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ANALOGY BETWEEN ELECTROSTATIC FIELD AND HEAT TRANSFER – SIMPLE MODELS

Prof. Rndr Ing. Miloslav Košek, CSc.
Ing. Jan Vodolan.

Abstract: *The perfect analogy allows solving of heat transfer problems by a lot of advanced methods of electrostatics or circuit theory for simple cases. Two quantities can be computed in principle: electric potential or electric field strength, which corresponds to two driving quantities: voltage or electric charge. The potential calculation in practical tasks needs the use of finite element method. However, if we consider the charge approach, the analytical solution can be found in several technical arrangements, or the integration can be applied successfully. This approach has many advantages against the potential one. As an important practical result, the correction factor for finite dimension plate capacitor was found. Its value is in good agreement with that given in literature. Practically, we can correct the permittivity or heat conductivity measurement in standard apparatus. In the future, the effect of material voids on thermal conductivity will be studied.*

Key words: *Analogy, Electrostatic field of plates, Electric condenser, Correct capacity of condenser, Permittivity correction*

Introduction

An analogy is used in many technical areas, since it allows a simple and effective solution of many practical tasks. Full analogy between heat transfer and electro-technical science exists, but to our knowledge, it was not used in its full extent. An example of simple analogy was found in Ref. [1]; equivalent circuit with concentrated parameters consisting of resistor and capacitor models simple heat transfer problem. More precise approach should consider equivalent circuit with distributed parameters. The most general approach, complete analogy between electrostatic field and heat transfer, is given in detail in Ref. [2]. Using suitable approach, technical problem of heat transfer can be solved simply and efficiently.

In this paper we present an example of specific application of general form of analogy. Important problem in heat conductivity measurement is finite dimensions of sample. The analogous of heat conductivity measured sample is the capacitor with dielectric. The basic simple formula for capacity should be corrected and the result can be used for correction in heat transfer measurement. Simple numerical method for the correction coefficient finding is presented in the paper.

The theoretical part focuses to methods for analytical solution of electrostatic field exited by charged fibers and simple plates. The numerical part shows how the speed of integration can increase. Then some interesting results are presented and discussed.

1 Theory

Depending on considered source, the electrostatic field of capacitor can be solved by two basic methods: potential or field strength. The potential approach should be preferred, if voltage is given, which corresponds to condenser with metallic plates. In this case the analytical solution cannot be found practically and finite element method (FEM) is necessary for approximate solution.

If the charge is given, the field strength calculation appears as the best decision. The corresponding technical task is the analogous model for heat transfer. In the experiment heat source of constant heat flux density usually acts on the specimen. In this case the solution of the electrical model can be analytical in some cases, otherwise numerical integration is necessary. It results in potential or field strength.

If the potential is given, the task to find field strength requires the numerical derivation, followed by big numerical errors. On the other hand, the potential can be simply and accurately calculated from field strength by numerical integration that smoothes errors. Therefore we use the field strength approach, although the formulae are more complicated.

The superposition principle is used for field strength calculation. Let us suppose that charge element $dQ=\sigma dS$

is at surface dS with position vector $\mathbf{r}_o$. Symbol σ denotes the surface charge density. The contribution of this elementary charge to electric field strength $\mathbf{E}$ at point of position vector $\mathbf{r}$ is given by formula

$$d\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sigma\frac{\mathbf{r}-\mathbf{r_0}}{|\mathbf{r}-\mathbf{r_0}|^3}dS \qquad (1)$$

where ε_o is the permittivity of vacuum.

In calculations the formula (1) has two types of coordinates. If field vector $\mathbf{r}$ has coordinates $(x, y, z) = \mathbf{r}$ and the material vector $\mathbf{r}_o$ is written as $\mathbf{r}_o = (x_o, y_o, z_o)$, the vector difference has the form

$$\mathbf{r}-\mathbf{r_0} = (x-x_0; y-y_0; z-z_0) \qquad (2)$$

and the distance between field and material points is

$$|\mathbf{r}-\mathbf{r_0}| = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} \qquad (3)$$

General formulae above will be used in three practical cases:

1. Uniformly charged fibre.
2. Rectangular plate of uniform surface charge density.
3. Uniformly charged circular plate.

The total field strength can be obtained by integration. In simple cases, analytical formula can be obtained.

1.1 Uniformly charged fibre.

If a thin straight fibre is charged by constant linear density η, the analytical integration is possible. The fibre is positioned at X axis, begins at point $-a$ and ends at point a, Fig. 1. The field has cylindrical symmetry; therefore it can be calculated in plane XZ. Field strength components E_x and E_z are given by analytical formulae

$$E_x = \frac{\eta}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+z^2}} - \frac{1}{\sqrt{(x+a)^2+z^2}}\right] \qquad (4)$$

$$E_z = \frac{\eta}{4\pi\varepsilon_0}\frac{1}{z}\left[\frac{x+a}{\sqrt{(x+a)^2+z^2}} - \frac{x-a}{\sqrt{(x-a)^2+z^2}}\right] \qquad (5)$$

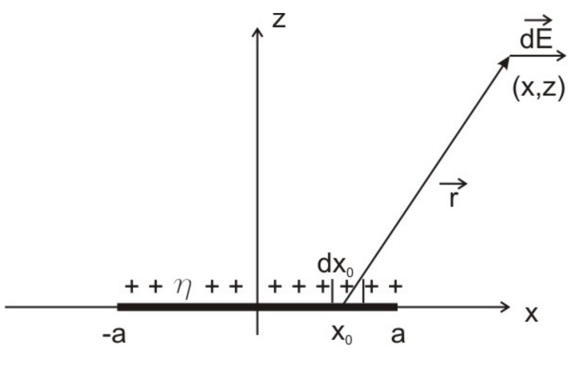

Fig. 1: Problem of charged fibre.

