University of West Bohemia
Faculty of Applied Sciences
Department of Mathematics

Properties of generalized trigonometric functions

Bachelor thesis

Pilsen 2016 Martina Vašinová

Declaration
I hereby declare that the entire bachelor thesis is my original work and that I have used only the cited sources.

In Pilsen ....

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Abstract
The main goal is to summarize known properties of generalized trigonometric functions and apply them on computation of integrals with these functions. Other important parts are code implementation in Matlab, visualization of the functions and analysis of the numerical part of the problem.

Contents

1 Introduction

1.1 Analytic point of view

A trigonometric function $\sin x$ is a 2π-periodic function bounded by 1 and -1. The inverse function, $\arcsin x$, is described by the following formula:

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}}\mathrm{d}t, 0 \le x \le 1. \tag{1}$$

We can generalize this function for $1 < p < \infty$:

$$\arcsin_p x = \int_0^x \frac{1}{\sqrt[p]{1-t^p}}\mathrm{d}t, 0 \le x \le 1, \tag{2}$$

which is an injective function from $\langle 0, 1\rangle$ to $\langle 0, \frac{\pi_p}{2}\rangle$.
We will describe π_p by the following equalities:

$$\begin{aligned} \int_0^1 \frac{1}{\sqrt{1-t^2}}\mathrm{d}t = [\arcsin t]_0^1 = \frac{\pi}{2} \Rightarrow \pi &= 2\int_0^1 \frac{1}{\sqrt{1-t^2}}\mathrm{d}t, \\ \pi_p &= 2\int_0^1 \frac{1}{\sqrt[p]{1-t^p}}\mathrm{d}t. \end{aligned} \tag{3}$$

We can tell from the relation above that $\pi_2 = \pi$.
The inverse function to $\arcsin_p x$ on $\langle 0, \frac{\pi_p}{2}\rangle$ is $\sin_p x$. Let $\sin_p x$ be extended on $x \in \langle \frac{\pi_p}{2}, \pi_p\rangle$ symetrically:

$$\sin_p x = \sin_p(\pi_p - x).$$

The function $\sin_p x$ is odd, that extends it for $x \in \langle -\pi_p, 0\rangle$ and a $2\pi_p$-periodic function, that defines it for whole $\mathbb{R}$. As we can see, $\sin_2 x = \sin x$.
These definitons for $\sin_p x$ and π_p are analogous to Edmunds' and Lang's [5] and to Elbert's [6].
But the definition of $\cos_p x$ is different: Edmunds' and Lang's [5] definition is

$$\cos_p x \overset{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}x}\sin_p x \tag{4}$$

and Elbert's [6] definition is

$$\cos_p x \overset{\text{def}}{=} \int_x^{\frac{\pi_p}{2}} \sin_p y \,\mathrm{d}y. \tag{5}$$

In this thesis the first definition will be used. The function $\cos_p x$ is even and $2\pi_p$-periodic and $\cos_2 x = \cos x$.
We will define $\tan_p x$ just as $\tan x$ is defined:

$$\tan_p x \overset{\text{def}}{=} \frac{\sin_p x}{\cos_p x}, \tag{6}$$

then $\tan_p x$ is odd and π_p-periodic and $\tan_2 x = \tan x$. We have to mention that $\tan_p x$ is not defined for $x = (k + \frac{1}{2})\pi_p, k \in \mathbb{Z}$.
The definitions of functions sec, cosec and cotan follow:

$$\begin{aligned} \mathrm{cosec}_p x &\stackrel{\text{def}}{=} \frac{1}{\sin_p x}, & (7)\\ \sec_p x &\stackrel{\text{def}}{=} \frac{1}{\cos_p x}, & (8)\\ \mathrm{cotan}_p x &\stackrel{\text{def}}{=} \frac{1}{\tan_p x}. & (9) \end{aligned}$$

Now we can examine if some well known properties of $\sin x$, $\cos x$ and $\tan x$ fit for generalized trigonometric functions too:

$$\begin{aligned} \sin_p(-x) &= -\sin_p x, x \in \mathbb{R}\ (\sin_p x \text{ is odd}) & (10)\\ \cos_p(-x) &= \cos_p x, x \in \mathbb{R}\ (\cos_p x \text{ is even}) & (11)\\ \tan_p(-x) &= -\tan x, x \in \mathbb{R}\ (\tan_p x \text{ is odd}) & (12)\\ |\sin_p x|^p + |\cos_p x|^p &= 1, x \in \mathbb{R}\ (\sin_p x \text{ and } \cos_p x \text{ are symmetric and periodic}) & (13)\\ |\sec_p x|^p - |\tan_p x|^p &= 1, x \in \mathbb{R} \setminus (k + \frac{1}{2})\pi_p, k \in \mathbb{Z} & (14) \end{aligned}$$

We will prove (14):
Let $x \in \mathbb{R} \setminus (k + \frac{1}{2})\pi_p, k \in \mathbb{Z}$:

$$|\sin_p x|^p + |\cos_p x|^p = 1 \Rightarrow |\cos_p x|^p = 1 - |\sin_p x|^p.$$

By dividing the equation by $|\cos_p x|^p$ and supposing $|\cos_p x|^p \neq 0$ we obtain

$$1 = \tfrac{1-|\sin_p x|^p}{|\cos_p x|^p} = \tfrac{1}{|\cos_p x|^p} - |\tan_p x|^p \Rightarrow |\sec_p x|^p - |\tan_p x|^p = 1.$$

1.2 Boundary and corresponding initial value problem

Let us consider following boundary value problem:

$$\begin{aligned} (|u'|^{p-2}u')' + \lambda|u|^{p-2}u &= 0, & (15)\\ u(0) = u(\pi_p) &= 0, & (16) \end{aligned}$$

where $p > 1, \lambda > 0$ are given real numbers. We will show the origin of $\mathrm{arcsin}_p(x)$ and $\sin_p(x)$ by rewriting (15):

$$\begin{aligned} (|u'|^{p-1})'u' + \lambda|u|^{p-2}uu' &= 0\\ (p-1)|u'|^{p-2}u''u' + \lambda|u|^{p-2}uu' &= 0\\ \frac{p-1}{p-1}(|u'(x)|^p - |u'(0)|^p) + \frac{\lambda}{p-1}(|u(x)|^p - |u(0)|^p) &= 0\\ |u'(x)|^p + |u(x)|^p\frac{\lambda}{p-1} &= 1 \end{aligned}$$

We assume $\lambda = p - 1$.

$$|u'(x)|^p + |u(x)|^p = 1$$

We assume $x \in \left[0, \frac{\pi_p}{2}\right]$.

$$\begin{aligned}
(u'(x))^p + (u(x))^p &= 1 \\
u'(x) &= (1 - u^p(x))^{\frac{1}{p}} \\
\frac{u'(x)}{(1 - u^p(x))^{\frac{1}{p}}} &= 1 \\
\int_0^t \frac{u'(x)}{(1 - u^p(x))^{\frac{1}{p}}} \, \mathrm{d}x &= t
\end{aligned}$$

Let us substitute $y = u(x), \mathrm{d}y = u'(x)\mathrm{d}x$.

