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NOISE MODELING IN RL CIRCUITS

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Abstract: *Paper deals with inductor-resistor electrical circuits. The deterministic model of the circuit is replaced by a stochastic model by adding a noise term in both the source and the resistance. The analytic solution of the resulting stochastic differential equations is presented. Numerical simulations are obtained using the stochastic Euler method. Computer programs in C\# are used to generate numerical solutions and their graphical representations. Simulation is verified by measurement of transient event on inductor-resistor electrical circuits*

INTRODUCTION

Physical systems are often modelled by ordinary differential equations. However, such models may represent idealized situations, as they ignore stochastic effects. By incorporating random elements into the differential equations, a system of stochastic differential equations (SDEs) arises. A general scalar SDE has the form

$$dX(t) = A(t, X(t))\, dt + B(t, X(t))\, dW(t),$$

where $A : [0,T] \times R \to R$ is the drift coefficient and $B : [0,T] \times R \to R$ is the diffusion coefficient. $W(t)$ is the so called Wiener process or Brownian motion, representing the noise. It is a stochastic process with independent increments, satisfying $W(0) = 0$ and $W(t) - W(s)$ distributed $N(0, t-s)$.

We can represent the SDE in the integral form as

$$X(t) = X_0 + \int_{t_0}^{t} A(s, X(s))\, ds + \int_{t_0}^{t} B(s, X(s))\, dW(s),$$

where the first integral is the Lebesgue integral and the second one is a stochastic integral, called the Itô integral (see [1]).

Although the Itô integral has some very convenient properties, the usual chain rule of classical calculus doesn't hold. Instead, the appropriate stochastic chain rule, known as Itô formula, contains an additional term.

The 1-dimensional Itô formula**.** Let the stochastic process $X(t)$ be a solution of the stochastic differential equation

$$dX(t) = u(t, X(t))\, dt + v(t, X(t))\, dW(t)$$

for some suitable functions u, v (see [4], p.44). Let $g(t,x) : (0,\infty) \times \mathbf{R} \to \mathbf{R}$ is a twice continuously differentiable function. Then

$$Y(t) = g(t, X(t))$$

is a stochastic process, for which

$$dY(t) = \frac{\partial g}{\partial t}(t, X(t))\, dt + \frac{\partial g}{\partial x}(t, X(t))\, dX(t) + \\ + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X(t))(\, dX(t))^2,$$

Where $(\, dX(t))^2 = (\, dX(t)) \cdot (\, dX(t))$ is computed according to the rules

$$dt \cdot dt = dt \cdot dW(t) = dW(t) \cdot dt = 0, \;\; dW(t) \cdot dW(t) = dt.$$

1 THE RL CIRCUIT WITH DETERMINI STIC PARAMETERS

Let us consider a simple RL electrical circuit (see [2]). The electrical current $i(t)$ at time t at the circuit satisfies the differential equation

$$L\, \frac{di(t)}{dt} + R\, i(t) = v(t), \quad i(0) = i_0,$$

where the resistance R and the inductance L are constants and $v(t)$ denotes the potential source at time t.

If $v(t)$ is a piecewise continuous function, the solution of this first order linear differential equation is

$$i(t) = i_0 \cdot e^{-\frac{Rt}{L}} + \frac{1}{L}\int_0^t e^{\frac{R(s-t)}{L}} \cdot v(s)\, ds.$$

2 THE RL CIRCUIT WITH A RANDOM SOURCE

Now we allow some randomness in the electrical source. Instead of $v(t)$ we have the non deterministic versions of this parameter:

$$v^*(t) = v(t) + \text{" noise "}.$$

Observations indicated that the " noise " can be described as a multiple of the so called "white noise process", denoted by $\xi(t)$. We get the following equation

$$L\,\frac{di(t)}{dt} + R\, i(t) = v(t) + \alpha\, \xi(t),$$

where α is a non negative constant. Its magnitude determine the deviation of the stochastic case from the deterministic one. Here we consider the initial condition and the current at time t as random variables and denote them by capital letters. Now we have $I(0) = I_0$ and for $t \in [0,T]$

$$\frac{dI(t)}{dt} + \frac{1}{L} RI(t) = \frac{1}{L}\left(v(t) + \alpha\, \xi(t)\right).$$

To get a stochastic differential equation we first multiply the equation by dt and then replace $\xi(t)\, dt$ by $dW(t)$, where $W(t)$ is the Wiener process. We got a stochastic differential equation:

$$dI(t) = \frac{1}{L}(v(t) - RI(t))\, dt + \frac{\alpha}{L}\, dW(t),$$

with the initial condition $I(0) = I_0$.