1.2 Rectangular plate

Let the plate parallel with plane XY is positioned at distance z_o from origin. Its centre is in the Z axis of coordinate system. The length in the direction of X axis is $2a$ and for the direction of Y axis it is $2b$, Fig. 2.

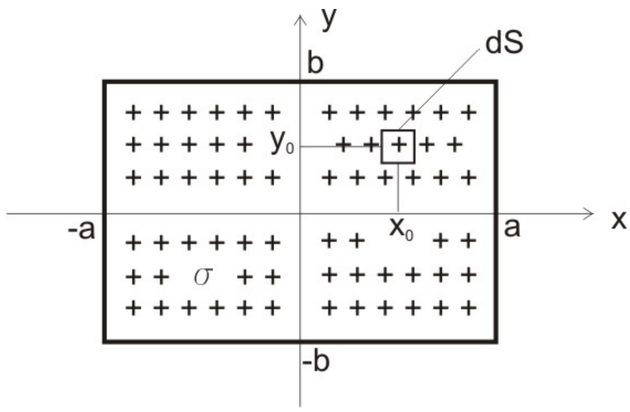

Fig. 2: Coordinates for charged plate.

If the charge density σ is constant, the formula for x-component of electric fields strength has the from

$$E_x = \frac{\sigma}{4\pi\varepsilon_0}\int_{-b}^{b}\left[\int_{a}^{a}\frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}dx_0\right]dy_0 \qquad (6)$$

The integration must be performed two times. For simplicity we rewrite the formula (6) into simpler form[1]

$$E_x = \frac{\sigma}{4\pi\varepsilon_0}\int_{-b}^{b}dE_x \qquad (7)$$

where the contribution from strips along X axis is given by formula

$$dE_x = \left[\int_{-a}^{a}\frac{x-x_0}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}}dx_0\right]dy_0 \qquad (8)$$

The analytical integration of (8) is possible and gives the result

$$dE_x = \frac{1}{\sqrt{(y-y_0)^2+\beta_-^2}}dy_0 - \frac{1}{\sqrt{(y-y_0)^2+\beta_+^2}}dy_0 \qquad (9)$$

where the following symbols were used for simplicity

$$\beta_+^2 = (x+a)^2+(z-z_0)^2$$
$$\beta_-^2 = (x-a)^2+(z-z_0)^2 \qquad (10)$$

[1] The contribution dE_x has not dimension of electric field strength. However, for simplicity and clearance, new symbol was not used. The same is true later for formula (16).

These symbols do not contain integrating variable y_0. Therefore, they are constant for integration according y_0.

After substitution from (9) and (10) into general formula (7), we get the general result

$$E_x = \frac{\sigma}{4\pi\varepsilon_0}\left[I_- - I_+\right] \tag{11}$$

where

$$\begin{aligned} I_+ &= \int_{-b}^{b} \frac{1}{\sqrt{(y-y_0)^2+\beta_+^{\;2}}} dy_0 \\ I_- &= \int_{-b}^{b} \frac{1}{\sqrt{(y-y_0)^2+\beta_-^{\;2}}} dy_0 \end{aligned} \tag{12}$$

Symbols I_+ and I_- were used for simplicity

The above integrals can be calculated analytically using the general formula

$$I = \int \frac{1}{\sqrt{u^2+\beta^2}} du = \operatorname{arcsinh}\frac{u}{\beta} \tag{13}$$

We have obtained an analytical formula for x component of electric field strength. The approach for y component is analogical, if we change the integration order in formula (6), i.e.

$$E_y = \frac{\sigma}{4\pi\varepsilon_0}\int_{-a}^{a}\left[\int_{-b}^{b}\frac{y-y_0}{\left[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right]^{3/2}}dy_0\right]dx_0 \tag{14}$$

The calculation of z component is more complicated. The basic formula is

$$E_z = \frac{\sigma}{4\pi\varepsilon_0}\int_{-b}^{b}\left[\int_{-a}^{a}\frac{z-z_0}{\left[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right]^{3/2}}dx_0\right]dy_0 \tag{15}$$

We will follow the derivation described for x component earlier. Again for simplicity we rewrite this formula into simpler form

$$E_z = \frac{\sigma}{4\pi\varepsilon_0}\int_{-b}^{b} dE_z \tag{16}$$

where the contribution from strips along X axis is given by formula

$$dE_z = \left[\int_{-a}^{a}\frac{z-z_0}{\left[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right]^{3/2}}dx_0\right]dy_0 \tag{17}$$

The analytical integration of (17) is possible and gives the result

$$\begin{aligned} dE_z &= \alpha(x+a)\frac{1}{(y-y_0)^2+\alpha^2}\frac{1}{\sqrt{(y-y_0)^2+\beta_+^{\;2}}}dy_0 \\ &- \alpha(x-a)\frac{1}{(y-y_0)^2+\alpha^2}\frac{1}{\sqrt{(y-y_0)^2+\beta_+^{\;2}}}dy_0 \end{aligned} \tag{18}$$

where the following symbols were used for simplicity

$$\begin{aligned} \alpha &= z - z_0 \\ \beta_+^{\;2} &= (x+a)^2 + (z-z_0)^2 \\ \beta_-^{\;2} &= (x-a)^2 + (z-z_0)^2 \end{aligned} \tag{19}$$

The definition of β are the same as for x component, see (10). These symbols do not contain integrating variable y_0. Therefore, they are constant for integration according y_0.