$$\int_0^{u(t)} \frac{1}{(1 - y^p)^{\frac{1}{p}}} \, \mathrm{d}y = t,$$

which we have already mentioned in (2).
Let us find the corresponding initial value problem to (15)-(16). We will apply following substitution:

$$\begin{aligned}
v &= u, \\
w &= |u'|^{p-2}u' = |v'|^{p-2}v' = |v'|^{p-1}\operatorname{sgn}(v').
\end{aligned}$$

Obviously $\operatorname{sgn}(w) = \operatorname{sgn}(v')$, then:

$$|w| = |v'|^{p-1} \Rightarrow |w|^{\frac{1}{p-1}} = |v'| \Rightarrow v' = |w|^{\frac{1}{p-1}}\operatorname{sgn}(w).$$

By substitution $w = |u'|^{p-2}u'$ in (15) we obtain:

$$w' + \lambda|v|^{p-2}v = 0 \Rightarrow w' = -\lambda|v|^{p-1}\operatorname{sgn}(v).$$

From boundary conditions (16) we obtain:

$$\begin{aligned}
v(0) = 0, \\
w(0) = \alpha,
\end{aligned}$$

where α must be chosen according to condititon $v(\pi_p) = 0$. This requirement meets $\alpha = 1$. Previously we have asuumed $\lambda = p - 1$. The final corresponding initial value problem follows:

$$\begin{aligned}
v' &= |w|^{\frac{1}{p-1}}\operatorname{sgn}(w), \\
w' &= (1-p)|v|^{p-1}\operatorname{sgn}(v), \\
v(0) &= 0, \\
w(0) &= 1.
\end{aligned} \tag{17}$$

Using Matlab function *ode45* we solved Equation (17).

1.3 Generalized trigonometric functions with two parameters

In this thesis we will also consider sine and cosine functions with two parameters, p and q. A version of the boundary value problem (15)-(16) for two parameters follows:

$$
\begin{aligned}
(|u'|^{p-2}u')' + \lambda|u|^{q-2}u &= 0, \qquad (18)\\
u(0) = u(\pi_{p,q}) &= 0, \qquad (19)
\end{aligned}
$$

where $p > 1, q > 1, \lambda > 0$ are given real numbers.
Using steps according to Section 1.2 we obtain following corresponding initial value problem:

$$
\begin{aligned}
v' &= |w|^{\frac{1}{p-1}} \operatorname{sgn}(w), \qquad (20)\\
w' &= \lambda|v|^{q-1} \operatorname{sgn}(v),\\
v(0) &= 0,\\
w(0) &= 1.
\end{aligned}
$$

Takeuchi [12] showed, that $\sin_{p,q} x$ is the solution of (18) for $\lambda = \frac{q(p-1)}{p}$.
Definition of $\pi_{p,q}$ follows:

$$
\pi_{p,q} \stackrel{\text{def}}{=} 2\int_0^1 \frac{1}{\sqrt[p]{1-t^q}}\mathrm{d}t. \qquad (21)
$$

1.4 Structure of the thesis

In Chapter 2, we present graphs of $\sin_p x$, whose origin is based on (15)-(16), and $\cos_p x$. We will also introduce $\pi_{p,q}$, whose value we found by experiment using (20). In Chapter 3, we study integrals of generalized trigonometric functions and assemble them into the form of classical "Table of Integrals" by e.g. Gradshteyn and Ryzhik [7] and/or Prudnikov, Brychkov, Marichev [10], [11]. In Chapter 4, we make a remark about the numerical part of the problem and in Chapter 5, we introduce our Matlab codes, which were a significant part of the research.

2 Graphs

We will introduce figures of our Matlab codes. As we know from Matlab documentation [9], the *plot* function connects points by line segments. This fact could affect final graph.
We will focus on Matlab codes and computation of values of $\pi_{p,q}$, $\sin_p x$ and $\cos_p x$ in the Implementation section.
As we can see, the curve of $\sin_p x$ has specific shape for p approaching one and infinity [3].

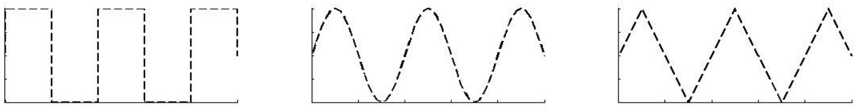

Figure 1: $\sin_p x$, $p \longrightarrow 1$ on the left, $p = 2$ in between, $p \longrightarrow \infty$ on the right