We use the Ito formula to get the solution of this equation. Then

$$I(t) = I_0\, e^{-\frac{R}{L}t} + \frac{1}{L}\int_0^t v(s)\, e^{\frac{R}{L}(s-t)}\, ds + \frac{\alpha}{L}\int_0^t e^{\frac{R}{L}(s-t)}\, dW(s).$$

For $E[I_0^2] < \infty$, the expectation $E[I(t)] = m(t)$ is the solution of the ordinary differential equation

$$m'(t) = \frac{1}{L}(v(t) - R\, m(t)), \quad m(0) = E[I_0].$$

We can easily compute that

$$E[I(t)] = e^{\frac{-Rt}{L}} \cdot E[I_0] + \frac{1}{L}\int_0^t e^{\frac{R(s-t)}{L}} \cdot v(s)\, ds,$$

for every $t > 0$. If the random variable $I(0) = I_0$ is constant, then the expectation of the stochastic solution is equal to the deterministic solution of the circuit. So the function $m(t) = E[I(t)]$ is independent of the fluctuational part of the SDE.

The second moment $D(t) = E[I^2(t)]$ is the solution of the ordinary linear equation

$$D'(t) = \left(-\frac{2R}{L}\right) D(t) + 2m(t)\frac{v(t)}{L} + \frac{\alpha^2}{L^2},\ D(0) = E[I_0^2],$$

where $m(t) = E[I(t)]$.

3 SIMULATIONS OF THE STOCHASTIC SOLUTION

To simulate $I(t)$ numerical techniques have to be used (see [4]). Here we use the simplest numerical scheme, the stochastic Euler scheme, which is based on numerical methods for ordinary differential equations.

The Euler scheme for the SDE describing the RL circuit has the form

$$I^{n+1} = I^n + \frac{1}{L}(v(t^n) - RI^n)h + \frac{\alpha}{L}\Delta W^n, \quad I^0 = i(0).$$

Here we consider an equidistant discretisation of the time interval $t^n = t^0 + nh$, where $h = \frac{T-t^0}{N} = t^{n+1} - t^n = \int_{t^n}^{t^{n+1}} dt$ and the corresponding discretisation of the Wiener process $\Delta W^n = W(t_{n+1}) - W(t_n) = \int_{t_n}^{t_{n+1}} dW(s)$. To be able to apply any stochastic numerical scheme, first we have to generate the random increments of the Wiener process W as independent Gauss random variables with mean $E[\Delta W^n] = 0$ and $E[(\Delta W^n)^2] = h$.

The Euler scheme converges with strong order $\gamma = \frac{1}{2}$. It means that there exist real constants $K > 0$ and $\delta > 0$, so that

$$E[|\, I(T) - I^N\, |] \le Kh^\gamma, \quad h \in (0, \delta)$$

where the numerical solution is denoted by I^N.

4 AN EXAMPLES

4.1 *Example 1*

Let us consider the RL electrical circuit, when L, R and $v(t) = V$ are constants, $I_0 = 0$. We graph the stochastic solution of the circuit with a stochastic source $(\alpha = 1)$. The stochastic solution $I(t)$ is a Gaussian process and so for every $t \in [0,T]$ the solution $I(t)$ is

$N(m(t), \sigma^2(t))$ distributed (again $m(t) = E[I(t)]$ and $\sigma^2(t) = E[I^2(t)] - m^2(t)$). We can compute that for every t

$$P(|I(t) - m(t)| < 1.96\sigma(t)) = 2\,\Phi(1.96) - 1 = 0.95,$$

where

$$\Phi(x) = \frac{1}{2\pi}\int_{-\infty}^{x} e^{-\frac{1}{2}s^2}\, ds.$$

Since we know $E[I^2(t)]$ and $m(t) = E[I(t)]$, we can predict that the trajectories of the stochastic solutions fall into $(m(t) - \varepsilon, m(t) + \varepsilon)$ with 95% probability. We compute the second moment $D(t) = E[I^2(t)]$ as a solution of the ordinary differential equation:

$$D'(t) = \left(-\frac{2R}{L}\right)D(t) + 2m(t)\frac{V}{L} + \frac{\alpha^2}{L^2}, \quad D(0) = 0.$$