After substitution from (18) and (19) into general formula (16), we get the general result

$$E_z = \frac{\sigma}{4\pi\varepsilon_0}\alpha\left[(x+a)I_+ - (x-a)I_-\right] \tag{20}$$

where

$$\begin{aligned} I_+ &= \int_{-b}^{b}\frac{1}{(y-y_0)^2+\alpha^2}\frac{1}{\sqrt{(y-y_0)+\beta_+^{\;2}}}dy_0 \\ I_- &= \int_{-b}^{b}\frac{1}{(y-y_0)^2+\alpha^2}\frac{1}{\sqrt{(y-y_0)+\beta_-^{\;2}}}dy_0 \end{aligned} \tag{21}$$

The above integrals can be calculated analytically using the general formula

$$I = \int\frac{1}{u^2+\alpha^2}\frac{1}{\sqrt{u^2+\beta^2}}du = \frac{arctg\dfrac{\sqrt{-\alpha^2+\beta^2}\,u}{\alpha\sqrt{\beta^2+u^2}}}{\alpha\sqrt{-\alpha^2+\beta^2}} \tag{22}$$

The result (22) was found by using system *Mathematica* for symbolic computation. Also, analytical solution was found for the z component of electric field strength, although it is much more complicated than the solution for two remaining components.

Exact analytical solution for electric field strength exists in the case of rectangular plate. Superposition of fields of two parallel plates leads to analytical solution for plate condenser.

1.3 Circular plate

In principle, for circular plate the analogical approach can be used as for rectangular one; from thin rings to the plate. However, any general analytical solution was not found, neither for rings. Therefore, here we only outline the basic formulae for numerical solution.

The circular plate is positioned at plane XY for simplicity. Since the charge distribution and field symmetry is cylindrical, cylindrical coordinates should be used in calculation. For material parameters the coordinates are radius ρ, azimuthally angle φ and z coordinate z_o. The relation between material Cartesian and cylindrical coordinates are given by formulae

$$\begin{aligned} x_0 &= \rho\cos\varphi \\ y_0 &= \rho\sin\varphi \\ dS &= \rho d\rho d\varphi \end{aligned} \tag{23}$$

The surface element in cylindrical coordinates is given in (23) for completeness.

The formulae for components of field strength suitable for numerical calculation are the following ones

$$\begin{aligned} E_x &= \frac{\sigma}{4\pi\varepsilon_0}\int_0^R\int_0^{2\pi}\frac{(x-\rho\cos(\varphi)\rho d\rho d\varphi}{([x-\rho\cos(\varphi)]^2+[y-\rho\sin(\varphi)]^2+z^2)^{3/2}} \\ E_y &= \frac{\sigma}{4\pi\varepsilon_0}\int_0^R\int_0^{2\pi}\frac{(y-\rho\cos(\varphi)\rho d\rho d\varphi}{([x-\rho\cos(\varphi)]^2+[y-\rho\sin(\varphi)]^2+z^2)^{3/2}} \\ E_z &= \frac{\sigma}{4\pi\varepsilon_0}\int_0^R\int_0^{2\pi}\frac{z\rho d\rho d\varphi}{([x-\rho\cos(\varphi)]^2+[y-\rho\sin(\varphi)]^2+z^2)^{3/2}} \end{aligned} \tag{24}$$

The only analytical solution of the set (24) is at the Z axis. In this case the z component is the nonzero one only and the following formula is valid for z component

$$E_z = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right] \tag{25}$$

The formula is given in each textbook, for instance [3]. For zero z coordinate the field of infinite plane is obtained, while for very long distances the field of point charge equal to the charge of the plate follows from formula (25).

2 Calculation

As for rectangular plate condenser the analytical solution has been derived for all cases. The final formulae are so complicated to be analysed in general. Numerical calculation is necessary in order to get necessary information. Since the programming is very simple, we do not focus to this case here. We only note that both the numerical integration and analytical formulae were used in order to find possible algorithm errors. Both the results were almost identical and optimisation of integration was found from comparison.

The numerical solution of electric field in circular plate condenser is based on the integration of field strength element given by formula (24). The cylindrical coordinate system is used. The only exact check is the field strength at condenser axis given by analytical formula (25).

In the calculation finite differences are used and defined by number of divisions N_r of radius and N_φ of full azimuth angle of 360°. Number of divisions of radius was fraction or multiple α of N_{ro} = 500, while the base for azimuth angle was usually the same fraction or multiple of $N_{\varphi o}$ = 360. The basic angle difference was therefore $\Delta\varphi$ = 1°.

Main problem of numerical integration is accuracy and speed. The accuracy depends on net finesse, the more net elements, the higher accuracy. On the other hand, the more elements, the lower speed. Relation between accuracy and speed should be found by the method of trials. We have investigated the relation for points on the circular desk axis, where exact values are known.

The dependence of result of numerical integration on the number of net elements is given in Fig. 3. The number of elements in logarithmic scale is on the X axis of the graph in Fig. 3. The point on Z axis was in distance of 0.001R from plate, where R is radius of the plate. It is evident that the numerical value of summing converges to the theoretical one monotonically. The detail of the convergence at high values of elements is in Fig. 4. In this graph the integration time is on the X axis. The relative difference of 0.1 % requires about 200 s of calculation time of relatively slow computer with AMD processor, 1.33 GHz.