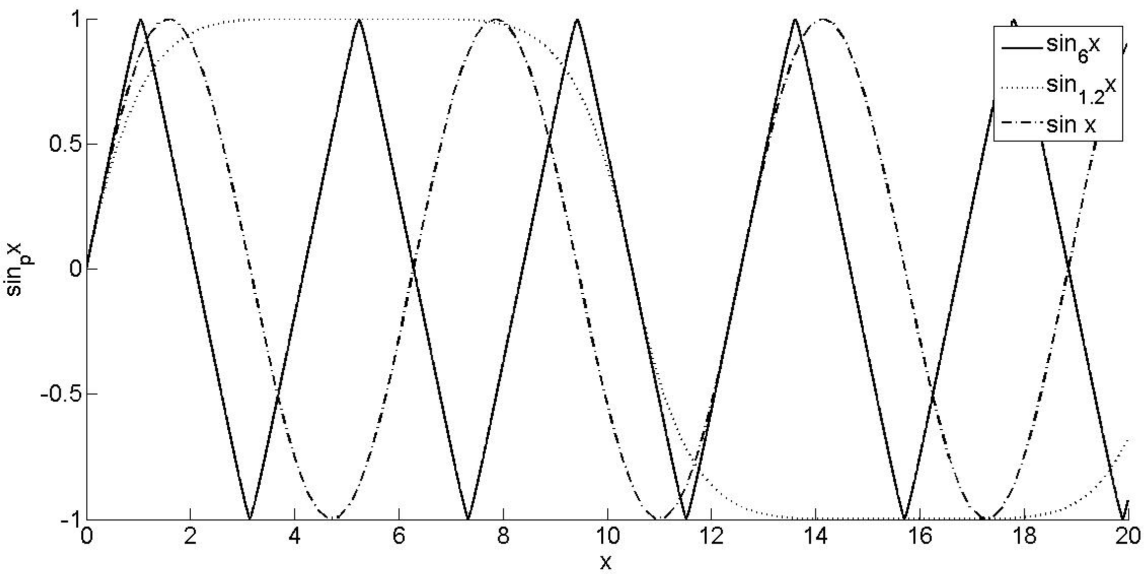

Figure 2: $\sin_p x$

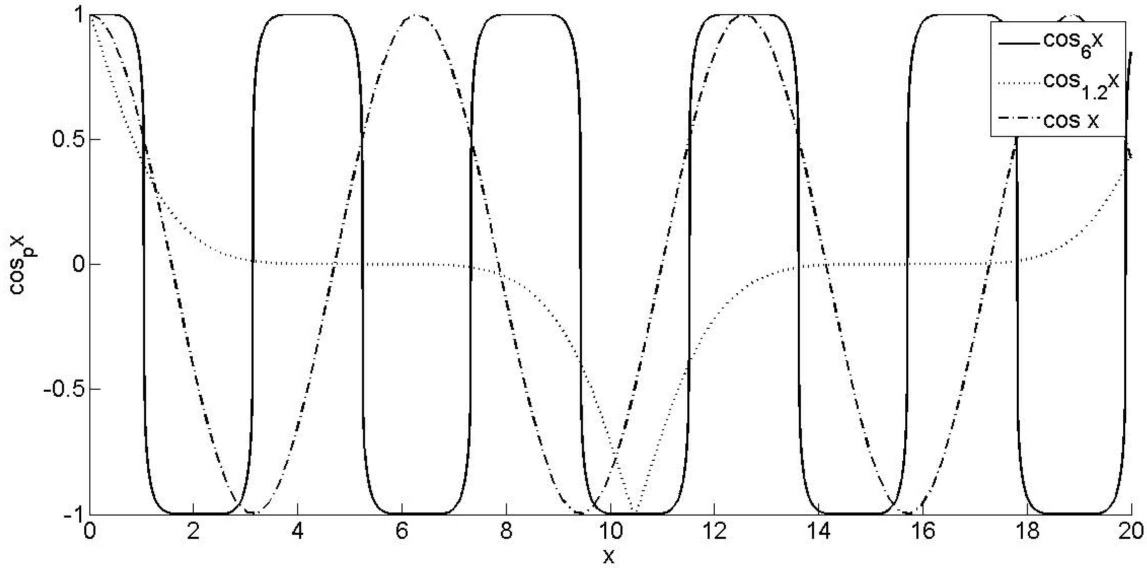

Figure 3: $\cos_p x$

It is not a surprise that for different p there is a different period of the trigonometric function. Generally, $\sin_p x$ and $\cos_p x$ have period $2\pi_p$. For the visualization of sine and cosine function above we have $\pi_6 = \frac{2\pi}{3}, \pi_{1.2} = \frac{10\pi}{3}$ and of course $\pi_2 = \pi$. In the code below you can see the computation in Matlab as well as the code for plotting π_p. We used the formula from (3).

Listing 1: Values and graph of π_p

```
syms x;
format long
pi6=2*int(sf(6),x,0,1)
%pi_p definition, p=6
pi12=2*int(sf(1.2),x,0,1)
%pi_p definition, p=1.2
pi2=2*int(sf(2),x,0,1)
%pi_p definition, p=2
Y=zeros(1,37);
%matrix will be used for saving pi_p values
i=1;
for p=2:0.5:20
    Y(1,i)=2*int(sf(p),x,0,1);
    i=i+1;
end
paxis=2:0.5:20;
%values on p-axis
axis([2 20 2 3.2]);
hold on;
plot(paxis,Y(1,:),'black--o','LineWidth',2.2),xlabel('p'),ylabel('\pi_p');
```

Listing 2: Function *sf*

```
function fun = sf(p)
%support function for computation of integral in pi_p definition
syms x;
fun=1/((1 - x^p)^(1/p));
end
```

We used codes in Listing 1 and Listing 2 to compute several values of π_p and to visualize π_p as a function of p. You can see the final graph in Figure 4 below.

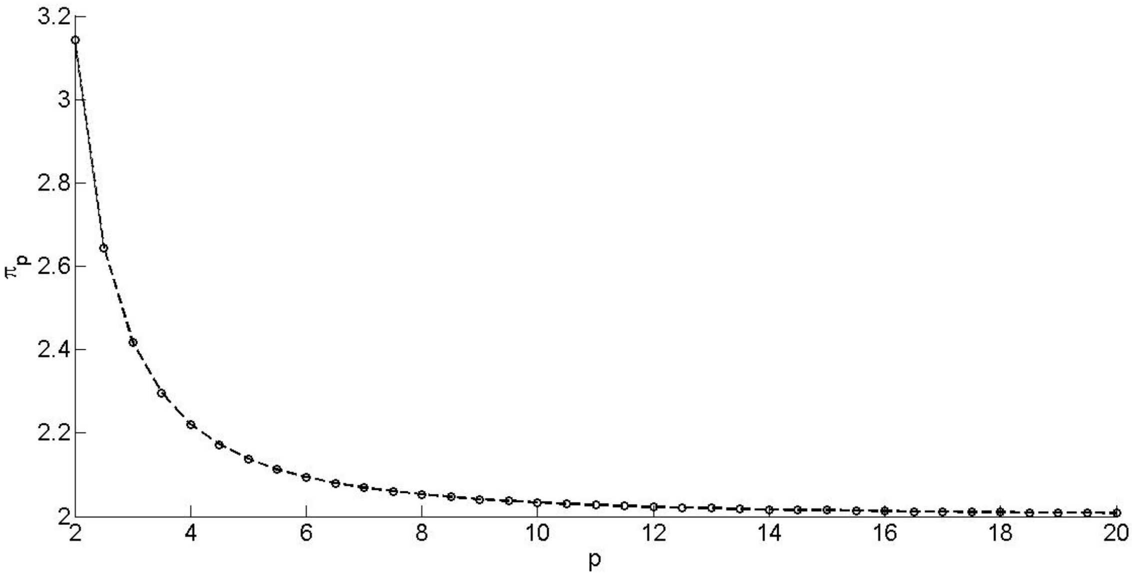

Figure 4: π_p

As we can see from the graph above, $\lim_{p\to 1}\pi_p = \infty$ and $\lim_{p\to\infty}\pi_p = 2$, see [5].
In this context, there is an interesting fact about generalized sine and cosine function. We know that $\sin(\frac{\pi}{4}) = \cos(\frac{\pi}{4})$. But for generalized sine and cosine the equation does not hold true. We can see it in Figure 5, where we chose $p = 6$, and in Figure 6, where we chose $p = 1.2$. The values of sine and cosine in $\frac{\pi_p}{4}$ are circled in the graph. We used *format long* with 15 digits after the decimal point and we obtained following values:

$$\sin_6(\frac{\pi_6}{4}) = 0.523341103807184,$$
$$\cos_6(\frac{\pi_6}{4}) = 0.996546124063566,$$
$$\sin_{1.2}(\frac{\pi_{1.2}}{4}) = 0.982849501600393,$$
$$\cos_{1.2}(\frac{\pi_{1.2}}{4}) = 0.039257643812817.$$

