega + \int_\Omega \tau u^- \varphi \, d\Omega \right) &= 0, \\ \int_\Omega d_2 \nabla v \nabla \varphi \, d\Omega - \left(\int_\Omega b_{2,1} u\varphi \, d\Omega + \int_\Omega b_{2,2} v\varphi \, d\Omega \right) &= 0, \end{aligned} \quad \forall \varphi \in W^{1,2}(\Omega). \tag{3.1}$$

a weak solution of (2.1). We already showed that integrals in (3.1) are finite.

Remark 3.1.
Let's remind that by Remark 1.2 the operator A shares its eigenfunctions with the Laplacian and the form of the eigenvalues of the operator A is $\mu_j = \frac{1}{1+\kappa_j}, j = 0, 1, 2, \ldots$.

We can rewrite system (3.1) to the system of operator equations

$$\begin{aligned} d_1(I - A)u - b_{1,1}Au - b_{1,2}Av + \tau\beta(u) &= 0, \\ d_2(I - A)v - b_{2,1}Au - b_{2,2}Av &= 0. \end{aligned} \tag{3.2}$$

We will consider arbitrary fixed d_2. We tend to express v from the second equation and insert it to the first one, as we did in Chapter 2. We need to assume that $\frac{d_2}{d_2+b_{2,2}} \notin \sigma(A)$, so that the operator $d_2I - d_2A - b_{2,2}A$ would be injective. Hence, it is necessary to have

$$\begin{aligned} \frac{d_2}{d_2 + b_{2,2}} &\neq \frac{1}{1+\kappa_j} \\ d_2 + d_2\kappa_j &\neq d_2 + b_{2,2} \\ \kappa_j &\neq \frac{b_{2,2}}{d_2}. \end{aligned}$$

Since $\frac{b_{2,2}}{d_2} < 0$, the previous condition is always satisfied. Hence, the operator $d_2I - d_2A - b_{2,2}A$ is linear and injective, that means the inverse exists. We can rewrite system (3.2) as

$$\begin{aligned} &d_1(I-A)u - b_{1,1}Au - b_{1,2}Av + \tau\beta(u) = 0, \\ &v = (d_2I - d_2A - b_{2,2}A)^{-1}b_{2,1}Au. \end{aligned}$$

After we insert v into the first equation, we denote

$$S := b_{1,1}A + b_{1,2}A(d_2I - d_2A - b_{2,2}A)^{-1}b_{2,1}A. \tag{3.3}$$

Hence we get the operator equation

$$d_1(I-A)u - Su + \tau\beta(u) = 0. \tag{3.4}$$

Observation 3.1:
The system

$$\begin{aligned} &d_1(I-A)u - Su + \tau\beta(u) = 0, \\ &v = (d_2I - d_2A - b_{2,2}A)^{-1}b_{2,1}Au, \end{aligned}$$

is equivalent with the system (3.2). *The similar equivalence is described by Observation* 2.1 *in the previous chapter.*

Remark 3.2. *We will call d_1 an eigenvalue of the problem* (3.4) *if and only if there exists a non-trivial solution of the problem* (3.4) *for d_1.*

Remark 3.3. *It can be proven in the same way as in Lemma* 2.2 *that the operator S is linear, continuous, symmetric and compact. Also in very similar way we can found out that the operator S share its eigenfunctions with the operator A and the Laplacian. The eigenvalues of the operator S corresponding to the eigenfunctions e_j are $\lambda_1^j = \frac{1}{\kappa_j+1}(\frac{b_{1,2}b_{2,1}}{d_2\kappa_j - b_{2,2}} + b_{1,1})$. Consequently the eigenvalues of the problem $d_1(I-A)u - Su = 0$ are $d_1^j = \frac{1}{\kappa_j}(\frac{b_{1,2}b_{2,1}}{d_2\kappa_j - b_{2,2}} + b_{1,1})$.*

We need to find the maximal eigenvalue d_1 of the problem (3.4). Multiplying (3.4) by u and expressing d_1, we get

$$d_1 = \frac{(Su,u) - \tau(\beta(u),u)}{((I-A)u,u)}. \tag{3.5}$$

There is a slight difference from Chapter 2. For the problem with mixed boundary conditions, it was sufficient to assume $u \neq o$ so that the fraction (2.29) would be finite. However in this case we need to assume $u \notin Ker(I-A)$, that is u non-constant. We will encounter this complication in the proof of the following theorem.

Theorem 3.1:
There exists a maximal eigenvalue $d_1^{MAX,\beta} = \max\limits_{\substack{u \notin Ker(I-A) \\ u \in W^{1,2}}} \frac{(Su,u)-\tau(\beta(u),u)}{((I-A)u,u)}$ *of the problem* (3.4) *and it is positive for sufficiently small* τ.

Proof.
Let's denote

$$M := \sup_{\substack{u \notin Ker(I-A) \\ u \in W^{1,2}}} \frac{(Su,u)-\tau(\beta(u),u)}{((I-A)u,u)}.$$

Let $u_c = constant$. Clearly $(I-A)u_c = 0$. At first we need to show that

$$u \to u_c \implies \frac{(Su,u)-\tau(\beta(u),u)}{((I-A)u,u)} \to -\infty. \tag{3.6}$$

For every u we have $((I-A)u,u) \geq 0$ and $-\tau(\beta(u),u) \leq 0$. Hence, if it is $(Su_c,u_c) < 0$, then (3.6) is satisfied. The constant eigenfunction corresponds to the eigenvalue $\kappa_0 = 0$ of the Laplacian. Hence from the form of the eigenvalues of S (see Remark 3.3) we get

$$\lambda^0 = b_{1,1} + \frac{b_{1,2}b_{2,1}}{-b_{2,2}} = \frac{-det(\mathbf{B})}{-b_{2,2}} < 0. \tag{3.7}$$

Hence, we can write

$$(Su_c,u_c) = (\lambda^0 u_c, u_c) = \lambda^0 \|u_c\|^2_{W^{1,2}} < 0 \tag{3.8}$$

and consequently (3.6) is true.

Let's show that there exists u such that (Su,u) is positive. From Remark 3.3 we can see that the eigenvalues of S have the form $\lambda^j = \frac{1}{\kappa_j+1}(\frac{b_{1,2}b_{2,1}}{d_2\kappa_j - b_{2,2}} + b_{1,1})$. Hence for some large j_0 is κ_{j_0} sufficiently large so that the eigenvalue λ^{j_0} is positive. Hence we have $(Se_{j_0}, e_{j_0}) = \lambda^{j_0}\|e_{j_0}\|^2_{W^{1,2}} > 0$ and $\frac{(Se_{j_0},e_{j_0})-\tau(\beta(e_{j_0}),e_{j_0})}{((I-A)e_{j_0},e_{j_0})}$ is (dependently on τ) positive.

In this moment it has a sense to look for maximizer of $\frac{(Su,u)-\tau(\beta(u),u)}{((I-A)u,u)}$ in $W^{1,2}\backslash Ker(I-A)$. We can choose a sequence $(u_n) \subset W^{1,2}$, $u_n \notin Ker(I-A)$ and $\|u_n\|_{W^{1,2}} = 1$ such that

$$\lim_{n\to\infty} \frac{(Su_n,u_n)-\tau(\beta(u_n),u_n)}{((I-A)u_n,u_n)} = M.$$

Due to Eberlain-Šmuljan theorem we can automatically assume $u_n \rightharpoonup u_0$. We use compactness of S, A and that β satisfies (2.7) to get $Su_n \to Su_0$, $Au_n \to Au_0$ and $\beta(u_n) \to \beta(u_0)$. Hence

$$\frac{(Su_n,u_n)-\tau(\beta(u_n),u_n)}{((I-A)u_n,u_n)} \to \frac{(Su_0,u_0)-\tau(\beta(u_0),u_0)}{((I-A)u_0,u_0)}, \tag{3.9}$$

where u_0 cannot be in $Ker(I-A)$. Indeed, if $u_0 \in Ker(I-A)$, then $\frac{(Su_n,u_n)-\tau(\beta(u_n),u_n)}{((I-A)u_n,u_n)} \to -\infty$ as $n \to \infty$, which is not possible.