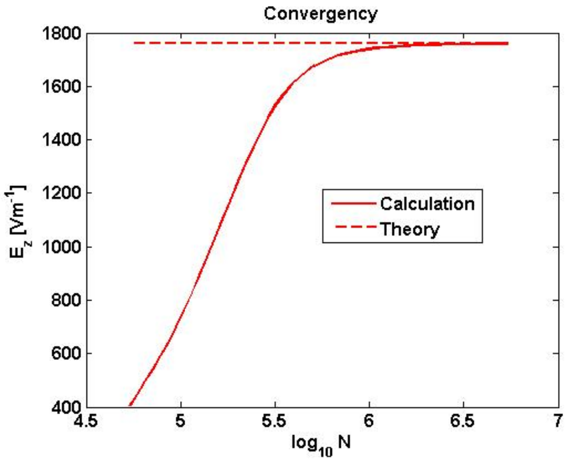

Fig. 3: Convergence of numerical integration.

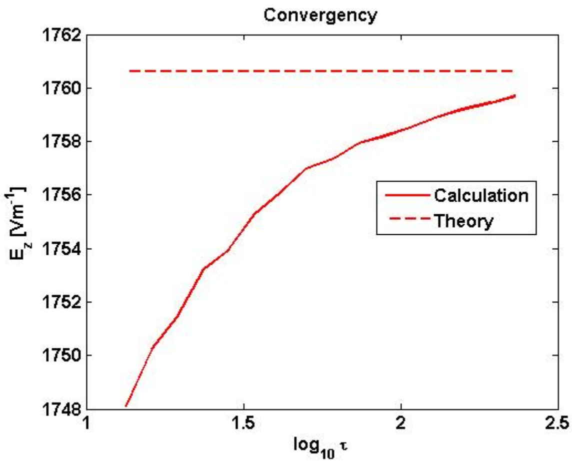

Fig.4: Detail of convergence.

The convergence for three different distances on Z axis (10, 1 and 0.1 mm) is in Fig. 5. The theoretical value is given by dashed line. The curves for given distance are distinguished by colour (green, blue and red respectively). It is evident from the Fig. 5 that the convergence depends on the distance from the plate, the closer the point, the worse convergence.

Another point of view on convergence is in Fig. 6, where the result of integration for points on Z axis near the plate is compared with theory. The approximate theory is the approximation of formula (25) for distance z small in comparison with radius R. For $z \ll R$ the following formula is valid

$$E_z = \frac{\sigma}{2\varepsilon_0} \frac{z}{R} \tag{26}$$

As it follows from Fig. 6, the difference between exact and approximate theory exists, but both of them converge to field strength for infinite plate as distance z converges to zero. However, the result of numerical integration is lower the theoretical value and the difference between theory and numerical calculation increases as the distance from plate decreases, the relative difference is shown in Fig. 7. The calculation in Fig. 6 and 7 used $N_r = 1500 = 3\ N_{ro}$ ($\alpha = 3$) elements for radius and $N_\varphi = 1080 = 3\ N_{\varphi o}$ elements for azimuth angle. If the number of elements decreases to half, i.e. $\alpha = 1.5$, the maximum of difference increases to about 4 %, as it is shown in Fig. 8.

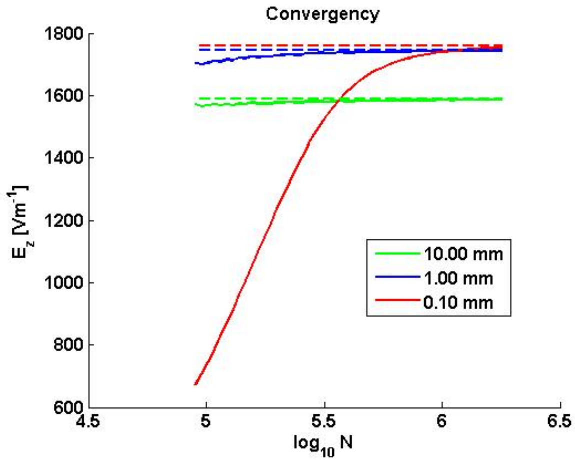

Fig.5: Convergence for different distances from plate.

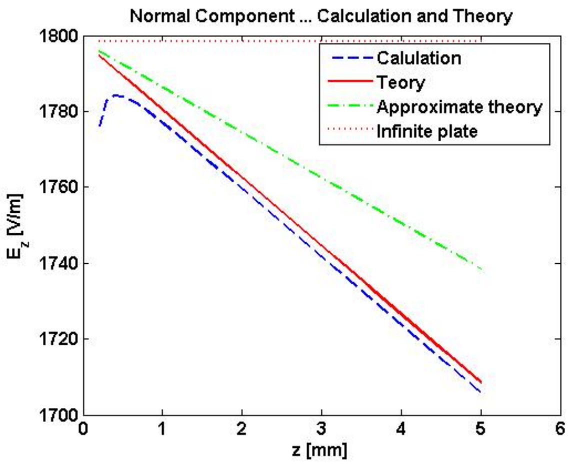

Fig.6: Comparison of numeric integration and theory. .