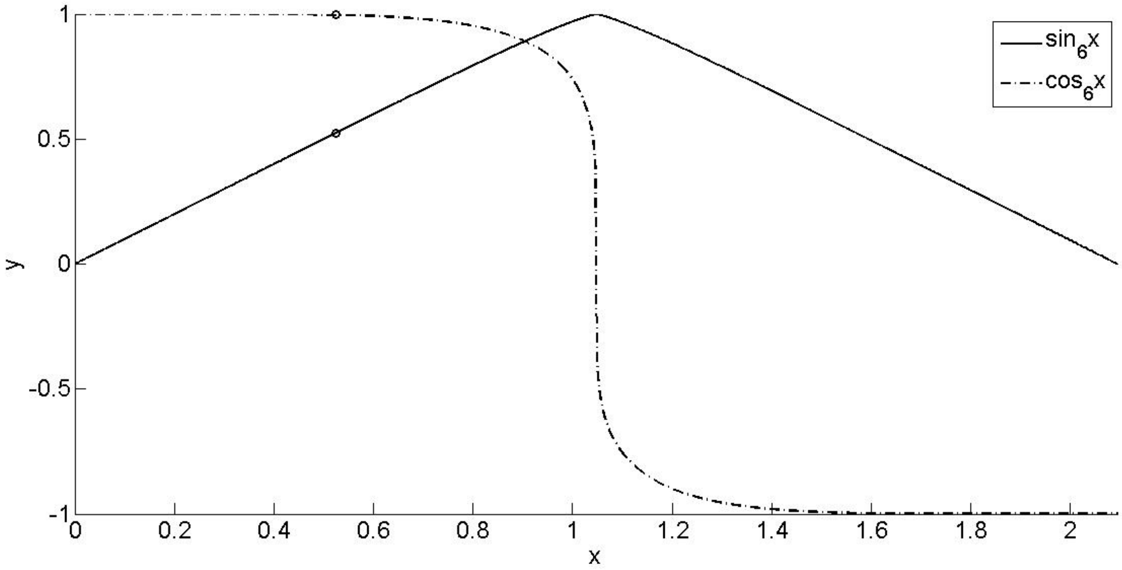

Figure 5: $\sin_6 x$ and $\cos_6 x$

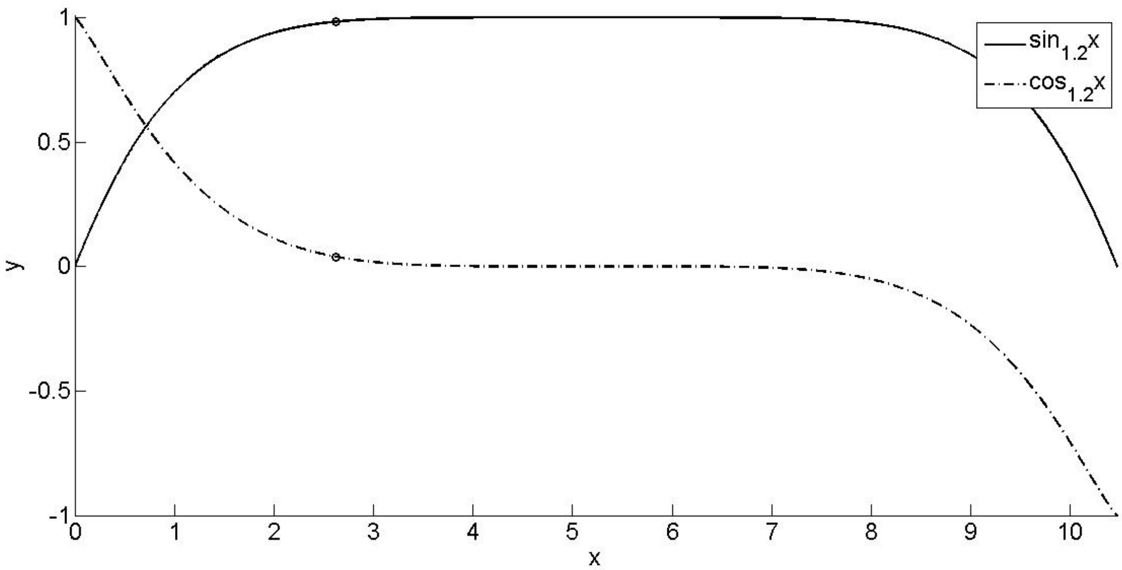

Figure 6: $\sin_{1.2} x$ and $\cos_{1.2} x$

In figures above we set the scale of the x-axis to π_p. According to this we can get an idea of the intersection point of sine and cosine functions. The exact computation could be an object of further research.

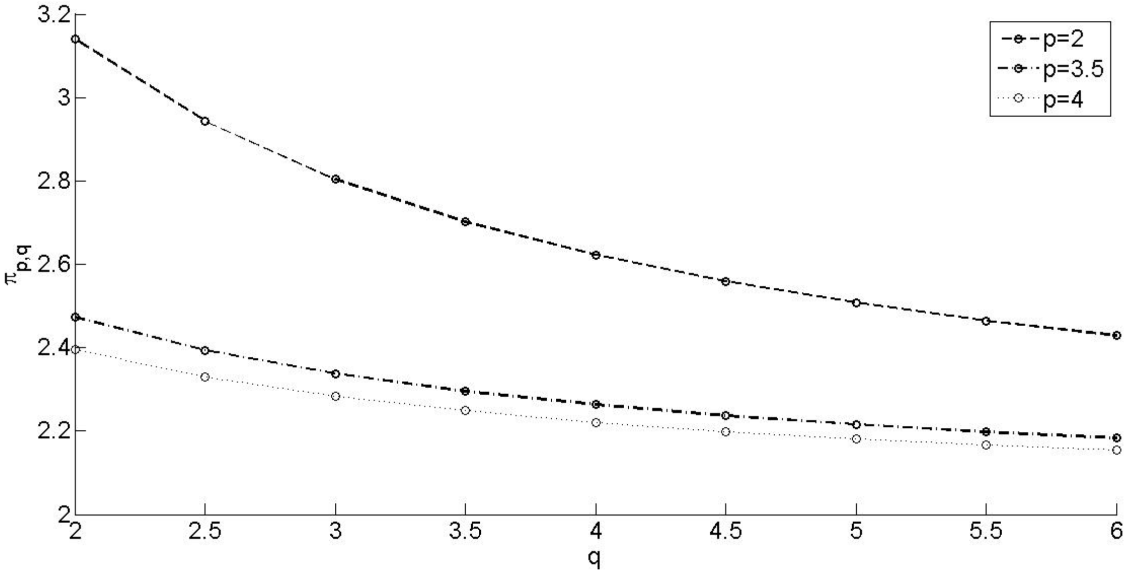

Figure 7: $\pi_{p,q}$ with fixed values of q

In Figure 7 we can see a graph of $\pi_{p,q}$ as a function of q. This means, that we fixed the value of p and plotted the values of $\pi_{p,q}$ for $q \in \{2, 2.5, 3, 3.5, 4, 4.5, 5, 5.5, 6\}$.