Hence the maximum exists and it is attained in the function u_0. The inner product

(Su_0, u_0) is positive, i.e. the fraction $\frac{(Su_0,u_0)-\tau(\beta(u_0),u_0)}{((I-A)u_0,u_0)}$ is positive too for sufficiently small τ.

To show that $d_1^{MAX,\beta}$ is the maximal eigenvalue of the problem (3.4) we use Theorem 2.3.1 with $T \equiv S$, $R \equiv \beta$ and $L \equiv A$. We have already proved the existence of the maximum. The conditions (2.28), (2.27) concerning the operator β were checked in the proof of Corollary 2.2. Hence $d_1^{MAX,\beta}$ is by Theorem 2.3.1 the maximal eigenvalue of the problem (3.4). □

Remark 3.4. *If the inner product (Su, u) would be negative for every $u \in W^{1,2}$, there would be no critical or bifurcation point in the whole positive quadrant $\mathbb{R}^2_+$ of the points $[d_1, d_2]$.*

We recall that the hyperbola corresponding to the eigenvalue κ_0 doesn't exist, i.e. $C_0 = \emptyset$. The most important difference from the Chapter 2 is that the eigenfunction e_1 corresponding to the eigenvalue κ_1 of the Laplacian (and to the hyperbola C_1) changes the sign. Hence eigenfunctions corresponding to all hyperbolas C_j change the sign on the domain Ω. There is a slight difference between two following Theorems and Theorems 2.3, 2.4 from the previous chapter.

Theorem 3.2:
All critical points $[d_1, d_2]$ of the problem (3.2) fulfil $d_1 \leq d_1^{MAX,\beta} < d_1^{MAX}$.

Proof.
The proof is mostly the same as the proof of Theorem 2.3. The only difference is that this time all eigenfunctions of the Laplacian corresponding to the hyperbolas $C_j, j = 1, 2, \dots$ change the sign, while in Chapter 2 the first eigenfunction e_1 corresponding to the hyperbola C_1 didn't change the sign. □

We will consider variable d_2.

Theorem 3.3:
For every part C_r^R of the envelope such that $0 < r < R$, there exists $\varepsilon > 0$ such that there are no critical points of (3.2) in $C_r^R(\varepsilon)$ (ε depending on τ).

Proof.
The proof of this Theorem is almost identical with the proof of Theorem 2.4. We only use here a different system, of course. □

Remark 3.5. *Please note that for bifurcation points of the appropriate system with non-linearities the same results hold as Theorem 3.3 states.*

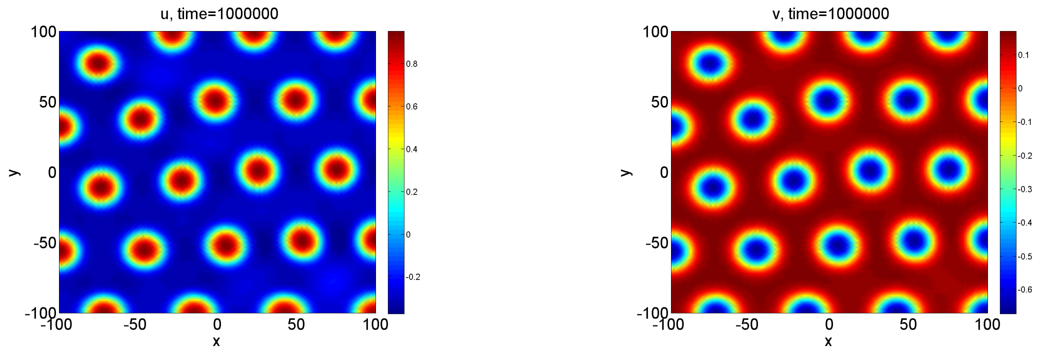
Chapter 4

Numerical experiments

This chapter focus on numerical experiments with a concrete model. The content of the chapter continues the article [12] by Vejchodský, Jaroš, Kučera and Rybář. From Chapter 1, we know that in classical problem (without any unilateral terms) there must be d_1 essentially lesser that d_2 in order to achieve the instability of the trivial stationary solution. It would seem natural to have the diffusion parameters approximately the same, i.e. $d_1 \approx d_2$. Therefore we try to get the ratio $\frac{d_1}{d_2}$ as close as possible to one by adding a unilateral term and also maintain the instability of the stationary solution. This problem is encountered in [12] by adding non-smooth unilateral term τv^- (and its smooth approximations or other variations) to the second equation of the appropriate system. This term is similar to the one we use in the first equation in the second chapter. This source term switch on if v falls under zero. The unilateral term is dependent on the parameter $\tau > 0$ by which can be controlled the strength of the unilateral term.

Since the function v^- is not smooth in zero, the linear analysis cannot be performed as we did in the first chapter. The Jacobi matrix $\mathbf{B}$ simply doesn't exist. Hence we can try to decide stability of the trivial stationary solution only by numerical methods.

In [12] there was tested the unbounded unilateral term. From the biological sense it is unrealistic. For this reason we will use in our experiments so called saturation terms, which are bounded. Besides the existence of patterns for the ratio $\frac{d_1}{d_2}$ close to one, we will observe also qualitative properties of these patterns. In [12], it can be seen that the unilateral terms can break the regularity of the pattern. An example of the regular pattern can be seen on Figure 4.1. As you can see the solutions u and v have just reversed maximum and minimum, so we will present only the solution u.



(a) Solution u (b) Solution v

Figure 4.1: Regular solutions

We will take over the system, the notation and the values of some parameters from [12]. Let's consider the special case of the system (1.1):

$$\begin{aligned} \frac{du}{dt} &= D\delta\Delta u + \alpha u + v - r_2uv - \alpha r_3 uv^2 + \overline{f}(u) \quad \text{in } \Omega, \\ \frac{dv}{dt} &= \delta\Delta v - \alpha u + \beta v + r_2uv + \alpha r_3 uv^2 + \overline{g}(v) \quad \text{in } \Omega. \end{aligned} \tag{4.1}$$

The diffusion parameters are $d_1 = D\delta$ and $d_2 = \delta$. The parameter D is then the ratio $\frac{d_1}{d_2}$. The functions $\overline{f}(u), \overline{g}(v)$ are the unilateral source terms such that $\overline{f}(0) = \overline{g}(0) = 0$. We will assume pure Neumann boundary conditions

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{4.2}$$

and the values of the parameters

$$\alpha = 0.899,\ \beta = -0.91,\ r_2 = 2,\ r_3 = 3.5.$$

The parameters tied to diffusion D, δ will be different in each section. The results of the PDE system can be best represented if $\Omega \subset \mathbb{R}^2$. We will choose domain $\Omega = [-100, 100]^2$.

In case that $\overline{f}(u) \equiv 0, \overline{g}(v) \equiv 0$, the system (4.1) is the usual system we discussed in the introduction. Then the matrix $\mathbf{B}$ would be

$$\mathbf{B} = \begin{bmatrix} \alpha & 1 \\ -\alpha & \beta \end{bmatrix} = \begin{bmatrix} 0.899 & 1 \\ -0.899 & -0.91 \end{bmatrix},$$

i.e. (4.1) is the system of "substrate depletion" type.

We will consider a noise as a initial condition in $t = 0$. The values of initial conditions will be in an interval $[-p, p], p \in \mathbb{R}_+$. We will assume sufficient end-time $T = 10^6$. The problem is solved by so called method of lines, which has been programmed by doc. RNDr.

Tomáš Vejchodský Ph.d in software MATLAB. In space the PDEs are approximated by the most usual linear version of FEM on triangle mesh and in time there are used ODE solvers preprogrammed in MATLAB.

