Connected even factors in the square of essentially 2-edge-connected graph

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Submitted: Aug 5, 2015; Accepted: Aug 22, 2017; Published: Sep 8, 2017
Mathematics Subject Classifications: 05C38, 05C48

Abstract

An essentially k-edge connected graph G is a connected graph such that deleting less than k edges from G cannot result in two nontrivial components. In this paper we prove that if an essentially 2-edge-connected graph G satisfies that for any pair of leaves at distance 4 in G there exists another leaf of G that has distance 2 to one of them, then the square G^2 has a connected even factor with maximum degree at most 4. Moreover we show that, in general, the square of essentially 2-edge-connected graph does not contain a connected even factor with bounded maximum degree.

Keywords: connected even factors; (essentially) 2-edge connected graphs; square of graphs

1 Introduction

We consider only finite undirected simple graphs. For terminology and notation not defined in this paper we refer to [15]. Let G be a connected graph. For vertices x, y of G,

*Supported by (i) the European Regional Development Fund (ERDF), Project, NTIS - New Technologies for Information Society, European Centre of Excellence, CZ.1.05/1.1.00/02.0090; (ii) project GA14-19503S of the Grant Agency of the Czech Republic.
†Supported by NSFC (No.11161046) and by Xinjiang Talent Youth Project (No.2013721012).
‡Supported by NSFC (No.11471037 and No.11671037).

let $N_G(x)$ denote the *neighborhood* of x in G, $d_G(x) = |N_G(x)|$ the *degree* of x in G, and $\text{dist}_G(x, y)$ the *distance* between x, y in G. The *square* of a graph G, denoted by G^2, is the graph with same vertex set as G in which two vertices are adjacent if their distance in G is at most 2. Thus $G \subseteq G^2$. There are several papers (e.g. see [2], [4], [5], [6], [7], [8], [9], and [10]) about hamiltonian properties in the square of a graph. This paper deals with connected even factors which generalize some previous known results.

A *factor* in a graph G is a spanning subgraph of G. A *connected even factor* in G is a connected factor in G in which every vertex has positive even degree. A $[2, 2s]$-*factor* of G is a connected even factor of G in which every vertex has degree at most $2s$. Some results for the existence of such kind factors by using forbidden subgraphs have been appeared, for examples see [1], [11], and [13]. Since a hamiltonian cycle is a $[2, 2s]$-factor with $s = 1$, the minimum s in a $[2, 2s]$-factor of a graph can be seen as a measure for how close a graph is to become hamiltonian. Furthermore we know from [14] that it is NP-complete to determine whether the square of a graph is hamiltonian. Therefore the determination of minimum s in a $[2, 2s]$-factor in the square of a graph is also NP-complete.

The result by Fleischner in [6] concerning the existence of a hamiltonian cycle (a $[2, 2]$-factor) in the square of 2-connected graph is well known. Recently, Müttel and Rautenbach in [12] gave a shorter proof of this result.

Theorem 1. [6] *If G is a 2-connected graph and v_1 and v_2 are two distinct vertices of G, then G^2 contains a hamiltonian cycle C such that both edges of C incident with v_1 and one edge of C incident with v_2 belong to G. Furthermore, if v_1 and v_2 are neighbors in C, then these are three distinct edges.*

Theorem 1 was a base for proving the following theorem by Abderrezzak et al. in [4] using forbidden subgraphs. The graph $S(H)$ is obtained from a graph H by subdividing each edge of H exactly once.

Theorem 2. [4] *If G is a connected graph such that every induced $S(K_{1,3})$ has at least three edges in a block of degree at most 2, then G^2 is hamiltonian.*

Theorem 2 was generalized by Ekstein et al. in [2] for $[2, 2s]$-factors.

Theorem 3. [2] *Let s be a positive integer and G be a connected graph such that every induced $S(K_{1,2s+1})$ has at least three edges in a block of degree at most two. Then G^2 has a $[2, 2s]$-factor.*

Let G be a connected graph. Recall that a graph G is *essentially k-edge connected* if deleting less than k edges from G cannot result in two nontrivial components. In this paper, we shall answer the question how it is for the existence of a $[2, 2s]$-factor in the square of a graph with 2-edge (or essentially 2-edge)-connectivity instead of (vertex) connectivity of a graph.