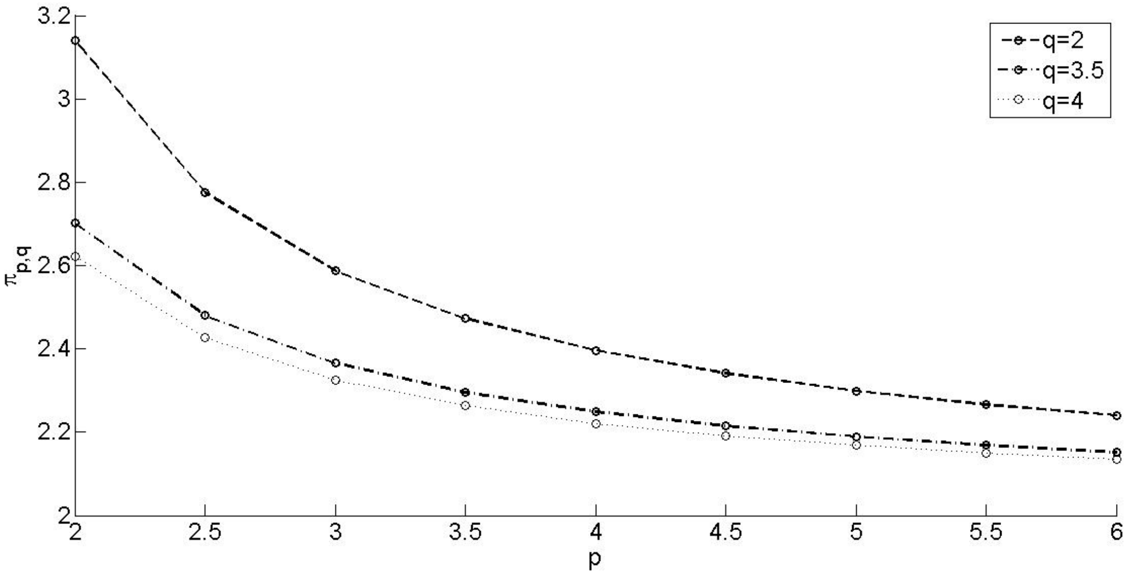

Figure 8: $\pi_{p,q}$ with fixed values of p

In figure 8 there is a graph of $\pi_{p,q}$ as a function of p. We found the values of both graphs above by experiment, which will be fully described in Chapter 5.

3 Integrals with generalized trigonometric functions

While solving integrals with trigonometric functions we usually use substitution. There are simple rules how to substitute in the table below.

Table 1: Basic substitutions

Type of integral	Substitution		
	z	$\mathrm{d}z$	$\mathrm{d}x$
$\int f(\sin x)\cos x\,\mathrm{d}x$	$\sin x$	$\cos x\,\mathrm{d}x$	$\frac{\mathrm{d}z}{\cos x}$
$\int f(\cos x)\sin x\,\mathrm{d}x$	$\cos x$	$-\sin x\,\mathrm{d}x$	$-\frac{\mathrm{d}z}{\sin x}$
$\int f(\tan x)\frac{1}{\cos^2 x}\,\mathrm{d}x$	$\tan x$	$\frac{\mathrm{d}x}{\cos^2 x}$	$\cos^2 x\,\mathrm{d}z$

We will study whether these methods are applicable for integrals with generalized trigonometric functions. Firstly, we will find solutions of several derivatives. We will use them later in the integral computations. These solutions have been already showed in [13] and [1], but we will show them step by step. We will also use the identity (13) and its variation with two parameters: $|\sin_p x|^p + |\cos_p x|^q = 1$, see [13]. We assume $x \in (0, \frac{\pi_p}{2})$.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\cos_p x &= \frac{\mathrm{d}}{\mathrm{d}x}(1-\sin_p^p x)^{\frac{1}{p}} = -\cos_p x \cdot p \cdot \sin_p^{p-1} x \cdot \frac{1}{p} \cdot (1-\sin_p^p x)^{\frac{1-p}{p}} = \\ &= -\cos_p^{2-p} x \cdot \sin_p^{p-1} x = -\frac{\sin_p^{p-1} x}{\cos_p^{p-1} x}\cos_p x = -\tan_p^{p-1}\cos_p x \end{aligned} \tag{22}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\tan_p x &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin_p x}{\cos_p x} = \frac{\cos_p^2 x - \sin_p x \cos_p' x}{\cos_p^2 x} = 1 - \frac{\sin_p x}{\cos_p x}\frac{-\tan_p^{p-1}\cos_p x}{\cos_p x} = \\ &= 1 + \tan_p^p x \end{aligned} \tag{23}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\cos_{p,q} x &= \frac{\mathrm{d}}{\mathrm{d}x}(1-\sin_{p,q}^p x)^{\frac{1}{q}} = -\frac{1}{q}(1-\sin_{p,q}^p x)^{\frac{1-q}{q}} \cdot p \cdot \sin_{p,q}^{p-1} x \cdot \cos_{p,q} x = \\ &= -\frac{p}{q}\cos_{p,q}^{2-q} x \sin_{p,q}^{p-1} x \end{aligned} \tag{24}$$

From the results above we assume that the methods of computing integrals with generalized trigonometric functions will be a bit different from those in the chart.

3.1 Basic substitutions

Let $p > 1, n > 0$. An equal sign with a star means computation by Wolfram Mathematica.

3.1.1 $\displaystyle\int \cos_p x \,\mathrm{d}x = \sin_p x$

[Holds for $x \in (0, \frac{\pi_p}{2})$ by definition, then for $x \in (-\frac{\pi_p}{2}, \frac{\pi_p}{2})$ by cosine property $\cos_p x = \cos_p(-x)$ (cosine is even) and for $\mathbb{R}$ by $2\pi_p$-periodicity.]

3.1.2 $\displaystyle\int f(\sin_p x)\cos_p x \,\mathrm{d}x \left| \begin{array}{l} z = \sin_p x, \\ \mathrm{d}z = \cos_p x \,\mathrm{d}x \end{array} \right| = \int f(z)\,\mathrm{d}z = F(\sin_p x)$

[Where $x \in (-\frac{\pi_p}{2}, \frac{\pi_p}{2}), f : (0,1) \to \mathbb{R}, F'(x) = f(x)$.]

3.1.2a $\displaystyle\int_0^{\frac{\pi_p}{2}} f(\sin_p x)\cos_p x \,\mathrm{d}x \left| \begin{array}{l} z = \sin_p x, \\ \mathrm{d}z = \cos_p x \,\mathrm{d}x \end{array} \right| = \int_0^{\sin_p \frac{\pi_p}{2}} f(z)\,\mathrm{d}z = \int_0^1 f(z)\,\mathrm{d}z =$
$= F(1) - F(0)$

[Where $x \in (-\frac{\pi_p}{2}, \frac{\pi_p}{2}), f : (0,1) \to \mathbb{R}, F'(x) = f(x)$.]

3.1.2b $\displaystyle\int_0^{\frac{\pi_p}{2}} \sin_p x \cos_p x \,\mathrm{d}x \left| \begin{array}{l} z = \sin_p x, \\ \mathrm{d}z = \cos_p x \,\mathrm{d}x \end{array} \right| = \int_0^1 z\,\mathrm{d}z = \left[\frac{z^2}{2}\right]_0^1 = \frac{1}{2}$

3.1.3 $\displaystyle\int \sin_p^n x \,\mathrm{d}x \left| \begin{array}{l} z = \sin_p x, \\ \mathrm{d}z = \cos_p x \,\mathrm{d}x, \\ \cos_p x = (1 - \sin_p^p x)^{\frac{1}{p}} \end{array} \right| = \int \frac{z^n}{(1-z^p)^{\frac{1}{p}}}\,\mathrm{d}z =^*$

$\displaystyle =^* \frac{z^{n+1}\, {}_2F_1\left(\frac{1}{p}, \frac{n+1}{p}; \frac{n+1}{p} + 1; z^p\right)}{n+1} = \frac{\sin_p^{n+1} x}{n+1}\, {}_2F_1\left(\frac{1}{p}, \frac{n+1}{p}; \frac{n+1}{p} + 1; \sin_p^p x\right)$

[Where $x \in (-\frac{\pi_p}{2}, \frac{\pi_p}{2})$.]

