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PIEZOELECTRIC FINITE ELEMENT BEAM FOR SMART STRUCTURES

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Abstract: The paper deals with finite beam element with piezoelectric layers and functionally graded material of core. In the process of homogenization of FGM core and piezoelectric layers direct integration method and multilayer method is used. There is also presented the derivation of individual submatrices of local stiffness and mass matrix, where concept of transfer constants is used.

Keywords: Piezoelectric Material; FGM; FEM; Beam Element

1 Introduction

Smart materials and structures belong between rapidly growing research area. This area is characterized by the strong influence of different subjects like material science, mathematical modeling, sensors, actuators, control, electronics and software engineering. Smart structures are from mathematical point of view systems with continuous parameters like beams and shells with locally distributed sensors and actuators. Typical representatives of such systems are elastic beam structures where piezoelectric sensors and piezoelectric actuators are located in order to reduce vibration of such structures excited by external loading. Mathematical modelling of the system is the first step in the process of reduce and control such system.

2 Constitutive equations of piezoelectric material

Piezoelectric constitutive equations describe the relationship between mechanical and electrical quantities [1, 2]. This relationship is derived in tensor notation, but for practical usage it can be rewritten into matrices notation.

The constitutive equations can be expressed by strain tensor components ε_{kl} and vector components of electric field intensity E_k and has a form

$$\sigma_{ij} = c^E_{ijkl}\varepsilon_{kl} - e_{ijk}E_k \quad (1)$$

$$D_i = e_{ikl}\varepsilon_{kl} + \epsilon^{\varepsilon}_{ik}E_k \quad (2)$$

where σ_{ij} are mechanical stress tensor components, D_i are components of electric displacement vector, c^E_{ijkl} are components of stiffness tensor under constant electric intensity, ϵ^{σ}_{ik} are components of permittivity tensor under constant mechanical stress and e_{ijk} are components of piezoelectric modulus tensor.

If we use symmetric properties of individual tensor in constitutive tensor equations, we can rewrite

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constitutive equations into matrix notation [3]. Then equations (1) and (2) have a form

$$\sigma_p = c^E_{pq}\varepsilon_q - e_{pk}E_k \tag{3}$$

$$D_i = e_{iq}\varepsilon_q + \epsilon^{\varepsilon}_{ik}E_k \tag{4}$$

D_i and E_k are vectors with three components, σ_q and ε_q are vectors with six components, matrices s^E_{pq} and c^E_{pq} have dimension 6×6, matrices d_{iq} and e_{pk} have dimension 3×6 and matrix $\epsilon^{\varepsilon}_{ik}$ has dimension 3×3.

Constitutive equations (3) and (4) written in a component form can be rewritten as

$$\boldsymbol{\sigma} = \mathbf{c}^E\boldsymbol{\varepsilon} - \mathbf{e}^T\mathbf{E} \tag{5}$$

$$\mathbf{D} = \mathbf{e}\boldsymbol{\varepsilon} + \boldsymbol{\epsilon}^{\varepsilon}\mathbf{E} \tag{6}$$

3 Finite element equations of piezoelectric material

To obtain basic FEM equations for piezoelectric structure, the Hamilton's principle is used and can be expressed in the form

$$\int_{t_1}^{t_2} (\delta L + \delta W)\, dt = 0 \tag{7}$$

where L is Lagrangian, W is work of external mechanical and electrical forces and t_1 and t_2 defined considered time interval.

Lagrangian of piezoelectric structure is given by

$$\begin{aligned} L =& T - U + W_e = \\ =& \int_V \frac{1}{2}\rho\dot{\mathbf{u}}^T\dot{\mathbf{u}}dV - \int_V \frac{1}{2}\boldsymbol{\varepsilon}^T\boldsymbol{\sigma}dV + \int_V \frac{1}{2}\mathbf{E}^T\mathbf{D}dV \end{aligned} \tag{8}$$

where T is kinetic energy of structure, U is potential energy of structure and W_e is electric energy stored in piezoelectric material. In kinetic energy term $\dot{\mathbf{u}}$ represents velocity vector.

Virtual work of external mechanical and electrical forces can be expressed as

$$\delta W = \sum\left(\delta\mathbf{u}^T\mathbf{F}\right) - \sum\left(\delta\boldsymbol{\phi}^T\mathbf{Q}\right) \tag{9}$$

where $\mathbf{F}$ and $\mathbf{Q}$ represents discrete forces and electric charges, respectively and $\mathbf{u}$ and $\boldsymbol{\phi}$ are displacement vector and scalar electric potential, respectively.