A vertex of degree 1 is called *a leaf*. A cut vertex y is *trivial* in G, if y is not a cut vertex in $G - M$, where M is a set of all leaves adjacent to y, otherwise is *non-trivial*. If $M = \{x\}$ and the neighbor of x is a trivial cut vertex of G, then x is called *a bad leaf*. A

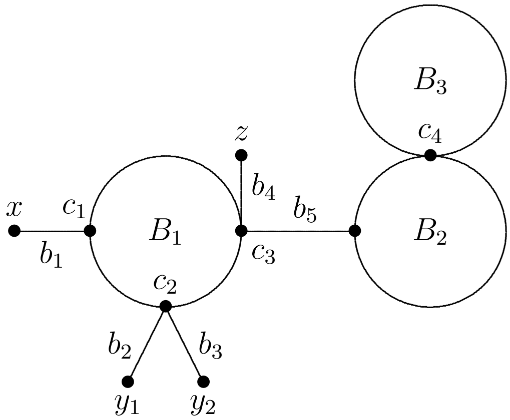

Figure 1: In this graph, c_1, c_2 are trivial cut vertices, c_3, c_4 are non-trivial cut vertices, x is a bad leaf, y_1, y_2, z are leaves, b_1 is a bad bridge, b_2, b_3, b_4 are trivial bridges, b_5 is a non-trivial bridge, and B_1, B_2, B_3 are cyclic blocks.

trivial bridge is a cut-edge of G containing a leaf, otherwise is *non-trivial*. *A bad bridge* is a trivial bridge of G adjacent to a bad leaf. For illustration see Fig. 1.

Firstly, we look at the graph in Fig. 2, from which one may see the following result.

Theorem 4. *For any fixed positive integer s, there exists an infinite class of essentially 2-edge-connected graphs G such that G^2 has no $[2, 2s]$-factor, even if the resulting graph obtained from G by deleting its all leaves is 2-connected.*

Proof. Note that the graph G in Fig. 2 is an essentially 2-edge-connected graph. Since every leaf v_i of G has degree exactly 3 in G^2, at least one edge of v_ix, v_iy have to be used in any possible $[2, 4]$-factor of G^2. Therefore, G^2 has no $[2, 2s]$-factor since G has $4s + 1$ such leaves. □

On the other hand, we may show the following result, which is the main result of this paper.

Theorem 5. *Let G be a connected graph without non-trivial bridges and without any two bad leaves at distance exactly 4. Then G^2 has a $[2, 4]$-factor.*

The following corollaries are immediate consequences of Theorem 5.

Corollary 6. *If G is a 2-edge connected graph, then G^2 contains a $[2, 4]$-factor.*

Corollary 7. *If G is an essentially 2-edge connected graph without bad leaves, then G^2 contains a $[2, 4]$-factor.*

Corollary 8. *Let G be a connected graph without non-trivial bridges. If any two bad leaves have distance at least 5 in G, then G^2 has a $[2, 4]$-factor.*

Note that the graph in Fig. 2 also shows that the distance 5 in Corollary 8 can not be replaced by distance 4.

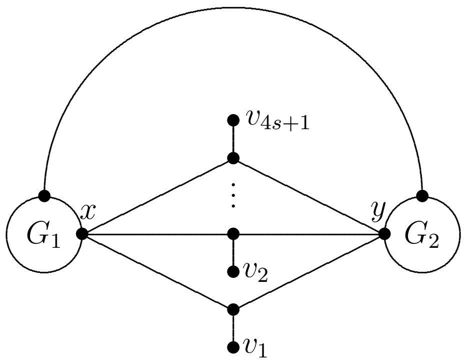

Figure 2: Essentially 2-edge connected graphs G such that their square contains no $[2, 2s]$-factor, where G_1 and G_2 are any essentially 2-edge connected graphs.

2 A *Useful lemma*

Before presenting this lemma, we need some additional notation. *Block graph* of a graph G, denoted by $BC(G)$, is the graph whose vertex set consists of all blocks and cut vertices of G, and two vertices are adjacent in $BC(G)$ if one of them is a block of G and the second one is its vertex. It is easy to see that $BC(G)$ is a tree for a connected graph G. Note that for any tree, we may choose any vertex as its root. Hence without loss of generality, we may assume that $B_1, \ldots, B_t$ be all blocks of G such that B_1 corresponds to the root of $BC(G)$. For a cut-vertex v of G, *the parent block* of v is the block containing v and its corresponding vertex in $BC(G)$ has the smallest distance to the root of $BC(G)$. The remaining blocks containing v are called *children blocks* of v with respect to the root of $BC(G)$.