3.1.3a $\displaystyle\int_0^{\frac{\pi_p}{2}} \sin_p^n x \,\mathrm{d}x \left| \begin{array}{l} z = \sin_p x, \\ \mathrm{d}z = \cos_p x \,\mathrm{d}x, \\ \cos_p x = (1 - \sin_p^p x)^{\frac{1}{p}} \end{array} \right| = \int_0^1 \frac{z^n}{(1-z^p)^{\frac{1}{p}}}\,\mathrm{d}z =^* \frac{\Gamma\left(\frac{p-1}{p}\right)\Gamma\left(\frac{n+1}{p}\right)}{n\Gamma\left(\frac{n}{p}\right)}$

3.2 Further computations

3.2.1 $\displaystyle\int \tan_p^{p-1} x \,\mathrm{d}x = \int \frac{\sin_p^{p-1} x}{\cos_p^{p-1} x}\,\mathrm{d}x \left| \begin{array}{l} z = \cos_p x, \\ \mathrm{d}z = -\cos_p^{2-p} x \sin_p^{p-1} x \,\mathrm{d}x, \\ \mathrm{d}z = -z^{2-p} \sin_p^{p-1} x \,\mathrm{d}x \end{array} \right| =$

$\displaystyle = \int \frac{\sin_p^{p-1} x}{z^{p-1}(-z^{2-p})\sin_p^{p-1} x}\,\mathrm{d}z = -\int \frac{1}{z}\,\mathrm{d}z = -\ln z = -\ln \cos_p x$

[Where $x \in (0, \frac{\pi_p}{2})$. See (22) and [2].]

3.2.2 $\displaystyle\int \tan_p^p x \,\mathrm{d}x = \tan_p x - x$

[Where $x \in (0, \frac{\pi_p}{2})$. See (23) and [2].]

3.2.3 $\displaystyle\int \cos_p^2 x \sin_p^{p-1} x \,\mathrm{d}x \left| \begin{array}{l} z = \cos_p x, \\ \mathrm{d}z = -\cos_p^{2-p} x \sin_p^{p-1} x \,\mathrm{d}x, \\ \mathrm{d}z = -z^{2-p} \sin_p^{p-1} x \,\mathrm{d}x \end{array} \right| = -\int z^2 \frac{\sin_p^{p-1} x}{z^{2-p} \sin_p^{p-1} x} \,\mathrm{d}z =$

$\displaystyle = -\int z^p \,\mathrm{d}z = -\frac{z^{p+1}}{p+1} = -\frac{\cos_p^{p+1} x}{p+1}$

[Where $x \in (0, \frac{\pi_p}{2})$. See (22).]

3.3 Functions with two parameters

3.3.1 $\displaystyle\int \tan_{p,q}^{p-1} x \,\mathrm{d}x = \int \frac{\sin_{p,q}^{p-1} x}{\cos_{p,q}^{p-1} x} \,\mathrm{d}x \left| \begin{array}{l} z = \cos_{p,q} x, \\ \mathrm{d}z = -\frac{p}{q} \cos_{p,q}^{2-q} x \sin_{p,q}^{p-1} x \,\mathrm{d}x, \\ \mathrm{d}z = -\frac{p}{q} z^{2-q} \sin_{p,q}^{p-1} x \,\mathrm{d}x \end{array} \right| =$

$\displaystyle = \int \frac{\sin_{p,q}^{p-1} x}{-z^{p-1} \frac{p}{q} z^{2-q} \sin_p^{p-1} x} \,\mathrm{d}z = -\frac{q}{p} \int \frac{1}{z^{p-q+1}} \,\mathrm{d}z = -\frac{q}{p} \int z^{q-p-1} \,\mathrm{d}z = -\frac{q z^{q-p}}{p(q-p)} =$

$\displaystyle = \frac{q}{p(p-q)} \cos_{p,q}^{q-p} x$

[Where $x \in (0, \frac{\pi_p}{2})$. See (24).]

3.3.2 $\displaystyle\int \cos_{p,q}^2 x \sin_{p,q}^{p-1} x \,\mathrm{d}x \left| \begin{array}{l} z = \cos_{p,q} x, \\ \mathrm{d}z = -\frac{p}{q} \cos_{p,q}^{2-q} x \sin_{p,q}^{p-1} x \,\mathrm{d}x, \\ \mathrm{d}z = -\frac{p}{q} z^{2-q} \sin_{p,q}^{p-1} x \,\mathrm{d}x \end{array} \right| = \int \frac{z^2 \sin_{p,q}^{p-1} x}{-\frac{p}{q} z^{2-q} \sin_{p,q}^{p-1} x} \,\mathrm{d}z =$

$\displaystyle = -\frac{q}{p} \int z^q \,\mathrm{d}z = -\frac{q}{p} \frac{z^{q+1}}{q+1} = -\frac{q}{p(q+1)} \cos_{p,q}^{q+1} x$

[Where $x \in (0, \frac{\pi_p}{2})$. See (24).]

4 Numerical part

For computing values of $\sin_p x$, $\cos_p x$ and $\sin_{p,q} x$ we used Matlab solver *ode45*. From Matlab documentation [9] we know that the solver *ode45* is based on Runge-Kutta (4,5) formula, the Dormand-Prince pair.

4.1 Runge-Kutta method

Runge-Kutta methods have properties similar to the Taylor expansion, but they do not require analytic derivation. General Runge-Kutta method can be described by following equations [8]:

$$\begin{aligned}
k_1 &= hf(x_n, y_n) \\
k_2 &= hf(x_n + \alpha_1 h, y_n + \beta_{11} k_1) \\
k_3 &= hf(x_n + \alpha_2 h, y_n + \beta_{21} k_1 + \beta_{22} k_2) \\
&\vdots \\
k_{j+1} &= hf(x_n + \alpha_j h, y_n + \beta_{j1} k_1 + \beta_{j2} k_2 + \ldots + \beta_{jj} k_j) \\
y_{n+1} &= y_n + \gamma_1 k_1 + \gamma_2 k_2 + \ldots + \gamma_{j+1} k_{j+1},
\end{aligned}$$

where $y' = f(x, y), y(x_0) = y_0$ is the initial value problem, h is step size.
Coefficients α_j and β_{jj} are usually given in the following form.

Table 2: Dormand-Prince coefficients scheme

$$\begin{array}{c|ccccccc}
0 & & & & & & & \\
\alpha_1 & \beta_{11} & & & & & & \\
\alpha_2 & \beta_{21} & \beta_{22} & & & & & \\
\alpha_2 & \beta_{31} & \beta_{32} & \beta_{33} & & & & \\
\vdots & & & & & & & \\
\alpha_6 & \beta_{61} & \beta_{62} & \beta_{63} & \beta_{64} & \beta_{65} & \beta_{66} & \\
\hline
 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 \\
 & \gamma_1^* & \gamma_2^* & \gamma_3^* & \gamma_4^* & \gamma_5^* & \gamma_6^* & \gamma_7^*
\end{array}$$

Coefficients γ_j give the fifth-order accurate method, coefficients γ_j^* give the fourth-order accurate method. Dormand's and Prince's [4] research brought three sets of coefficients α_j and β_{jj}. For *ode45* following coefficients are used.