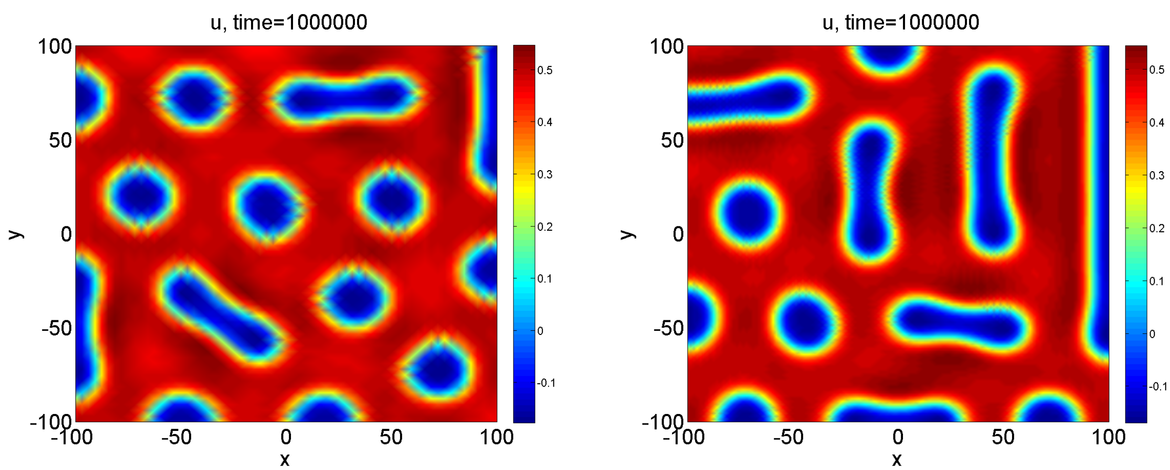
It can be problematic to specify the optimal value p. Since the stability is a local concept, it is necessary to take p as small as possible. Due to limited computer precision we will take p in interval $[0.001, 0.1]$ depending on the type of the experiment.

We will distinguish two approaches to the experiments:

1.) We study Turing effect, i.e. whether the trivial stationary solution is unstable. If for $p = 10^{-3}$ model generates the pattern, we consider the trivial stationary solution unstable.

2.) We study existence of the patterns, i.e. we don't say anything about stability of the trivial solution. The maximal value of D such that the model generates patterns we denote D_{THR} (THR means threshold).

Since our model is non-linear, the computation is sensitive to change of spatial discretization. On Figures 4.2 are solution for different mesh refinement. The patterns are slightly different. In edge case the refinement of mesh can determine if we get the pattern or the constant solution.



(a) Solution using mesh of 2113 vertices

(b) Solution using mesh of 8321 vertices

Figure 4.2: Comparison of solutions generated using different mesh refinement

4.1 Simplest non-smooth unilateral term in the first equation

Let's consider the unilateral terms

$$\overline{f}(u) = \tau u^{-}, \quad \overline{g}(v) \equiv 0, \tag{4.3}$$

where $\tau \in \mathbb{R}_0^+$. We will try to show that for fixed d_2 there is an interval of d_1 on the left side of hyperbolas, where the model doesn't generate the pattern. This result should agree with non-existence of bifurcation points of the system (4.1) with (4.2) in such interval and is similar to the results from Chapter 2.

Let's assume fixed d_2. Since we work in dimension $N = 2$, we can get the eigenvalues $\lambda_{n,m}$ of the Laplacian using Fourier method (see [2]) in the form:

$$\lambda_{n,m} = \left(\frac{n\pi}{100}\right)^2 + \left(\frac{m\pi}{100}\right)^2, \quad n, m = 0, 1, \ldots.$$

If we sort the sequence $\lambda_{n,m}$ in ascending order and denote it κ_j, then the sequence κ_j is the sequence of the eigenvalues of the Laplacian. From the equation $H_d(\kappa_j) = 0$ (see (1.24)), where coefficients $b_{i,j}$ follow from the linear analysis of (4.1), we can compute the maximal d_1 such that $[d_1, d_2] \in C_E$ and denote it $d_1^{\tau=0}$. Let's note that $d_1^{\tau=0}$ is the maximal diffusion parameter d_1 such that the system (4.1) with $\overline{f}(u) = \overline{g}(v) \equiv 0$ generates patterns.

We will discuss twenty different choices of d_2 and we will experimentally determine an approximate values of the maximal d_1 for two values of τ, such that the system (4.1) with (4.3) generate patterns. For $\tau = 0.1$ we denote $d_1^{\tau=0.1}$ the maximal d_1 and for $\tau = 0.2$ we denote it $d_1^{\tau=0.2}$. All of these experiments are executed with the range of the initial values $p = 0.001$. If the model generates pattern for this p, then we consider the trivial solution unstable. The numerical results are in Table 4.1 along with differences of the values $d_1^{\tau=0.1}$ and $d_1^{\tau=0.2}$ from $d_1^{\tau=0}$.

If we look at the second, third and fifth column of the Table 4.1 it is clear that

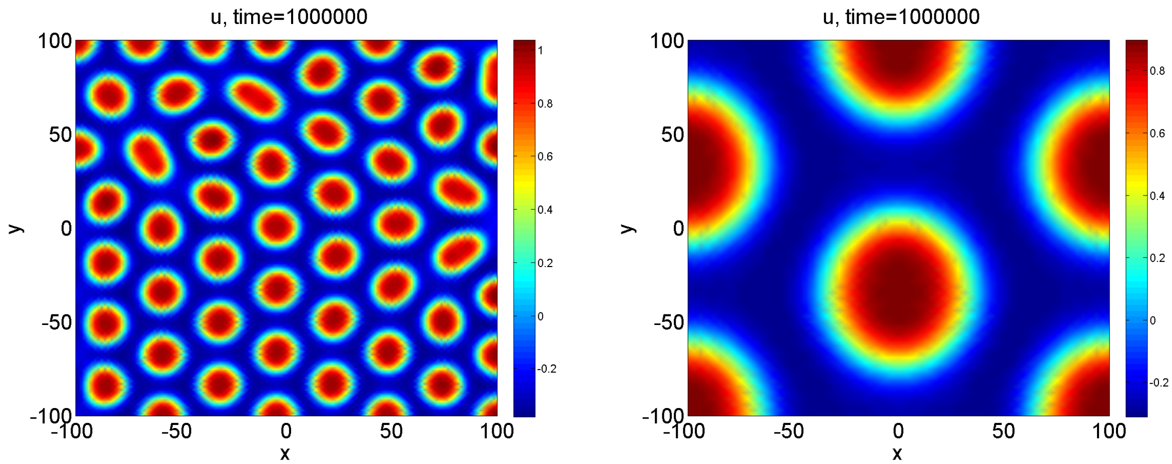
$$d_1^{\tau=0.2} < d_1^{\tau=0.1} < d_1^{\tau=0}, \tag{4.4}$$

which is the expected result. Naturally we would expect that if we double the parameter τ, the difference from the original value $d_1^{\tau=0}$ would double too. But if we compare the differences in the fourth and the sixth column, the results doesn't confirm this expectation. Since finding these estimates requested large amount of experiments, we used a bit rougher discretization to speed up the process. That could affect the results too, especially for smaller values of d_2. In the conclusion the exact values are not that important. More important is that (4.4) holds.

On Figures 4.5 and 4.6 is illustrated comparison of the results. The parameter $d_2 = \delta$ influences the density of the patterns. The relation of the diffusion parameters and the region size is described in section 2.4 of [7]. We illustrate this property on Figures 4.3. The model with this type of the unilateral term also cause irregularity of the patterns. However it happens "deeper" in former region of instability, i.e. for smaller d_1. Figures 4.4 show such patterns for two values of d_2.