The following lemma, we call it a *Useful lemma*, is a key for the proof of our main result (Theorem 5).

Lemma 9. *(Useful lemma) Let G be a connected graph without non-trivial bridges and without bad leaves (except $K_{1,2}, K_{1,3}$) and u be a vertex of G that is neither a cut vertex nor a leaf (if any).*
Then G^2 has a $[2, 4]$-factor F such that

a) $d_F(x) = 2$ for any vertex x that is not a cut vertex of G;

b) both edges of F incident with u belong to G;

c) for each cut vertex y of G it holds that $d_F(y) = 4$ and at least two edges of F incident with y belong to G, moreover if y is a trivial cut vertex, then these two edges are trivial bridges;

d) for any cut vertex y of G, the two edges incident with u in F are distinct from the two edges incident with y in F as specified in (c);

e) *for any two cut vertices y_1 and y_2 of G, the two edges of F incident with y_1 as specified in (c) are distince from those with y_2 as specified in (c).*

Proof. If G is $K_{1,s}$, for $s \geqslant 4$, then G^2 is a complete graph and the result is obvious. Now we assume that G contains at least one cyclic block and $G' = G - M$, where M is a set of all leaves adjacent with all trivial cut vertices of G.

Let $\mathbb{O} = B_1, B_2, \ldots, B_k$ be an ordering of all blocks of G' such that either $u \in V(B_1)$, if any, or we choose arbitrary cyclic block as B_1, satisfying the following properties:

- for any cut vertex v of G', all children blocks of v with respect to the root r of $BC(G')$ corresponding to B_1 appear consecutively in $\mathbb{O}$ such that bridges containing v are in $\mathbb{O}$ before cyclic blocks containing v;
- $\text{dist}_{BC(G')}(r, v_i) < \text{dist}_{BC(G')}(r, v_j)$ implies $i < j$, where v_i, v_j are vertices of $BC(G')$ corresponding to B_i, B_j, respectively.

Then G' is a connected graph without non-trivial bridges and without bad leaves and we prove by induction on k that $(G')^2$ contains a $[2, 4]$-factor F' such that

1) $d_{F'}(x) = 2$ for any vertex x that is not a cut vertex of G;
2) both edges of F' incident with u, if any, belong to B_1;
3) for each cut-vertex y of G', it holds that $d_{F'}(y) = 4$ and at least two edges of F' incident with y belong to G'. Moreover,
   - if y belongs to exactly two blocks of G', then at least two edges of F' incident with y are edges from the children block of y with respect to r (the root of $BC(G')$ corresponding to B_1);
   - if y belongs to more than two blocks of G', then at least two edges of F' incident with y are edges from two different children blocks of y with respect to r.

For $k = 1$, $G' = B_1$ and $(G')^2$ even has a hamiltonian cycle C such that both edges of F' incident with u, if any, belong to B_1 by Theorem 1.

Let $k > 1$ and assume that Lemma 9 is true for all integers less than k. By the definition of G' and $\mathbb{O}$, B_k is an end cyclic block of G' and let v_0 be the cut vertex of G' with $v_0 \in V(B_k)$.

If $B_{k-1} = v_0 l$ (i.e. B_{k-1} is a bridge) and B_{k-1}, B_k are only children blocks of v_0 with respect to r, then we set $G_1 = G' - \{V(B_k) \cup \{l\} \setminus \{v_0\}\}$, otherwise we set $G_2 = G' - \{V(B_k) \setminus \{v_0\}\}$. Hence G_1, G_2 are connected graphs without non-trivial bridges and without bad leaves and have $k-2, k-1$ blocks, respectively. Hence by the induction hypothesis, $(G_1)^2, (G_2)^2$ have a $[2, 4]$-factor F_1, F_2 with properties 1), 2), and 3), respectively.

By Theorem 1, there is a Hamiltonian cycle C in $(B_k)^2$ such that two edges f_1, f_2 of C incident with v_0 belong to B_k and thus belong to G'.

Case 1: G_1 exists.

Let $f_1 = v_0v_k$. Then $F' = ((F_1 \cup C) \cup \{v_0l, v_kl\}) \setminus \{f_1\}$ is the $[2,4]$-factor of $(G')^2$ with properties 1), 2), and 3).

Case 2: G_1 does not exist and v_0 is not a cut vertex in G_2.