Table 3: Dormand-Prince coefficients

$$
\begin{array}{c|ccccccc}
0 & & & & & & & \\
\frac{1}{5} & \frac{1}{5} & & & & & & \\
\frac{3}{10} & \frac{3}{40} & \frac{9}{40} & & & & & \\
\frac{4}{5} & \frac{44}{45} & -\frac{56}{15} & \frac{32}{9} & & & & \\
\frac{8}{9} & \frac{19372}{6561} & -\frac{25360}{2187} & \frac{64448}{6561} & -\frac{212}{729} & & & \\
1 & \frac{9017}{3168} & -\frac{355}{33} & \frac{46732}{5247} & \frac{49}{176} & -\frac{5103}{18656} & & \\
1 & \frac{35}{384} & 0 & \frac{500}{1113} & \frac{125}{192} & -\frac{2187}{6784} & \frac{11}{84} & \\
\hline
 & \frac{35}{384} & 0 & \frac{500}{1113} & \frac{125}{192} & -\frac{2187}{6784} & \frac{11}{84} & 0 \\
 & \frac{5179}{57600} & 0 & \frac{7571}{16695} & \frac{393}{640} & -\frac{92097}{339200} & \frac{187}{2100} & \frac{1}{40}
\end{array}
$$

4.2 Matlab *ode* solvers comparison

There are other solvers for numerical computing differential equations in Matlab. We will compare *ode23*, *ode45*, *ode113* and *ode23t*. We will solve differential equation

$$
\begin{aligned}
v' &= |w|^{\frac{1}{p-1}} \operatorname{sgn}(w), \\
w' &= (p-1)|v|^{p-1} \operatorname{sgn}(v), \\
v(0) &= 0, \\
w(0) &= 1
\end{aligned}
$$

and we will compare values of $\sin_p x$ for $p = 6$ and $x = \pi_6$ and values of $\sin x$ for $x = \pi$. The exact solutions are $\sin_6(\pi_6) = 0$ and $\sin(\pi) = 0$. Matlab solutions follow in the tables below. We used *format long* with 15 digits after the decimal point.

Table 4: Matlab ode solutions for $\sin_6(\pi_6)$

Solver	$\sin_6(\pi_6)$	Error
ode23	$-2.832196283053534 \cdot 10^{-14}$	$2.832196283053534 \cdot 10^{-14}$
ode23t	$-9.167072030227972 \cdot 10^{-9}$	$9.167072030227972 \cdot 10^{-9}$
ode45	$1.508248387294131 \cdot 10^{-12}$	$1.508248387294131 \cdot 10^{-12}$
ode113	$7.151747709928367 \cdot 10^{-11}$	$7.151747709928367 \cdot 10^{-11}$

Table 5: Matlab ode solutions for $\sin(\pi)$

Solver	$\sin(\pi)$	Error
ode23	$-3.965825335228851 \cdot 10^{-13}$	$3.965825335228851 \cdot 10^{-13}$
ode23t	$1.655499162146904 \cdot 10^{-7}$	$1.655499162146904 \cdot 10^{-7}$
ode45	$-2.307345287055895 \cdot 10^{-12}$	$2.307345287055895 \cdot 10^{-12}$
ode113	$-1.738804369060178 \cdot 10^{-10}$	$1.738804369060178 \cdot 10^{-10}$

Difference between exact values and computed values is quite small. The least precise solver is *ode23t*. The reason might be the fact, that *ode23t* is determined to solve stiff equations. Other used solvers, *ode23*, *ode45* and *ode113*, are set to solve non-stiff equations. We will not discuss stiffness of the equations in this thesis.
We also compared values of $\sin x$ for $x = \frac{\pi}{4}$. The exact solution is $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$.

Table 6: Matlab ode solutions for $\sin(\frac{\pi}{4})$

Solver	$\sin(\frac{\pi}{4})$	Error
ode23	0.707106781122869	$6.367839588961033 \cdot 10^{-11}$
ode23t	0.707106751914025	$2.927252262807656 \cdot 10^{-8}$
ode45	0.707106781180183	$6.364797577873560 \cdot 10^{-12}$
ode113	0.707106781176126	$1.042177455445881 \cdot 10^{-11}$

According to the tables above we can note that the error of *ode45* solutions is for each example on the 12th digit after the decimal point. In the first two examples, *ode23* is the most precise solver, but in the third example the most precise one is *ode45*.

5 Implementation

We will introduce codes which were used to visualize generalized trigonometric functions in this thesis. We used MATLAB R2014a, version 8.3 on processor Intel(R) Core(TM)i3-3110M CPU @ 2.40 GHz. By using *ode45* we solved the initial value problem (17).
Using *odeset* function we set relative tolerance

$$\mathrm{RelTol} = \frac{|X - Y|}{\min(|X|, |Y|)}$$

and absolute tolerance

$$\mathrm{AbsTol} = |X - Y|$$

to 10^{-16}. By choosing the tolerances we control the errors:

$$|e(i)| \leq \max(RelTol \cdot |y(i)|, AbsTol(i)),$$

where $e(i)$ is error at step i and $y(i)$ is current solution [9].

5.1 Values and graph of $\sin_p x$ and $\cos_p x$

The code in Listing 3 computes values of $\sin_p x$ on $\langle 0, 20\rangle$ for different values of p. Functions $p1$ (Listing 5) and $p2$ (Listing 6) stand for the initial value problem (17) with $p = 6$ and $p = 1.2$.
The code in Listing 4 is analogous to the code in Listing 3, but computes values of $\cos_p x$ on $\langle 0, 20\rangle$. By solving first order differential equations we obtain two columns. The first column $Y(:, 1)$ stands for $\sin_p x$. As we have defined in Chapter 1, $\cos_p x \stackrel{\mathrm{def}}{=} \frac{\mathrm{d}}{\mathrm{d}x} \sin_p x$. The second column is $Y(:, 2)$, we will name it y_2. Then we obtain:

$$\cos_p x = \mathrm{sgn}(y_2)|y_2|^{\frac{1}{p-1}}.$$

The final graphs we can see in Figure 2 and Figure 3.

Listing 3: Values and graph of $\sin_p x$

```
% initial value problem:
% y1'=sign(y2)|y2|^(1/(p-1))
% y2'=-lambda*sgn(y1)|y1|^(p-1)
% y1(0)=0,y2(0)=1
options=odeset('RelTol',1e-16,'AbsTol',1e-16);
t=[0,20];
init_y1=0;
init_y2=1;
figure('name', 'Sine');
axis([0 20 -1 1]);
hold on;
[T,Y]=ode45(@p1,t,[init_y1 init_y2],options);
%function p1 sets p=6
plot(T,Y(:,1),'black-','LineWidth',2.2),ylabel('sin_px'),xlabel('x');
%graph of sin_6(x)
[T,Y]=ode45(@p2,t,[init_y1 init_y2],options);
%function p2 sets p=1.2
plot(T,Y(:,1),'black:','LineWidth',2.2),ylabel('sin_px'),xlabel('x');
%graph of sin_1.2(x)
x=linspace(0,20,100);
y=sin(x);
plot(x,y,'black-.','LineWidth',2.2),ylabel('sin_px'),xlabel('x');
%graph of sin(x)
legend('sin_6x','sin_{1.2}x','sin x','Location','northeast');
hold off;
```