Using classical shape functions for solid (2D or 3D) finite elements and previous defined constitutive piezoelectric law, equations (7)-(9) can be expressed in matrix form

$$\begin{bmatrix} \mathbf{M}^e_{uu} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}^e \\ \ddot{\boldsymbol{\phi}}^e \end{bmatrix} + \begin{bmatrix} \mathbf{C}^e_{uu} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}}^e \\ \dot{\boldsymbol{\phi}}^e \end{bmatrix} + \begin{bmatrix} \mathbf{K}^e_{uu} & \mathbf{K}^e_{u\phi} \\ \mathbf{K}^e_{\phi u} & \mathbf{K}^e_{\phi\phi} \end{bmatrix} \begin{bmatrix} \mathbf{u}^e \\ \boldsymbol{\phi}^e \end{bmatrix} = \begin{bmatrix} \mathbf{F}^e \\ \mathbf{Q}^e \end{bmatrix} \tag{10}$$

where $\mathbf{u}^e$, $\dot{\mathbf{u}}^e$ and $\ddot{\mathbf{u}}^e$ is vector of nodal displacement, velocity and acceleration, respectively, $\boldsymbol{\phi}^e$ is vector of nodal electric potential, $\mathbf{F}^e$ is vector of nodal structural forces and $\mathbf{Q}^e$ is vector of nodal electric charge. Individual submatrices represent the mass, damping, stiffness, permitivity and piezoelectric coupling of finite element.

4 FEM equations of piezoelectric beam element

2D beam element with piezoelectric layers, where beam core is made from functionally graded material is shown in Figure 1, where all degrees of freedom are depicted. Mechanical degrees of freedom in each node are two displacements (in direction x a y) and rotation (in plane $x - y$) [4]. Electric degrees of freedom are electric potentials ϕ_i on 4 electrodes. The height of beam core made from FGM is h, the

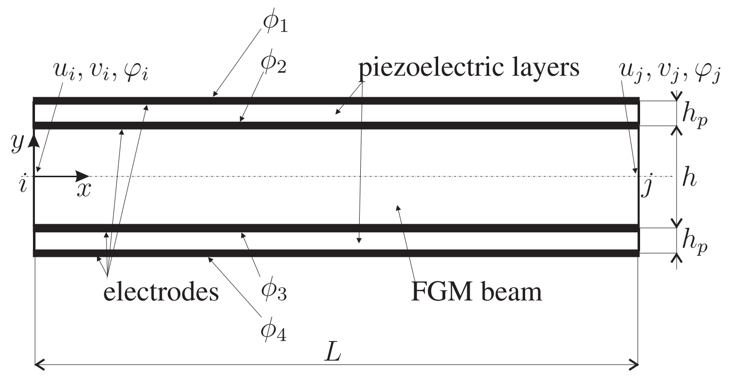

Figure 1: Electric DOF in 2D beam element

height of piezoelectric layer is h_p, the depth and the length of the beam element are b and L, respectively. Material properties of FGM core are function of longitudinal and transversal coordinate x and y, material properties of piezoelectric layers are constants.

In order to derive individual submatrices of the beam element with piezoelectric layers and FGM core, two steps in homogenization process have to be performed. At first, homogenization of material properties of FGM core have to be performed, where direct integration method is used [4]. In the next step, homogenization of piezoelectric layers and homogenized FGM core (from step one) is performed. After these two operations, homogenized material properties of the beam vary through the length of the beam as a function of longitudinal coordinate x and are constant in transversal direction.

4.1 Equations for structural analysis

The structural submatrix $\mathbf{K}_{uu}^e$ for the beam element with piezoelectric layers can be expressed in a form

$$\mathbf{K}_{uu}^e = \begin{bmatrix} k_u^{'} & 0 & 0 & -k_u^{'} & 0 & 0 \\ & k_{v2}^{'} & k_{v3}^{'} & 0 & -k_{v2}^{'} & k_{v2} \\ \mathrm{S} & & k_{v33}^{'} & 0 & -k_{v3}^{'} & k_{v3} \\ & \mathrm{Y} & & k_u^{'} & 0 & 0 \\ & & \mathrm{M} & & k_{v2}^{'} & -k_{v2} \\ & & & & & k_{v23} \end{bmatrix} \tag{11}$$

where individual components contain the influence of FGM core stiffness and also the influence of piezoelectric layers stiffness. The calculation of components is identical for classical multilayer or FGM beam without piezoelectric layer and is described in [4]. Mass matrix $\mathbf{M}_{uu}^e$ can be calculated numerically using classical shape functions and homogenized density of FGM beam with piezoelectric layers.