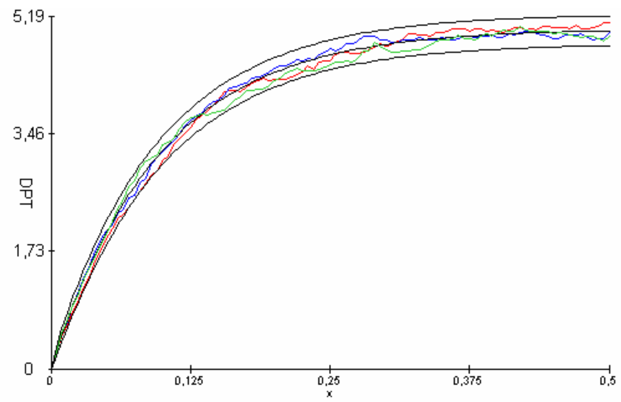

Fig. 1. Three trajectories of the stochastic solution and a 95%prediction interval

4.2 Example 2

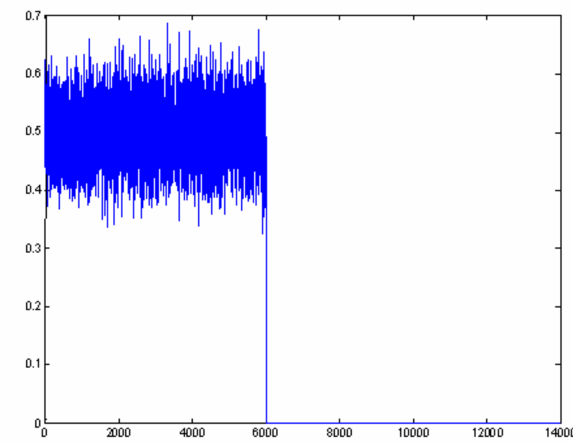

Fig. 2. The source with the "noise"

For this experiment we used the arbitrary function generator HP 33120A and the oscilloscope Agilent DSO 6052A. Experiment configuration is shown on Fig. 4

We used a coil with inductance $L = 210mH, R_s = 89\Omega$. We connected a resistor of $R = 5k\Omega$ and a source of $U = 0.522V$. Then we added a "noise" to the signal. The results of the measurements were processed in Matlab and the confidence intervals were computed as in section 5. We plotted the computed expectation and the prediction interval together with the measured current to the following picture. One can see, that the measured data agrese with the above described theory.

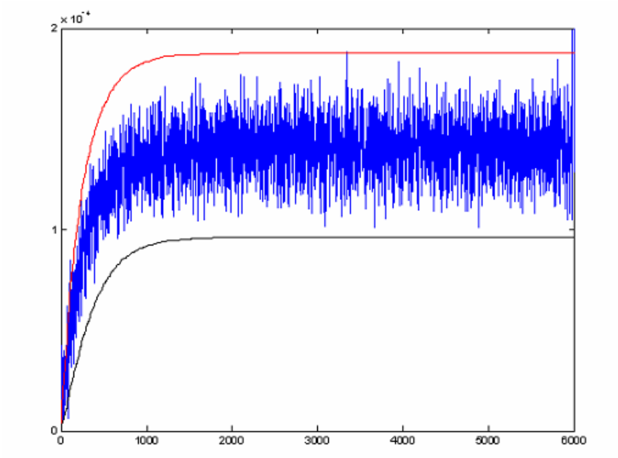

Fig. 3. The results of the experiment

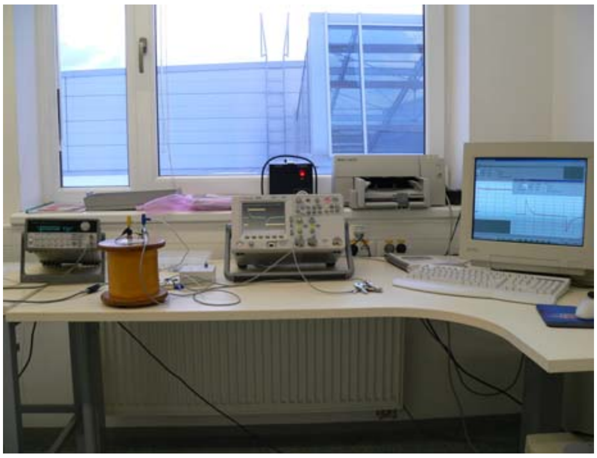

Fig. 4. The results of the experiment

5 CONCLUSION

This paper shows an application of the Itô stochastic calculus to the problem of modelling inductor-resistor electrical circuits. The deterministic model of the circuit is replaced by a stochastic model by adding a noise term in the source. The analytic solution of the resulting stochastic differential equation is obtained using the Itô formula. Statistical estimates of the stochastic solutions are examined and confidence intervals are found for the trajectories of the solution. The programming language C#, a part of the new MS .NET platform, is used for numerical simulations. The results were verified in an experiment by measurements on inductor-resistor electrical circuits.

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