It is evident from Fig. 3 to 8 that in order to get a good accuracy also near the plate, many elements must be used and the computer time will be very large. Therefore, another integration method is necessary

Probably optimum solution of this problem is to divide plate area into two parts, as it is sketched in Fig. 9:

1. The main contribution for a given point is from circle centred in its projection to plate, i.e. empty circle[2] in Fig. 9. For this case exact analytical formula exists.
2. The small correction from the rest of plate is obtained by numerical integration. The suitable integration grid is sketched in Fig. 9.

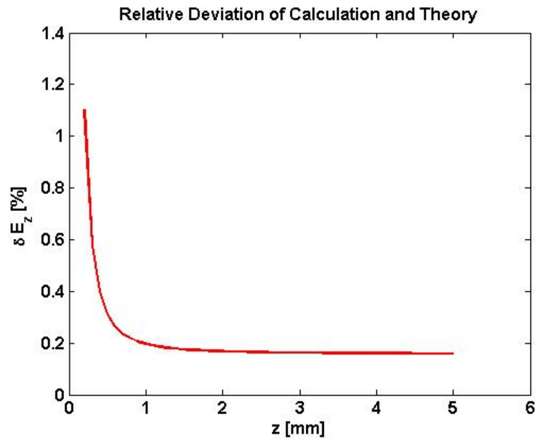

Fig.7: Relative difference between numeric integration and theory, finer step.

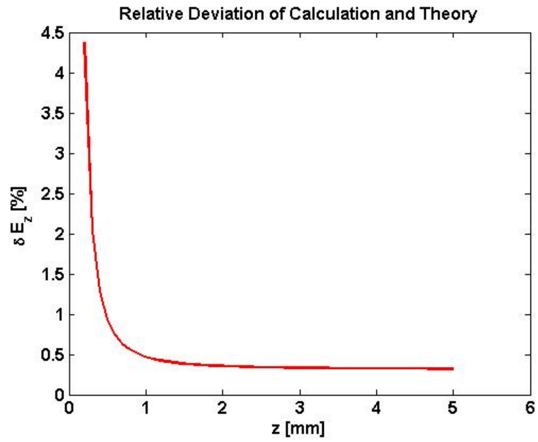

Fig 8: Relative difference between numeric integration and theory, more coarse step.

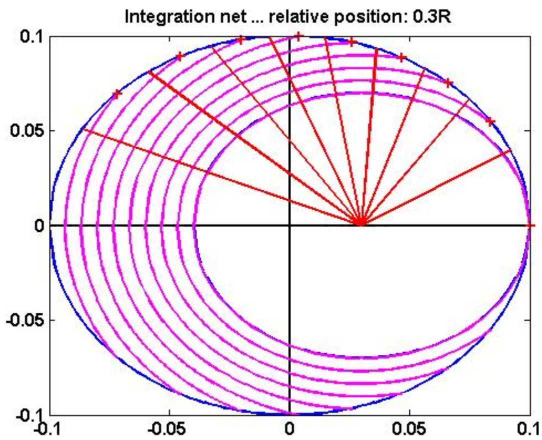

Fig. 9: Optimum method for integration

[2] Due to the incorrect aspect ratio, the circles have form of ellipses.

The comparison of standard and optimized method is in Fig. 10. The field strength is calculated on a vertical line parallel with Z axis at distance of 0.85R from it, where R is radius. For long vertical distances z from plate there is practically no difference between standard and optimized method, see also Fig. 11, where the relative difference between two methods is shown. However for short distances the optimized method is in good agreement with theory (Fig. 10), while the error of standard method is big (Fig. 11).

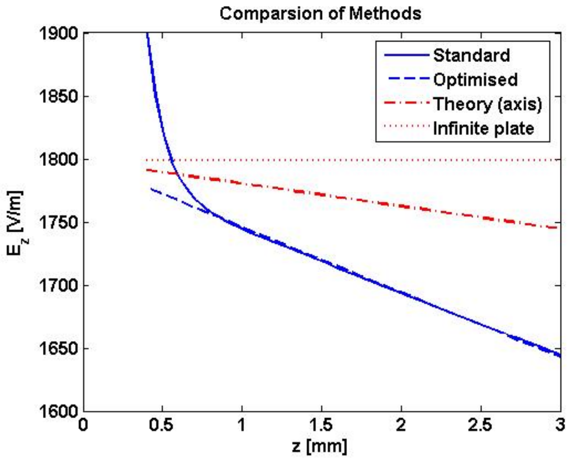

Fig. 10: Comparison of standard and optimised methods

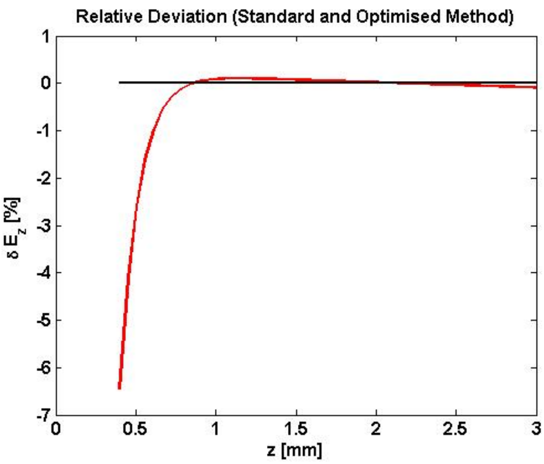

Fig. 11: Relative difference between standard and optimised methods

The fast increase of field strength from standard method in Fig. 10 is in contradiction to its decrease in Fig. 6. The discrepancy is due to the position of points, where the field is calculated. The results in Fig. 6 are for Z axis, while the results in Fig 10 are for vertical line at distance of 0.85R (R is the radius) from Z axis, i. e. near the plate edge. The theoretical curves in Fig 10 are not for verification of calculations. They only show that optimised method is correct for small distances from the plate.