4.2 Equations for electric analysis

Electric field intensity in piezoelectric layer is constant and for top layer can by expressed as [5, 6]

$$E_1 = -\frac{\partial \phi}{\partial y} = \frac{\phi_2 - \phi_1}{h_p} \tag{12}$$

and for bottom layer as

$$E_2 = -\frac{\partial \phi}{\partial y} = \frac{\phi_4 - \phi_3}{h_p} \tag{13}$$

Both components of electric field intensity can be written in a form

$$\mathbf{E} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = -\begin{bmatrix} 1/h_p & -1/h_p & 0 & 0 \\ 0 & 0 & 1/h_p & -1/h_p \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = -\mathbf{B}_\phi \boldsymbol{\phi}^e \tag{14}$$

For 1D problems the matrix of material properties for electric field – permittivity is reduced to only one material property ϵ^ε, but because the beam element contains two identical layers, we can write

$$\boldsymbol{\epsilon}^{\boldsymbol{\varepsilon}} = \begin{bmatrix} \epsilon^\varepsilon & 0 \\ 0 & \epsilon^\varepsilon \end{bmatrix} \tag{15}$$

Then the perimitivity submatrix has a form

$$\mathbf{K}^e_{\phi\phi} = -\int_V \mathbf{B}^T_\phi \boldsymbol{\epsilon}^\varepsilon \mathbf{B}_\phi dV = -\int_L \mathbf{B}^T_\phi \boldsymbol{\epsilon}^\varepsilon \mathbf{B}_\phi A_p dx \tag{16}$$

where A_p is cross-section of one piezoelectric layer, i.e. $A_p = bh_p$.
After some mathematical manipulations the equation (16) can be expressed as

$$\mathbf{K}^e_{\phi\phi} = \begin{bmatrix} -\frac{A_p L \epsilon^\varepsilon}{h_p^2} & \frac{A_p L \epsilon^\varepsilon}{h_p^2} & 0 & 0 \\ \frac{A_p L \epsilon^\varepsilon}{h_p^2} & -\frac{A_p L \epsilon^\varepsilon}{h_p^2} & 0 & 0 \\ 0 & 0 & -\frac{A_p L \epsilon^\varepsilon}{h_p^2} & \frac{A_p L \epsilon^\varepsilon}{h_p^2} \\ 0 & 0 & \frac{A_p L \epsilon^\varepsilon}{h_p^2} & -\frac{A_p L \epsilon^\varepsilon}{h_p^2} \end{bmatrix} \tag{17}$$

4.3 Coupling of structural and electrical analysis

Piezoelectric material properties express coupling between mechanical and electrical field - matrices $\mathbf{e}$ or $\mathbf{d}$. The relationship between these two matrices can be expressed by elasticity matrix. In 1D problem in $x - y$ plane (in index notation $x_1 - x_2$) we have only one material property – e_{21} or d_{21}, where index 2 represents direction of piezoelectric layer polarization and also the direction of electric field intensity vector and index 1 defines direction of mechanical deformation. The relationship between these two quantities is reduced to expression $e_{21} = d_{21}E_p$ [7, 8], where E_p is Young modulus of elasticity of piezoelectric material. In reality, relationship between matrices $\mathbf{e}$ and $\mathbf{d}$ is more complicated and the quantity e_{21} computed from matrix $\mathbf{d}$ and elastic matrix for 3D system and the quantity e_{21} computed from d_{21} and E_p have different values. Therefore if we have quantities e_{21} and d_{21} computed from matrix relationship, it is better to used them then simplified relationship.
Piezoelectric material properties of both piezoelectric layers are defined as

$$\mathbf{e} = \begin{bmatrix} e_{21} & 0 & -ye_{21} \\ e_{21} & 0 & -ye_{21} \end{bmatrix} \tag{18}$$

The expression $\mathbf{e}^T\mathbf{E}$ defines mechanical stress caused by piezoelectric effect. In the beam elements, internal quantities are not mechanical stress but internal forces and moments, then the first and the third column of matrix (18) multiplied by corresponding components of $\mathbf{B}_u$ and $\mathbf{B}_\phi$ as well as corresponding components of displacement $\mathbf{u}$ and potential $\boldsymbol{\phi}$ represents axial forces and bending moments, respectively.