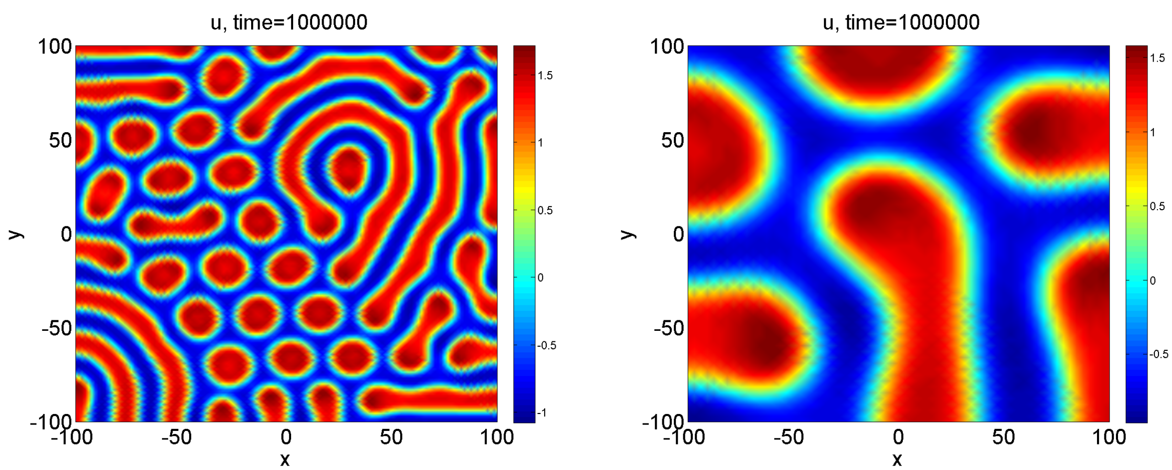
$d_2 = \delta$	$d_1^{\tau=0} = D^{\tau=0}\delta$	$d_1^{\tau=0.1} = D^{\tau=0.1}\delta$	$d_1^{\tau=0} - d_1^{\tau=0.1}$	$d_1^{\tau=0.2} = D^{\tau=0.2}\delta$	$d_1^{\tau=0} - d_1^{\tau=0.2}$
5	2.65	2.39	0.26	2.3	0.35
10	5.32	4.76	0.56	4.53	0.79
15	7.98	7.16	0.82	6.81	1.17
20	10.64	9.54	1.09	9.1	1.53
25	13.28	11.96	1.32	11.39	1.89
30	15.96	14.29	1.66	13.63	2.32
35	18.53	16.73	1.8	15.96	2.57
40	21.28	19.11	2.16	18.23	3.04
45	23.87	21.51	2.36	20.51	3.36
50	26.6	23.95	2.64	22.86	3.73
55	29.16	25.99	3.17	25.12	4.04
60	31.58	28.56	2.44	27.28	3.72
65	34.16	30.94	3.21	29.59	4.56
70	37.07	33.4	3.6	31.89	5.11
80	42.55	38.16	4.39	36.51	6.04
90	47.64	43.04	4.6	41.03	6.61
96	50.52	45.91	4.61	43.74	6.78
110	56.78	52.31	4.47	50.03	6.75
120	60.9	57.35	3.55	54.74	6.16
130	65.04	62.17	2.83	59.47	5.53

Table 4.1: Approximate estimates of maximal d_1 for some τ, such that the model generates the pattern.



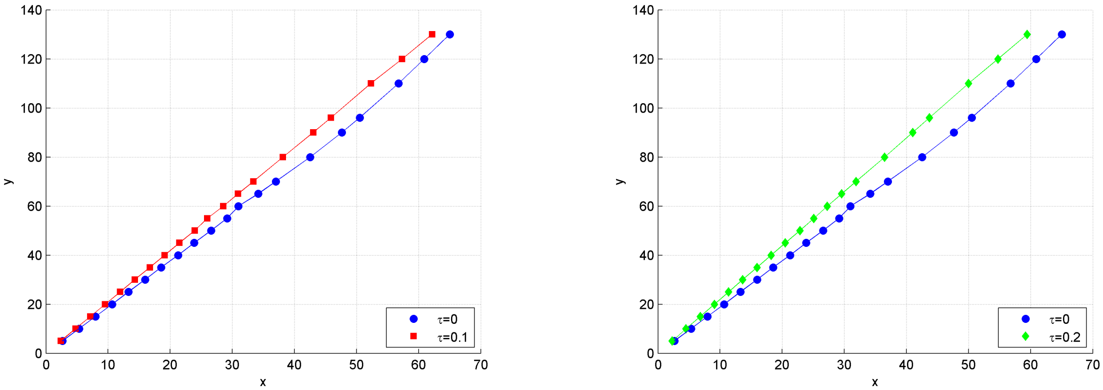
(a) High density pattern for $d_2 = 10$ (b) Low density pattern for $d_2 = 130$

Figure 4.3: Comparison of density of the patterns for two values of d_2



(a) High density degenerated pattern (b) Low density degenerated pattern

Figure 4.4: Examples of degenerated patterns for two values of d_2



(a) Comparison of $d_1^{\tau=0}$ and $d_1^{\tau=0.1}$ (b) Comparison of $d_1^{\tau=0}$ and $d_1^{\tau=0.2}$

Figure 4.5: Comparison of two sets of results to the original values

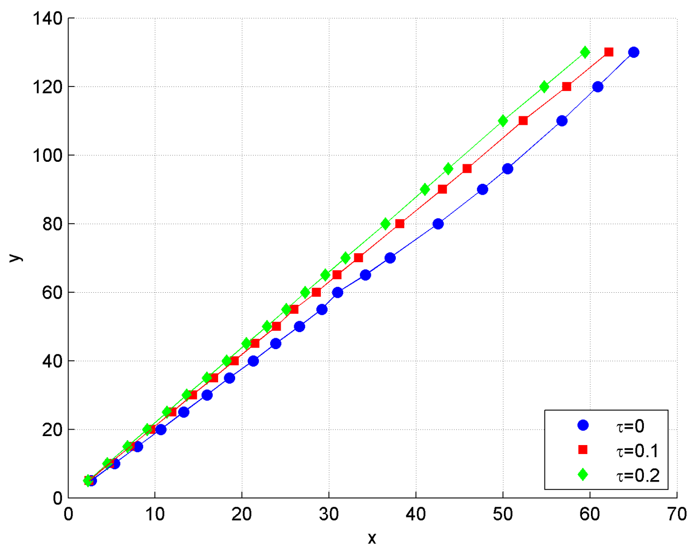


Figure 4.6: Comparison of $d_1^{\tau=0}$,$d_1^{\tau=0.1}$ and $d_1^{\tau=0.2}$

4.2 Unilateral term with saturation in the second equation

Let's consider the unilateral terms

$$\overline{f}(u) \equiv 0, \quad \overline{g}(v) = \frac{\tau v^-}{1 + \varepsilon v^-}, \tag{4.5}$$

where $\tau, \varepsilon \in \mathbb{R}_0^+$. This term differs from the term τv^- in [12] by its boundedness and more variability thanks to more parameters. Obviously if $\varepsilon = 0$ then $\overline{g}(v)$ coincide with τv^-.

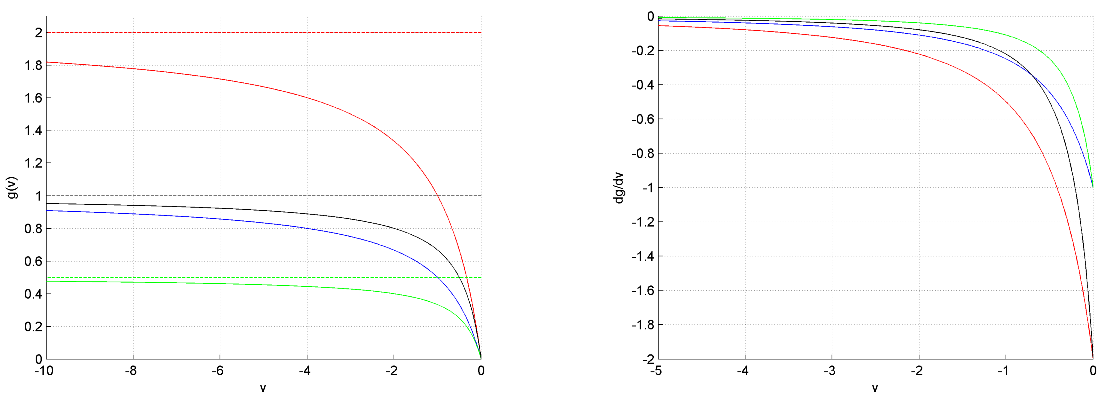
The limit is

$$\lim_{v\to-\infty} \frac{\tau v^-}{1+\varepsilon v^-} = \frac{\tau}{\varepsilon}. \tag{4.6}$$

The limit value $\frac{\tau}{\varepsilon}$ is increasing if τ is increasing or if ε is decreasing. The unilateral term is again non-smooth, because the derivative in zero doesn't exist. However we can compute derivative at least for all $v < 0$:

$$\frac{d}{dv}\left(\frac{-\tau v}{1-\varepsilon v}\right) = -\frac{\tau}{(1-\varepsilon v)^2} \quad \forall v < 0. \tag{4.7}$$

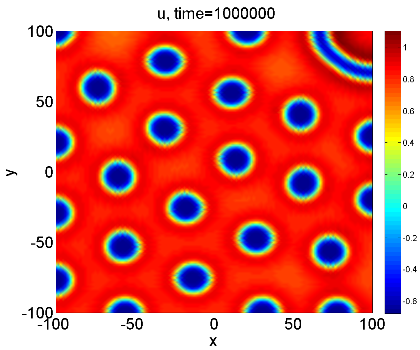
Since we take the range of initial values $p \leq 0.1$ and the values of solutions (see for example Figure 4.1) are usually in interval $(-1, 1)$, the effect of the unilateral term should be most notable in the small neighbourhood of zero. On Figures 4.7 is illustrated function $\overline{g}(v)$ and its derivation for $v < 0$.