Listing 4: Values and graph of $\cos_p x$

```
% initial value problem:
% y1'=sign(y2)|y2|^(1/(p-1))
% y2'=-lambda*sgn(y1)|y1|^(p-1)
% y1(0)=0,y2(0)=1
options=odeset('RelTol',1e-16,'AbsTol',1e-16);
t=[0,20];
init_y1=0;
init_y2=1;
figure('name','Cosine');
axis([0 20 -1 1]);
hold on;
[T,Y]=ode45(@p1,t,[init_y1 init_y2],options);
y2=Y(:,2);
%root y2, see initial value problem
c=(sign(y2)).*(abs(y2)).^(1/(6-1));
%c is equivalent to cosine, see y1' in initial value problem
plot(T,c,'black-','LineWidth',2.2),ylabel('cos_px'),xlabel('x');
%graph of cos_6(x)
[T,Y]=ode45(@p2,t,[init_y1 init_y2],options);
y2=Y(:,2);
c=(sign(y2)).*(abs(y2)).^(1/(1.2-1));
plot(T,c,'black:','LineWidth',2.2),ylabel('cos_px'),xlabel('x');
%graph of cos_1.2(x)
y=cos(x);
plot(x,y,'black-.','LineWidth',2.2),ylabel('cos_px'),xlabel('x');
%graph of cos(x)
legend('cos_6x','cos_{1.2}x','cos x','Location','northeast');
hold off;
```

Listing 5: Function *p1*

```
function dy = p1(t,y)
%support function for ode45
p=6;
lambda=p-1;
dy=[(sign(y(2))).*(abs(y(2))).^(1/(p-1));
    -1*lambda*(sign(y(1))).*(abs(y(1))).^(p-1)];
end
```

Listing 6: Function *p2*

```
function dy = p2(t,y)
%support function for ode45
p=1.2;
lambda=p-1;
dy=[(sign(y(2))).*(abs(y(2))).^(1/(p-1));
    -1*lambda*(sign(y(1))).*(abs(y(1))).^(p-1)];
end
```

5.2 Values and graph of $\pi_{p,q}$

We wanted to find values of $\pi_{p,q}$ and plot the graph for several fixed values of p and q. We know that $\sin_{p,q} x = 0$ for $x = k\pi_{p,q}, k \in \mathbb{Z}$. We used bisection method with accuracy 10^{-3}. The boundaries *s* and *t* were set to 1 and 4. The *while* method continues till the current accuracy ($mis = \frac{|t-s|}{2}$) is bigger than 10^{-3}. The variable *tEnd* (current *x*) is set to $\frac{s+t}{2}$. The boundaries change according to value of $\sin_{p,q} x$ in current *tEnd*.
In Listing 7, $p \in \{2, 2.5, 3, 3.5, 4\}$, $q \in \{2, 2.5, 3, 3.5, 4, 4.5, 5, 5.5, 6\}$.
In Listing 8, $q \in \{2, 2.5, 3, 3.5, 4\}$, $p \in \{2, 2.5, 3, 3.5, 4, 4.5, 5, 5.5, 6\}$.
Both codes use the same functions $search$, containing the bisection method, and pq, the support function for *ode45*. We chose the figures for only $p, q \in \{2, 3.5, 4\}$. The final graphs are in Figure 7 and Figure 8. Other (easier) method of plotting graph of $\pi_{p,q}$ is to visualize it according to definition (21). However, the method we used would be suitable if the definition was unknown.

Listing 7: Values and graph of $\pi_{p,q}$ with fixed p

```
global p;
global q;
M=zeros(5,9);
k=1;
%k values are used for indexing rows of M
for j=2:0.5:4
    %for each p is made a vector of pi_p(q)
    l=1;
    %l values are used for indexing columns of M
    for i=2:0.5:6
        p=j;
        q=i;
        tEnd=search(1,4);
        M(k,l)=tEnd;
        l=l+1;
    end
    k=k+1;
end
m=2:0.5:6;
axis([2 6 2 3.2]);
hold on;
%values on x-axis
plot(m,M(1,:),'black--o','LineWidth',2.2);
%graph of pi_p,q as a function of q, p=2
hold on
plot(m,M(4,:),'black-.o','LineWidth',2.2);
%graph of pi_p,q as a function of q, p=3.5
hold on
plot(m,M(5,:),'black:o'),xlabel('q'),ylabel('\pi_{p,q}','LineWidth',2.2);
%graph of pi_p,q as a function of q, p=4
legend('p=2','p=3.5','p=4')
hold off
```

Listing 8: Values and graph of $\pi_{p,q}$ with fixed q

```
global p;
global q;
M=zeros(5,9);
k=1;
%k values are used for indexing rows of M
for j=2:0.5:4
    %for each q is made a vector of pi_q(p)
    l=1;
    %l values are used for indexing columns of M
    for i=2:0.5:6
        p=i;
        q=j;
        tEnd=search(1,4);
        M(k,l)=tEnd;
        l=l+1;
    end
    k=k+1;
end
m=2:0.5:6;
axis([2 6 2 3.2]);
hold on;
%values on x-axis
plot(m,M(1,:),'black--o','LineWidth',2.2);
%graph of pi_p,q as a function of p, q=2
hold on
plot(m,M(4,:),'black-.o','LineWidth',2.2);
%graph of pi_p,q as a function of p, q=3.5
hold on
plot(m,M(5,:),'black:o'),xlabel('p'),ylabel('\pi_{p,q}','LineWidth',2.2);
%graph of pi_p,q as a function of p, q=4
legend('q=2','q=3.5','q=4')
hold off
```

Listing 9: Function *search*

```
function [tEnd] = search(s,t)
%finds value of pi_p,q by bisection method
tEnd=0;
acc=0.001;
%accuracy
mis=10;
%mistake
options = odeset('RelTol',1e-16,'AbsTol',1e-16);
while mis>acc
    tEnd=(s+t)/2;
    [T,Y]=ode45(@pq,[0,tEnd],[0,1],options);
    mis=abs(t-s)/2;
    if Y(length(Y),1)<0
        %Y(length(Y),1) is a value of sin_p,q in tEnd
        t=tEnd;
    else s=tEnd;
    end
end
end
```

Listing 10: Function *pq*

```
function dy = pq(t,y)
%support function for ode45
global p;
global q;
lambda=q*(p-1)/p;
dy=[(sign(y(2))).*(abs(y(2))).^(1/(p-1));
    -1*lambda*(sign(y(1))).*(abs(y(1))).^(q-1)];
end
```

6 Conclusion

We explained the terms $\sin_p x$, $\cos_p x$, $\sin_{p,q} x$, π_p and $\pi_{p,q}$ and showed them from different points of view, especially in graphs. We also compared properties of generalized trigonometric functions with properties of their basic forms and tried to apply well known methods of computing integrals with trigonometric functions to computations of integrals with generalized trigonometric functions. According to the results of Chapter 3 we can say that the only rule, which can be applied on integrals with both functions, basic and generalized, is the rule for $\int f(\sin x) \cos x \, \mathrm{d}x$ (Table 1). We also showed the implementation of the problem in Matlab and described it in the numerical context.

This thesis might initiate further research of the intersection of generalized sine and cosine function for different values of p. Other follow-up might be a discussion of stiffness of the initial value problem. Finally, more extensive research of the *ode* solvers in this context might be also useful.

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