Hence v_0 belongs to exactly two blocks of G' and $F' = F_2 \cup C$ is the $[2,4]$-factor of $(G')^2$ with properties 1), 2), and 3).

Case 3: G_1 does not exist and v_0 is a cut vertex in G_2.

Let $f_1 = v_0v_k$. We consider two possibilities depending on the property 3).

If exactly two blocks of G_2 contain v_0, then by the induction hypothesis $d_{G_2}(v_0) = 4$ and there are two edges of F_2 incident with v_0 from a children block B_{k-1} of v_0. (Note that B_{k-1} is a cyclic block, since G_1 does not exist.) Let $e_{k-1} = v_0v_{k-1}$ be such an edge of F_2. Since $\mathrm{dist}_{G'}(v_{k-1}, v_k) = 2$, the edge $v_{k-1}v_k$ is an edge of $(G_2)^2$. Thus $F' = ((F_2 \cup C) \cup \{v_{k-1}v_k\}) \setminus \{e_{k-1}, f_1\}$ is the $[2,4]$-factor of $(G')^2$ with properties 1), 2), and 3).

If there are more than two blocks of G_2 containing v_0, then by the induction hypothesis $d_{G_2}(v_0) = 4$ and there are two edges e_{k-2}, e_{k-1} of F_2 incident with v_0 in B_{k-2}, B_{k-1}, respectively. Note that it could be $B_{k-2} = e_{k-2}$ or $B_{k-1} = e_{k-1}$. Let $e_{k-2} = v_0v_{k-2}$. Since $\mathrm{dist}_{G'}(v_{k-2}, v_k) = 2$, the edge $v_{k-2}v_k$ is an edge of $(G_2)^2$. Thus $F' = ((F_2 \cup C) \cup \{v_{k-2}v_k\}) \setminus \{e_{k-2}, f_1\}$ is the $[2,4]$-factor of $(G')^2$ with properties 1), 2), and 3).

Now we extend F' to a $[2,4]$-factor F in G^2 with required properties. Note that the properties 1), 2), a nd 3) imply the properties a)-e) in Lemma 9.

Let $u_1, u_2, \ldots, u_t$ be all trivial cut vertices of G and $l_i^1, l_i^2, \ldots, l_i^{s_i}$ be all leaves incident with u_i, for $i = 1, 2, \ldots, t$. Note that $s_i \geqslant 2$, otherwise we have a bad bridge in G, a contradiction. For $i = 1, 2, \ldots, t$, let $C_i = u_il_i^1l_i^2 \ldots l_i^{s_i}u_i$ be cycles in G^2 and $C' = \cup_{j=1}^{t} C_j$. Since $d_{F'}(u_i) = 2$ and $u_il_i^1, l_i^{s_i}u_i$ are edges from G, $F = F' \cup C'$ is the $[2,4]$-factor of G^2 with properties a)-e). □

Note that clearly the square of $K_{1,2}$, $K_{1,3}$ is hamiltonian but there is no $[2,4]$-factor with a vertex of degree 4 in the square of $K_{1,2}$, $K_{1,3}$, respectively.

3 Proof of Theorem 5

In this section we prove Theorem 5.

Proof. Firstly if G is $K_{1,2}$ or $K_{1,3}$, then clearly G^2 is even hamiltonian.

Now let X be a set of all bad leaves of G and $G' = G - X$. For $x_i \in X$, we denote y_{x_i} or only y_i its unique neighbor in G. By Lemma 9, there is a [2,4]-factor F' of $(G')^2$ with properties a)-e). Note that $d_{F'}(y_i) = 2$ for each y_i.

By the definition, any two bad leaves have a distance at least 3. Let $X_0 \subseteq X$ be the set of all bad leaves that has a bad leaf at the distance exactly 3 in G. Then, for all $x_i \in X_0$, corresponding y_i's induce a subgraph of G' in which all components (denoted by $H_1, H_2, \ldots, H_s$) are complete graphs, otherwise we have in G two bad leaves at distance 4, a contradiction.