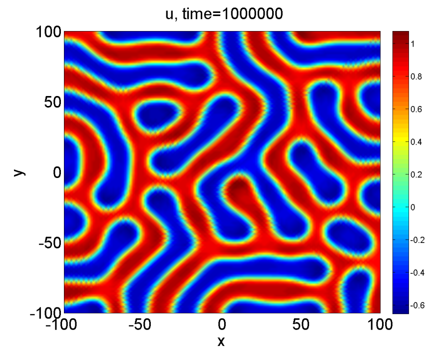
(a) Function $\overline{g}(v)$ (b) Derivative of $\overline{g}(v)$

Figure 4.7: Illustration of the unilateral term and its derivation for $v < 0$ and several value of τ and ε. The plot is blue for $\tau = 1, \varepsilon = 1$, red for $\tau = 2, \varepsilon = 1$, green for $\tau = 1, \varepsilon = 2$ and black for $\tau = 2, \varepsilon = 2$.

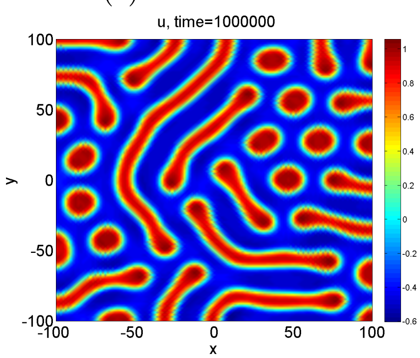
The important property of $\overline{g}(v)$ is that increasing both τ and ε in the same proportion result in same limit of $\overline{g}(v)$ but noticeably larger derivative (in absolute value) close to zero. The function is then more efficient in the small neighbourhood of zero, where we expect the biggest effect. That is the major advantage of $\frac{\tau v^-}{1+\varepsilon v^-}$. This way we can boost effect of the unilateral term in the small neighbourhood of zero, but maintain reasonable limit.



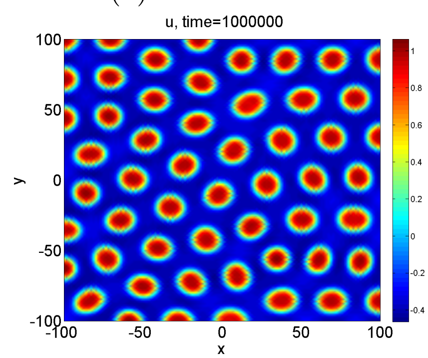
(a) Result for $\varepsilon = 2$



(b) Result for $\varepsilon = 3$



(c) Result for $\varepsilon = 5$



(d) Result for $\varepsilon = 10$

Figure 4.8: Examples of patters generated by the model with saturation term in (4.5). Parameters are $D = 0.5$,$\tau = 0.2$.

We will perform our experiments for $\delta = 6$ as in [12]. Unlike the previous section 3.1, here we will study only existence of the pattern, not the stability of the trivial solution. The goal is to find maximal value of D such that the model with the saturation term (4.5) generates pattern. We can experiment with different values of τ and ε. In the article [12] were used "small" values of τ. For term τv^- was experimentally determined maximal $D_{THR} = 0.71$. We will try to use larger τ and in combination with the saturation term (4.5) reach better results, i.e. larger D_{THR}.

Figures 4.8 illustrates qualitatively different examples of patterns that the model with saturation term (4.5) usually generate. For "large" $\varepsilon = 10$ Figure 4.8d shows, that the pattern is qualitatively same as regular pattern on Figure 4.1. That makes sense, because the term $\frac{\tau v^-}{1+\varepsilon v^-} \to 0$ as $\varepsilon \to \infty$. As $\varepsilon \to 0$, the pattern tends to be more and more disrupted. Actually the term $\frac{\tau v^-}{1+\varepsilon v^-} \to \tau v^-$ as $\varepsilon \to 0$.

For the range of initial values $p = 0.1$ we ran tests for different values of ε and τ. We were able to find the best results for $\tau = 0.1$. In case of τ for example 0.2 or 0.5, the critical ratio D_{THR} usually didn't even exceed 0.7, which makes these test uninteresting. The values of D_{THR} with precision on two significant digits are in table 4.2.

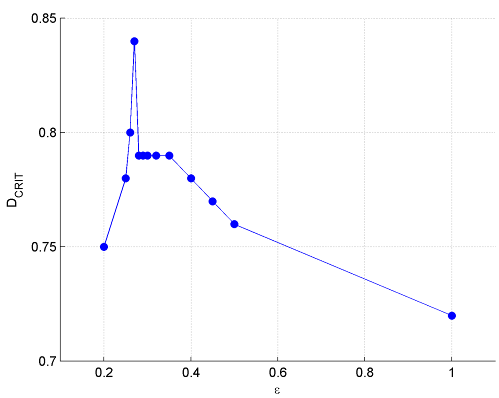
ε	0.20	0.25	0.26	0.27	0.28	0.29	0.30	0.32	0.35	0.40	0.45	0.50	1.00
D_{THR}	0.75	0.78	0.80	0.84	0.79	0.79	0.79	0.79	0.79	0.78	0.77	0.76	0.72

Table 4.2: Maximal value of D for ε such that the model with saturation term (4.5) generates patterns. The range of initial values is $p = 0.1$ and $\tau = 0.1$.

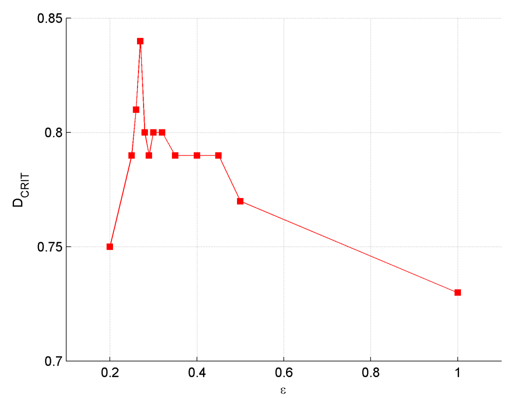
The experiments for the same parameters and the range of the initial values $p = 0.001$ lead to very similar result as can be seen in Table 4.3 and Figure 4.9.

ε	0.20	0.25	0.26	0.27	0.28	0.29	0.30	0.32	0.35	0.40	0.45	0.50	1.00
D_{THR}	0.75	0.79	0.81	0.84	0.80	0.79	0.80	0.80	0.79	0.79	0.79	0.77	0.73

Table 4.3: Maximal value of D for ε such that the model with saturation term (4.5) generates patterns. The range of initial values is $p = 0.001$ and $\tau = 0.1$.



(a) Range of initial values $p = 0.1$



(b) Range of initial values $p = 0.001$

Figure 4.9: D_{THR} for $\tau = 0.1$ in the dependence of ε

In the conclusion, the largest D_{THR} we were able to reach by the terms (4.5) is $D_{THR} = 0.84$. In comparison with $D_{THR} = 0.71$ for model with τv^-, it is a considerable improvement.

The patterns for large D are sometimes trivial (not always), for example there can be only one spot on the region Ω. The pattern for $\tau = 0.1, \varepsilon = 0.27$ and $D_{THR} = 0.84$ looks like on Figure 4.10.