The optimised method also exhibits an improvement for tangential component of electric fields strength as it is illustrated in Fig. 12. However, the relative difference between results of both methods is less than 1 %.

3 Results

A lot of graphs mapping electric field strength in various directions for charged plates and condensers are the graphical and numerical outputs of the programs written in MATLAB. We will focus here to the field and capacity of circular plate capacitor. Its radius R has value R = 100 mm and distance between plates is d = 2 mm, therefore the plate distance is 1 % of diameter $D = 2R$.

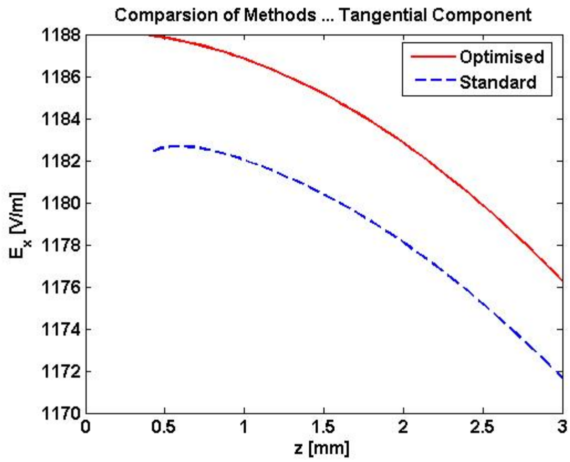

Fig. 12: Tangential component of field stretch by standard and optimised methods.

The normal component, or Z axis component, of electric field strength in the direction parallel to Z axis (normal to plates) and for different distances from the Z axis is in Fig. 13.

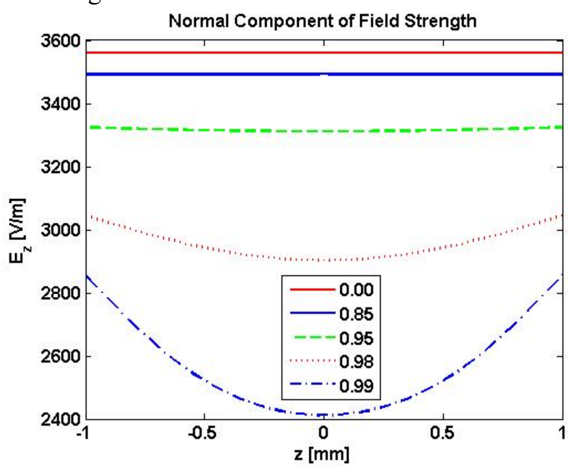

Fig. 13: Normal component of field strength E_z of plate capacitor

As it follows from Fig. 12, the normal component of electric field strength is practically uniform up to distance of 0.95R from Z axis of condenser; it means 90 % of plate area. Near the plate edges strong end effects exist.

As it was shown in section Calculation, the optimised method calculates the electric field strength at given point from two parts: main part and correction. Main part E_{zo} is excited from the circle around the point projection and is given by analytical formula. The correction ΔE_z is obtained by numeric integration from the rest of plate. The correction is in Fig. 14. As we could expect, for central part (about 90 % of plate area) the correction is small and uniform. Near the edges the correction is large and non-uniform. Again end effects take a place.

The tangential component of electric field strength for the same points as the normal component is in Fig. 15. The tangential component is negligible up to distance about 0.85 R from the condenser axis, e. i. at more than 70 % of plate area. On the other hand, near plate edges the tangential component is high and comparable with normal one.

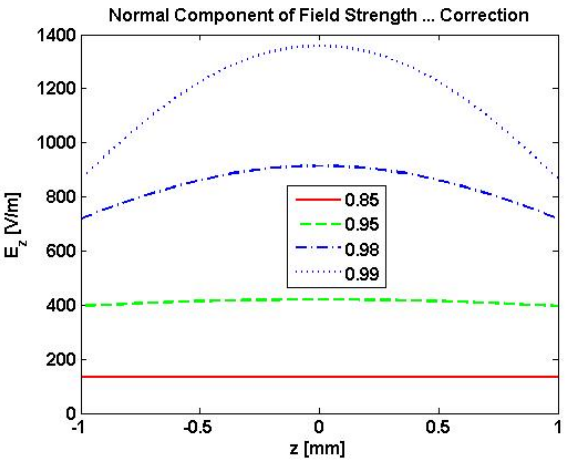

Fig. 14: Correction of normal component of field strength ΔE_z of plate capacitor

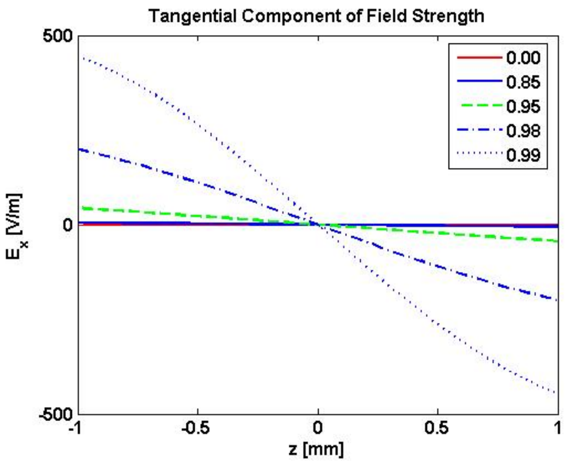

Fig. 15: Tangential component of field strength E_x of plate capacitor

The comparison of calculation and theory is in Fig. 16. The theoretical values on condenser axis are about 1 % lower than those of infinite plate or ideal condenser. The field at the distance of 0.85R from condenser axis is lower by about 2 % in comparison with axis. At higher distances the difference increases rapidly.