Let $V(H_i) = \{y_{i,1}, y_{i,2}, \dots, y_{i,t_i}\}$, $t_i \geqslant 2$ for $i = 1, 2, \dots, s$. Then we set

$$M_i = \bigcup_{j=1}^{t_i-1} \{x_{i,j}y_{i,j+1}, x_{i,j+1}y_{i,j}\} \bigcup \{x_{i,1}y_{i,1}, x_{i,t_i}y_{i,t_i}\}.$$

All bad leaves of $X \setminus X_0$ are pairwise at distance at least 5 and we divide them into the following three disjoint classes by the following way (see Fig. 3 for illustration):

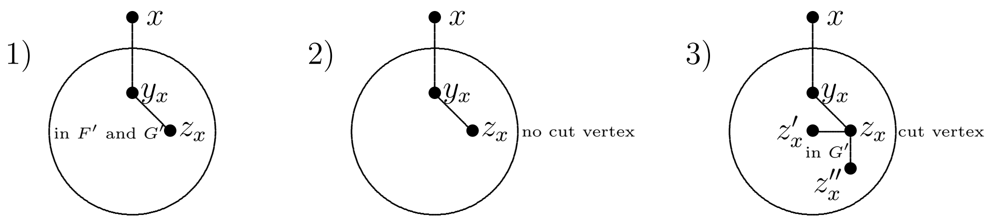

Figure 3: Three cases in an ordering of all bad leaves of $X \setminus X_0$ in G.

1) Let X_1 be the set of all vertices $x \in X \setminus X_0$ such that there exists a vertex z_x with $y_x z_x \in E(F') \cap E(G')$;

2) Let X_2 be the set of all vertices $x \in X \setminus (X_0 \cup X_1)$ such that there exists z_x, which is not a cut vertex of G', with $y_x z_x \in E(G')$ (and $y_x z_x \in E(F')$);

3) Let X_3 be the set of all vertices $x \in X \setminus (X_0 \cup X_1 \cup X_2)$ (it means that there exists only a cut vertex z_x of G' with $y_x z_x \in E(G')$ (and $y_x z_x \in E(F')$).

Note that by Lemma 9 we have

- $d_{F'}(z_x) = 2$ for $x \in X_2$;
- $d_{F'}(z_x) = 4$ and at least two edges incident with z_x (namely $z_x z'_x, z_x z''_x$) are in $E(E') \cap E(G')$ for $x \in X_3$.

Now set

$$E_0 = \bigcup_{i=1}^{s} M_i, \quad E_1 = \bigcup_{x \in X_1} \{xy_x, xz_x\}, \quad E'_1 = \bigcup_{x \in X_1} \{y_x z_x\},$$

$$E_2 = \bigcup_{x \in X_2} \{xy_x, xz_x, y_x z_x\},$$

$$E_3 = \bigcup_{x \in X_3} \{xy_x, xz_x, y_x z'_x\}, \quad E'_3 = \bigcup_{x \in X_3} \{z_x z'_x\}.$$

For all x, z_x's are different, otherwise if $z_x = z_{x'}$, for $x \neq x'$, then $xy_x z_x (= z_{x'}) y_{x'} x'$ is a path of length 4 in G joining two bad leaves, a contradiction. Similarly, none of z_x's is a neighbor of a bad leaf in G.

Possibly, $z_{x_{i_1}} z_{x_{i_2}} \dots z_{x_{i_k}}$ is a path in F' for $\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\} \subseteq X_3$. In order to have different edges in E_3 and E'_3 we set $z'_{x_j} = z_{x_{j+1}}$, for $j = i_1, i_2, \dots, i_{k-1}$, and $z'_{x_{i_k}}$ as arbitrary neighbor of $z_{x_{i_k}}$ in F' and in G different from $z_{x_{i_{k-1}}}$. Note that by 3) and Lemma 9 such a vertex exists and could be some z_{x_j}, for $j \in \{i_1, i_2, \dots, i_{k-2}\}$.

Hence we conclude that $F = (F' \cup (E_0 \cup E_1 \cup E_2 \cup E_3)) \setminus (E'_1 \cup E'_3)$ is a [2,4]-factor of G^2. □

4 Conclusion

Now we can answer the question from the Introduction. By Theorem 1 we know that the square of a 2-connected graph has a $[2, 2s]$-factor for $s = 1$. In this paper we proved that the square of a 2-edge-connected graph has a $[2, 2s]$-factor for $s = 2$ (Corollary 6) and that the square of a essentially 2-edge-connected graph without bad leaves has a $[2, 2s]$-factor also for $s = 2$ (Corollary 7). In general, there exist essentially 2-edge-connected graphs whose square have no $[2, 2s]$-factor for every s. This example of G even exists under an additional condition that the graph obtained from G by deleting all leaves is 2-connected (Theorem 4).

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