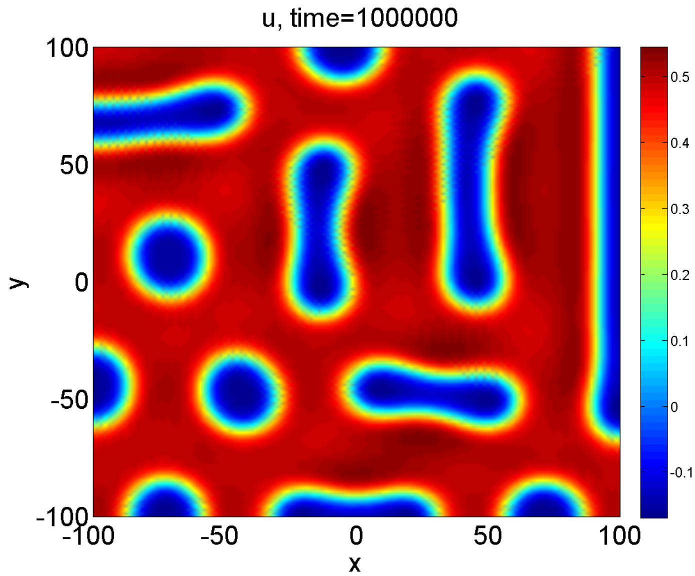


Figure 4.10: Pattern for $D_{THR} = 0.84, \tau = 0.1, \varepsilon = 0.27$ and initial range $p = 0.1$

4.3 Quadratic unilateral term with saturation

Let's consider the unilateral terms

$$\overline{f}(u) \equiv 0, \quad \overline{g}(v) = \frac{\tau(v^-)^2}{1+\varepsilon(v^-)^2}, \tag{4.8}$$

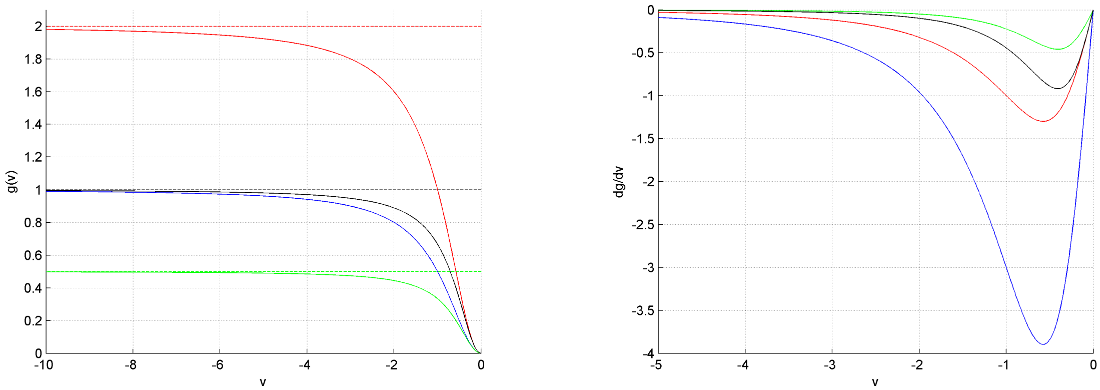
where $\tau, \varepsilon \in \mathbb{R}_0^+$. Like in the previous section we have the unilateral term with two parameters and the limit

$$\lim_{v \to -\infty} \frac{\tau(v^-)^2}{1+\varepsilon(v^-)^2} = \frac{\tau}{\varepsilon}. \tag{4.9}$$

The limit (4.9) has the same properties as (4.6) of course. Again if $\varepsilon \to \infty$ then $\frac{\tau(v^-)^2}{1+\varepsilon(v^-)^2} \to 0$. Hence for large ε, the patterns tend to be more regular (compare with the previous section). Opposite to the saturation term (4.5) of the section 3.2, the unilateral term from (4.8) is smooth:

$$\frac{d}{dv}\left(\frac{\tau(v^-)^2}{1+\varepsilon(v^-)^2}\right) = -\frac{2\tau v^-}{(1+\varepsilon(v^-)^2)^2} \quad \forall v \in \mathbb{R}. \tag{4.10}$$

The derivative in zero is actually zero, the linear analysis wouldn't be influenced by this term at all. Hence the trivial solution will be unstable for the values under some critical D, which is around 0.5. Here we will not study the stability of the trivial solution, but again the existence of the patterns. The parameter D_{THR} will be the maximal D such that model with unilateral term generates pattern. The function $\overline{g}(v)$ and its derivative is illustrated on Figures 4.11 along with its behaviour for different τ and ε.

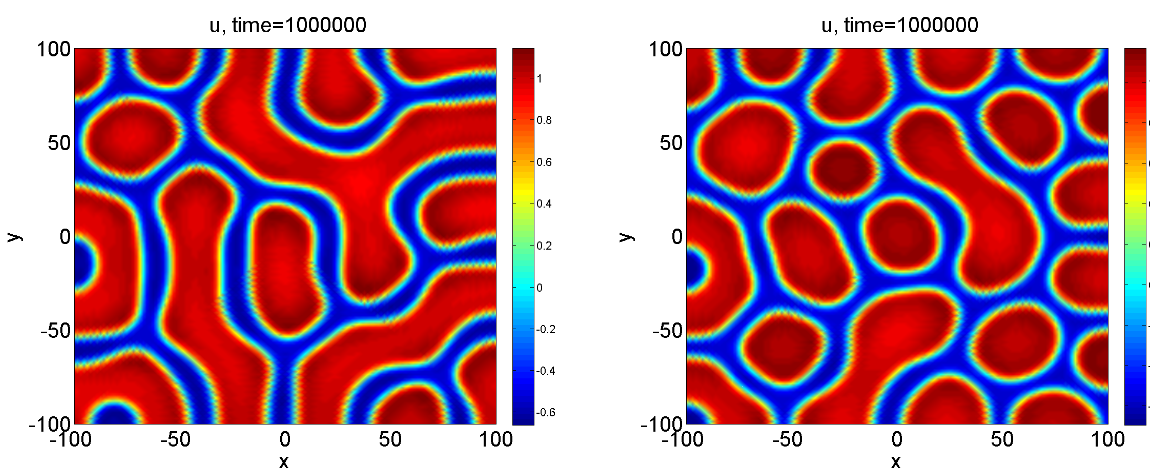


(a) Function $\overline{g}(v)$ (b) Derivative of $\overline{g}(v)$

Figure 4.11: Illustration of the unilateral term and its derivation for several values of τ and ε. The plot is blue for $\tau = 1, \varepsilon = 1$, red for $\tau = 2, \varepsilon = 1$, green for $\tau = 1, \varepsilon = 2$ and black for $\tau = 2, \varepsilon = 2$.

The experiments have shown that the model with quadratic saturation term generates qualitatively similar patterns as the saturation term (4.5). Moreover the patterns of type illustrated on Figures 4.12 are generated in some situations. These patterns are more widespread than the ones we have encountered before.

We will experiment with two values of the parameter τ. The trivial solution will be stable



(a) Example 1 (b) Example 2

Figure 4.12: Examples of the patterns generated by the model with unilateral term $\overline{g}(v)$

for values of D larger than approximately 0.53. The only way how to get some patterns for larger D is to take larger p, in our case $p = 0.1$. Since we study the existence of the patterns and not the stability of the trivial solution, taking larger p is all right. In the tables 4.4 and 4.5 are the values D_{THR} depending on ε for two values of $\tau = 0.2$ and $\tau = 0.3$.

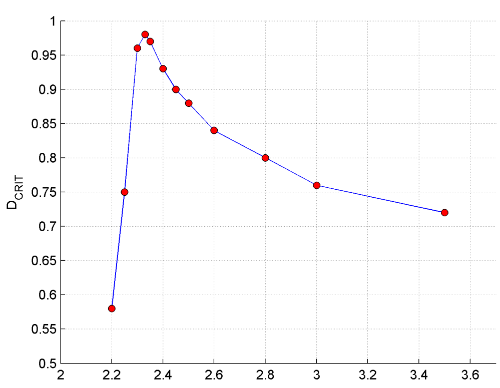
ε	2.2	2.25	2.30	2.33	2.35	2.4	2.45	2.5	2.6	2.8	3	3.5
D_{THR}	0.58	0.75	0.96	0.98	0.97	0.93	0.9	0.88	0.84	0.8	0.76	0.72

Table 4.4: Maximal value of D for ε such that the model with quadratic saturation term generates patterns. Tested for $\tau = 0.2$.