Detail of relative deviation of the field at the distance of 0.85R from condenser axis and the field at the axis is in Fig. 17. The z component of the field is practically uniform. Fig. 18 presents the similar difference, but for distance of 0.95R from condenser axis. The difference is much higher, near 7 %, but the z component of the field strength is practically uniform again, its change is less than 0.5 %..

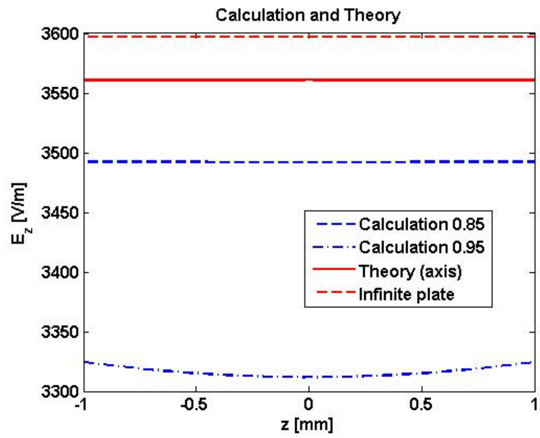

Fig. 16: Theoretical and real electric field strength E_z of plate capacitor

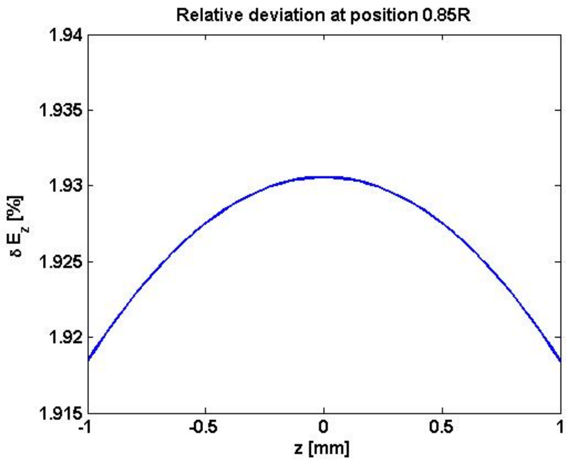

Fig. 17: Relative difference between field on condenser axis and at distance of 0.85R from it.

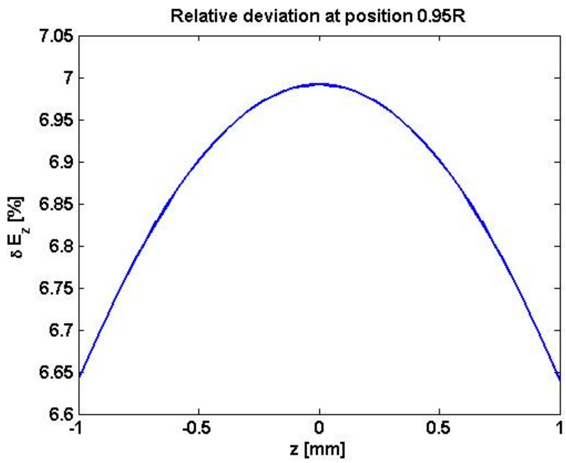

Fig. 18: Relative difference between field on condenser axis and at distance of 0.95R from it.

Probably, the most important result is the distribution of voltage along the plate radius, which is given in Fig. 19. The voltage between plates can be calculated by the numerical integration (simple summation) of calculated values of the z component of electric field strength. The ratio of distance between plates and diameter is relatively high, 5 %, therefore it does not correspond to previous graphs. The choice of these capacitor dimensions was to enhance the graphs. The theoretical value for ideal capacitor axis is given by the horizontal dashed line.

Relatively large distance between plates results in a big difference between real voltage and voltage of ideal capacitor, which has very large (theoretically infinite) dimensions of plates. Especially large end effects are obvious in Fig. 19. The voltage between plate edges is less than half of the value on condenser axis.

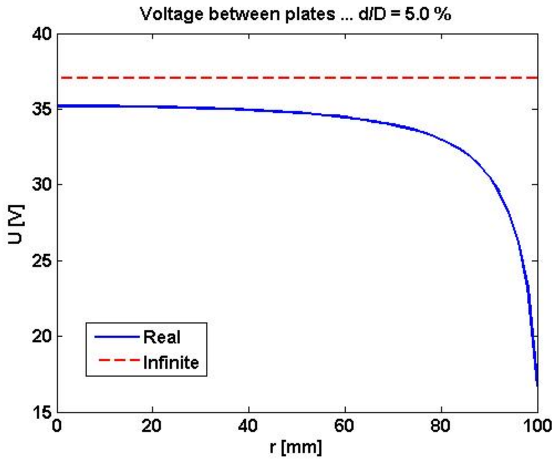

Fig. 19: Voltage between circular plates

The effective numerical integration allows calculating the capacity of finite dimension capacitors. Since the voltage in non-uniform, we suppose that the condenser consists of small parallel connected elementary capacitors and the resulting capacity is their sum. The correction factor δC necessary to multiply the result from standard formula for plate capacitor is in Tab. 1. In the ratio d/D the symbol d is the distance between plates and $D = 2R$ is plate diameter. The values are in satisfactory agreement with those given in technical literature [3] (last row), if we take into account that the ratio of dimensions in not defined precisely in the literature.

d/D [%]	0.5	1	2.5	5	10	15
δC [%]	1.9	3.2	7.0	12.4	20.8	27.2
δC Liter.		~2			~20	

Tab. 1: Correction factor for capacitance

4 Discussion

In the paper we have focused to the determination of electric field of technical charged objects by the use of integral form of Maxwell equations of electrostatics. The solution was to use numerical integration. Another approach is to use the differential form of Maxwell equations and apply the finite element method (FEM).