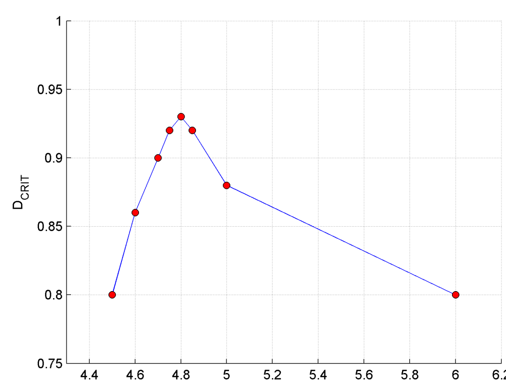
ε	4.5	4.6	4.7	4.75	4.8	4.85	5	6
D_{THR}	0.8	0.86	0.9	0.92	0.93	0.92	0.88	0.8

Table 4.5: Maximal value of D for ε such that the model with quadratic saturation term generates patterns. Tested for $\tau = 0.3$.

It can be seen that the values of D_{THR} are in some cases really close to one. The closest we have got is $D_{THR} = 0.98$. That is again improvement with respect to the largest $D_{THR} = 0.84$ of the previous section. The patterns for very large D mostly $D > 0.9$ are in some cases really trivial and qualitatively uninteresting. Example of such a pattern is Figure 4.14. However unless the corresponding solutions are constant or "almost" constant, we consider them the pattern. On Figures 4.13 the values of D_{THR} in the dependence on ε are illustrated for two values of τ.



(a) D_{THR} for $\tau = 0.2$



(b) D_{THR} for $\tau = 0.3$

Figure 4.13: D_{THR} for the range of the initial values $p = 0.1$

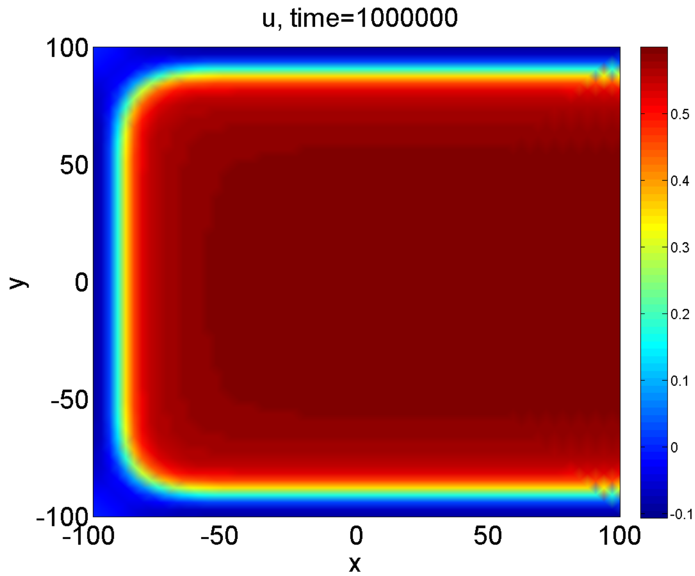


Figure 4.14: Pattern for $D_{THR} = 0.98, \tau = 0.2, \varepsilon = 2.33$ and initial range $p = 0.1$

4.4 Saturation term with τ spatially dependent

Until this section, we have considered unilateral terms on the whole region Ω. This time we will consider τ spatially dependable. Let's consider unilateral term

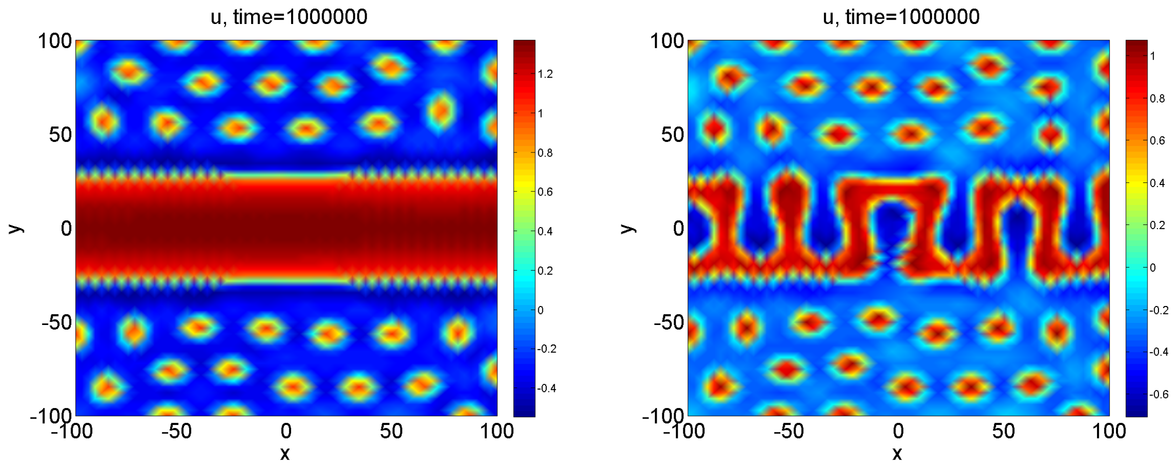
$$\tau(x,y)\frac{v^-}{1+\varepsilon v^-} \tag{4.11}$$

added to the first equation of (4.1). The function $\tau(x,y)$ is defined as

$$\tau(x,y) = \begin{cases} \theta & y < 25 \wedge y > -25, \\ 0 & \text{else}, \end{cases} \tag{4.12}$$

where $\theta > 0$. It is actually constant θ-rectangle around x-axis and zero on the rest of the Ω. We are multiplying the characteristic function by saturation term (4.5) from section 3.2. The aim of this section is to find out what qualitative properties the patterns generated by model with term (4.11) posses.

The experiments show that patterns are outside the rectangle regular and on the rectangle similar to patterns, which occur in section 3.2. It means that for "small" ε the pattern is very irregular. On Figure 4.15*a* it can be seen that the pattern on rectangle completely disappeared. For larger values of ε the pattern is irregular like on Figure 4.15*b* and for "large" ε, it gets regular as on Figure 4.16. We have already observed this behaviour in section 3.2 on the whole region Ω. All of the experiments were run for initial range $p = 0.1$, because the computation time was already like 8 or more hours per experiment. This complication also prevented us from experimenting with larger D.



(a) Annihilated pattern for $\theta = 0.1, \varepsilon = 0.1$ (b) Heavily disrupted pattern for $\theta = 0.2, \varepsilon = 3$

Figure 4.15: Irregular patterns for $D = 0.5$ and different θ, ε

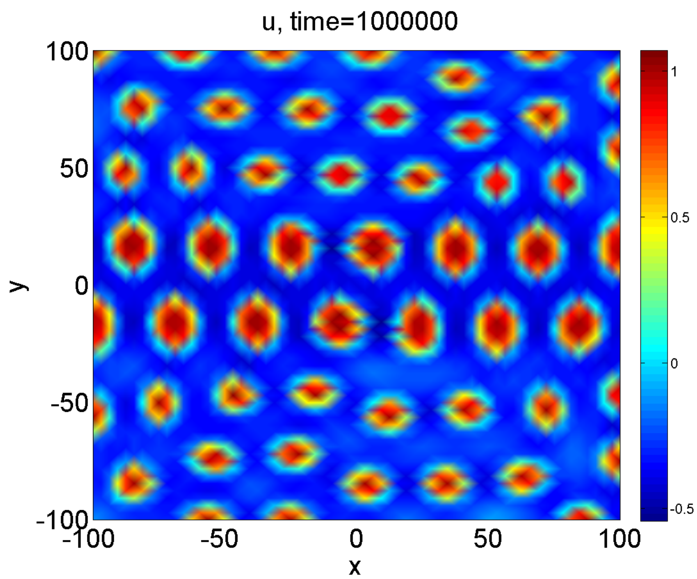


Figure 4.16: Regular pattern for $\theta = 0.1, \varepsilon = 3, D = 0.5$

Conclusion

We have summarized a theory concerning Turing effect and the system of reaction-diffusion equations. The most important was an introduction of the hyperbolas C_j and the envelope C_E, which divide the positive quadrant $\mathbb{R}^2_+$ of the plane of diffusion parameters $[d_1, d_2]$ on the region of stability D_S and the region of instability D_U. We also found out that in case of the classical problem without any unilateral terms there must be d_1 essentially lesser than d_2, so that the spatially non-homogeneous solutions would arise from small perturbations of the stationary solution.

In Chapter 2 we have achieved new results concerning critical points of the system (2.8) and bifurcation points of the system (2.23), both with mixed boundary conditions. Theorems 2.3, 2.4 and Corollary 2.4 are the main results of this chapter. We have proved that there are no critical or bifurcation points in $C_r^R(\varepsilon)$ (ε-neighbourhood of the envelope C_E). Hence the region where there are no critical or bifurcation points is larger than at the classical problem. The stationary spatially non-homogeneous solutions describing spatial patterns don't arise in the region $C_r^R(\varepsilon) \cup D_S$.