Several advantages of our approach, using integral form of Maxwell equations, exist:

1. Integral approach leads to analytical solution for practically important charged objects, while the analytical solution of differential approach is only in very simple cases of low technical interest.
2. The analytical solution is correct in every point.
3. The integration can be performed in every point and the accuracy can increase according to demand.
4. The FEM solution is in the set of points that cannot be selected and its accuracy is not known.
5. The use of FEM results in the potential and then the field strength must be calculated by numerical derivation, which is imprecise.
6. The FEM cannot respect the condition of zero potential in infinity[3], while integration formulae include it automatically.
7. The numeric integration results in field strength and the potential can be obtained by numerical integration, which reduces errors.

On the other hand the disadvantage of integration approach is in these facts

1. Constant charge density is supposed in order to get analytical formulae. The FEM uses constant potential, which is typical boundary condition in electrostatic field.
2. The method of integration is limited to relatively simple structures, while the use of FEM is practically unlimited.

Fortunately, the heat conduction analogy supposes constant charge density, which corresponds to constant heat flux used in typical experiments. The task of constant potential can be also solved. In this case the charge density changes, therefore the analytical solution cannot be applied. Since the charge distribution is not known, the semiautomatic method of corrections should be used in principle, but it requires a lot of computer time. In present time it is not a serious limitation, since parallel computer cooperation (cluster) is possible and the integration is probably the simplest task for the use of cluster.

Also the limitation to relatively simple structures and models is not a strict one. In the measurement simple structures are necessary in order to get precise results. It is also well-known that simple models usually lead to surprisingly good agreement with experiment, irrespective of their simplicity. Furthermore, next improvement is possible by adding corrections to the simple model, but its initial simplicity is lost.

In present work we limited to the calculation of vacuum condenser with constant charge density. The second limitation (constant charge density) was discussed above. The dielectrics can be included into model by several ways depending on its shape. If we consider thin layer of very large, theoretically infinite, lateral dimensions along X and Y axis, the electric field strength can be found by:

1. Insertion of relative permittivity of dielectrics into formulae.
2. Using the surface and volume coupled charges. They are proportional to electric field strength for supposed soft dielectrics.

While the first approach can be used with high accuracy only inside the condenser, the second approach is valid everywhere. The simplest approximation is to suppose

[3] Commercial FEM systems have possibility to simulate this condition.

the uniform polarisation, which leads to uniform coupled surface charge and no coupled volume charge. Then the method given above can be used without modification. It is similar to the use of relative permittivity.

The use of non-uniform surface and volume coupled charges complicates the integration as in the case of constant potential. But the corrections are simpler in this case, since the charges are proportional to electric field change, especially the surface one. For the volume coupled charge the numeric derivation of polarisation is necessary, which can lead to numerical errors.

If the dielectric has finite dimensions that usually copy the shape of the electrode, boundary conditions must be satisfied on the free (not under the electrode) face of dielectrics:

1. The tangential component of electric field strength must be continuous.
2. The normal component of electric flux density must be continuous.

Both these conditions can be satisfied approximately by the suitable choice of electric charges as in above cases. However, the calculation is considerably more complicated in this case.

It should be stressed that the use of FEM for the condenser with dielectrics has the same problems and the inclusion of general boundary conditions, given above, is not simple.

The correction factor in Tab. 1 derived for constant charge density is valid also for condenser of constant potential, since it is of no matter, what is constant. For the case of heat conduction analogy it corresponds to the case of large and thin layer of measured material. Side dimensions of material layer should be very large in comparison with dimensions of warming elements. The correction is approximately valid also for material of low heat conductivity and with the same dimensions as warming elements.

The correction factor has relatively high values. It means that only very thin layers of dielectric or material in general should be used for correct measurement of permittivity or heat transfer coefficient.

5 CONCLUSION

The paper shows that the method of analytical or numerical integration is very useful for solution of basic problems of hest transfer using analogy with electrostatic field. It has many advantages in comparison with finite element method. In relatively simple arrangement analytical formulae for electric field strength calculation were derived.

As a practical result the correction factor for capacity or heat conduction measurement was obtained. The value of the correction factor requires a very this layer for accurate measurement. In the case of circular electrodes or heating element the layer thickness should be less than 1 % of diameter.

This approach is valid for material of very low heat conductance. Otherwise the dielectric must be included into the electric condenser model. The dielectrics can be modelled by coupled surface or volume charges. Therefore, the solution is similar, but more complicated.

Another case that has simple solution is the elliptic hole in dielectrics. Some preliminary results are in Ref. [2]. It can by used for the modelling of heat conductivity of porous medium.

6 ACKNOWLEDGEMENT

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Prof. RNDr Ing. Miloslav Košek, CSc
Ing. Jan Vodolan
Technical University of Liberec
Hálkova 6
46117 Liberec 1, Czech Republic
miloslav.kosek@tul.cz
jan,.vodolan@tul.cz