In Chapter 3 we have formulated and proved Theorems 3.2 and 3.3 for the system (3.2) with pure Neumann boundary conditions, which are alternatives of Theorems 2.3, 2.4.

In the last chapter we have experimented with the concrete model and various unilateral terms. We were able to reach the value $D_{THR} = 0.98$ of the ratio $D = \frac{d_1}{d_2}$, so that the model still generates patterns. This value was reached by using the unilateral term $\frac{\tau(v^-)^2}{1+\varepsilon(v^-)^2}$. Unilateral terms can disturb the regularity of the patterns. We have discovered some new and interesting shapes of these patterns. The amount of performed experiments is approximately 1200.

In the future work, we would like to use the term $\tau_1(x)u^- - \tau_2(x)u^+$ instead of the unilateral term τu^- from Chapters 2,3 and generalize our results.

Appendix

A Definitions

Definition A.1 (Positively homogeneous operator):
Let H be a Hilbert space and an operator $T : H \mapsto H$. Then T is a positively homogeneous operator if

$$T(tu) = tT(u), \quad \forall t \in \mathbb{R}_0^+, \forall u \in H.$$

Definition A.2 (Sobolev conjugate):
Let N be the space dimension and $1 \leq p < N$. Then

$$p^* = \frac{Np}{N-p} > p$$

is the Sobolev conjugate of p.

Definition A.3 (Caratheodory conditions):
Let $f : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}$ be a real function. Then f satisfies Carathéodory conditions if
i.) $F(\cdot, \xi)$ is measurable for all $\xi \in \mathbb{R}^N$,
ii.) $F(x, \cdot)$ is continuous almost everywhere on Ω.

Definition A.4 (Nemytskii operator):
Let $n : \Omega \times \Omega \mapsto \mathbb{R}$ be a real function satisfying Caratheodory conditions. We define an operator $\mathcal{N}$ by

$$\mathcal{N}(u, v)(x) = n(u(x), v(x)) \text{ almost everywhere on } \Omega$$

and call it Nemytskii operator.

B Staments

Theorem B.1 (Stability of ODE system):
Consider dynamical system defined by

$$\frac{dx}{dt} = f(x), \ x \in \mathbb{R}^N$$

where f is smooth. Suppose it has an equilibrium $\overline{x}$ and denote $\mathcal{J}$ the Jacobian matrix of f evaluated at $\overline{x}$. Then $\overline{x}$ is stable if all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ of $\mathcal{J}$ satisfy $Re(\lambda) < 0$.

Lemma B.1 (Hölder's inequality):
Let $p \in [1, \infty]$ and set

$$p' = \begin{cases} \frac{p}{p-1} & 1 < p < \infty, \\ 1 & p = \infty, \\ \infty & p = 1. \end{cases}$$

Let $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$. Then $u \cdot v \in L^1(\Omega)$ and Hölder's inequality holds:

$$\|u \cdot v\|_1 \leq \|u\|_p \|v\|_{p'}.$$

Theorem B.2 (Riesz representation theorem):
Let H be a Hilbert space and $F : H \mapsto \mathbb{R}$ a continuous linear operator. Then there exists exactly one $v \in H$ such as for all $u \in H$

$$F(u) = (u, v).$$

Furthermore following identity holds:

$$\|F\|_* = \|v\|.$$

Theorem B.3 (Rellich-Kondrachov):
Let Ω be a bounded open set in $\mathbb{R}^n$ with a locally Lipschitz boundary, $k \in \mathbb{N}$, $p \in [1, \infty)$.
(i.) Let $k < \frac{n}{p}$ and $q \in [1, p^*)$ where

$$p^* := \frac{np}{n - kp}.$$

Then the embedding $W^{k,p}(\Omega)$ into $L^q(\Omega)$ is compact.
(ii.) If $k = \frac{n}{p}$, then $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q \in [1, \infty)$.
(iii.) If $0 \leq \gamma \leq k - \frac{n}{p}$, then $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow C^{0,\gamma}(\overline{\Omega})$.

Theorem B.4 (Eberlain-Šmuljan):
Banach space is reflexive $\Leftrightarrow$ every bounded sequence has weakly convergent subsequence.

Theorem B.5 (Trace theorem):
Let $\Omega \in C^{0,1}$ be a bounded domain in $\mathbb{R}^N$. There exists one and only one continuous linear operator T which assigns every function $u \in W^{1,p}(\Omega)$, $(p > 1)$ a function $Tu \in L^p(\partial\Omega)$ and has following property:

$$\text{"}\forall u \in C^\infty(\overline{\Omega}) \text{ we have } Tu = u|_{\partial\Omega}\text{"}.$$

Following identity holds:

$$W_0^{1,p} = \{u \in W^{1,p}(\Omega) : Tu = o \text{ in } L^p(\partial\Omega)\}.$$

Instead of $Tu = o$ in $L^p(\partial\Omega)$ we say that $u = 0$ on $\partial\Omega$ in the sense of traces.

Theorem B.6 (Inverse operator):
Let X, Y be real vector spaces. Let $L : \mathcal{D}(L) \mapsto Y$ be a linear operator with domain $\mathcal{D}(L) \subset X$ and range $\mathcal{R}(L) \subset Y$. Then:
(i.) The inverse $T^{-1} : \mathcal{R}(L) \mapsto \mathcal{D}$ exists if and only if

$$Lx = 0 \implies x = 0. \tag{4.13}$$

(ii.) If T^{-1} exists, it is a linear operator.
(iii.) If $dom(\mathcal{D}(L)) = n < \infty$ and T^{-1} exists, then $dom(\mathcal{D}(L)) = dom(\mathcal{R}(L))$.

Theorem B.7 (Bounded inverse):
Let X, Y be Banach spaces and $T : X \mapsto Y$ be a bijective linear and bounded operator. Then inverse T^{-1} is also bounded.

Theorem B.8 (Fredholm alternative):
Let H be a Hilbert space and $T : H \mapsto H$ be a self-adjoint compact operator.
(i.) For any $\lambda \neq 0$, equation

$$Tu - \lambda u = f \tag{4.14}$$

has a solution for every $f \in H$ if λ is not an eigenvalue of T. The solution is then unique.
(ii.) If $\lambda \neq 0$ is an eigenvalue of T, equation (4.14) has a solution if and only if the function f is orthogonal to every solution of homogeneous equation $Tu - \lambda u = 0$.

Lemma B.2 (Compactness of product):
Let $T : X \mapsto X$ be a compact operator and $S : X \mapsto X$ a bounded linear operator on a normed space X. Then TS and ST are compact.

Lemma B.3 (Self-adjointness of product):
A product of two linear bounded self-adjoint operators T, S on a Hilbert space is self-adjoint if and only if the operators commute:

$$ST = TS.$$

Theorem B.9 (Nemytskii theorem):
Let $\mathcal{N}$ be a Nemytskii operator satisfying Carathéodory conditions and $\Omega \subset \mathbb{R}^N$. If $p < \infty$, and a non-linear function n satisfies

$$|n(\chi, \xi)| \leq g_1(\mathbf{x}) + c(\mathbf{x})(|\chi|^{\frac{p}{q}} + |\xi|^{\frac{p}{q}}) \quad \forall \chi, \xi \in \mathbb{R}, \text{ almost everywhere on } \Omega,$$

where $g_1(\mathbf{x}) \in L^q(\Omega)$ and $c(\mathbf{x}) \in L^\infty(\Omega)$, then Nemytskii operator $\mathcal{N}$ is well defined and continuous operator from the space $L^p \times L^p$ to L^q.

Theorem B.10 (Sobolev embedding theorem):
Let $k \in \mathbb{N}$ and let $p \in [1, \infty)$.
(i.) If $k < \frac{N}{p}$, then $W^{k,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{k}{N}$.
(ii.) If $k = \frac{N}{p}$, then $W^{k,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N) \quad \forall r \in [p, \infty)$ and
$W^{k,p}(\mathbb{R}^N) \hookrightarrow L^r_{loc}(\mathbb{R}^N) \quad \forall r \geq 1$.

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