Description of piezoelectric behaviour by e_{21} is more suitable for sensor equation – matrix $\mathbf{K}^e_{\phi u}$, description by d_{21} is more suitable for actuator equation – matrix $\mathbf{K}^e_{u\phi}$, i.e.

$$\mathbf{e} = \begin{bmatrix} d_{21}E_p & 0 & -yd_{21}E_p \\ d_{21}E_p & 0 & -yd_{21}E_p \end{bmatrix} \tag{19}$$

Using equations (18) and (19) we can write piezoelectric coupling submatrices in form

$$\mathbf{K}^e_{u\phi} = \begin{bmatrix} -\frac{A_p d_{21} E_p}{h_p} & \frac{A_p d_{21} E_p}{h_p} & -\frac{A_p d_{21} E_p}{h_p} & \frac{A_p d_{21} E_p}{h_p} \\ 0 & 0 & 0 & 0 \\ \frac{A_y d_{21} E_p}{h_p} & -\frac{A_y d_{21} E_p}{h_p} & \frac{A_y d_{21} E_p}{h_p} & -\frac{A_y d_{21} E_p}{h_p} \\ \frac{A_p d_{21} E_p}{h_p} & -\frac{A_p d_{21} E_p}{h_p} & \frac{A_p d_{21} E_p}{h_p} & -\frac{A_p d_{21} E_p}{h_p} \\ 0 & 0 & 0 & 0 \\ -\frac{A_y d_{21} E_p}{h_p} & \frac{A_y d_{21} E_p}{h_p} & -\frac{A_y d_{21} E_p}{h_p} & \frac{A_y d_{21} E_p}{h_p} \end{bmatrix} \tag{20}$$

and

$$\mathbf{K}^e_{\phi u} = \begin{bmatrix} -\frac{A_p e_{21}}{h_p} & 0 & \frac{A_y e_{21}}{h_p} & \frac{A_p e_{21}}{h_p} & 0 & -\frac{A_y e_{21}}{h_p} \\ \frac{A_p e_{21}}{h_p} & 0 & -\frac{A_y e_{21}}{h_p} & -\frac{A_p e_{21}}{h_p} & 0 & \frac{A_y e_{21}}{h_p} \\ -\frac{A_p e_{21}}{h_p} & 0 & \frac{A_y e_{21}}{h_p} & \frac{A_p e_{21}}{h_p} & 0 & -\frac{A_y e_{21}}{h_p} \\ \frac{A_p e_{21}}{h_p} & 0 & -\frac{A_y e_{21}}{h_p} & -\frac{A_p e_{21}}{h_p} & 0 & \frac{A_y e_{21}}{h_p} \end{bmatrix} \tag{21}$$

where parameter A_y represents first moment of cross-section of piezoelectric layer

$$A_y = \frac{1}{2} A_p (h + h_p) \tag{22}$$

4.4 FEM equations for the beam element with piezoelectric layers

FEM equations for beam element with piezoelectric layers and FGM core for transient analysis have classical form

$$\begin{bmatrix} \mathbf{M}^e_{uu} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}^e \\ \ddot{\boldsymbol{\phi}}^e \end{bmatrix} + \begin{bmatrix} \mathbf{C}^e_{uu} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}}^e \\ \dot{\boldsymbol{\phi}}^e \end{bmatrix} + \begin{bmatrix} \mathbf{K}^e_{uu} & \mathbf{K}^e_{u\phi} \\ \mathbf{K}^e_{\phi u} & \mathbf{K}^e_{\phi\phi} \end{bmatrix} \begin{bmatrix} \mathbf{u}^e \\ \boldsymbol{\phi}^e \end{bmatrix} = \begin{bmatrix} \mathbf{F}^e \\ \mathbf{Q}^e \end{bmatrix} \tag{23}$$

where individual submatrices are defined by (11), (17), (20) and (21), vector of nodal unknowns is defined as

$$\begin{bmatrix} \mathbf{u}^e \\ \boldsymbol{\phi}^e \end{bmatrix} = \begin{bmatrix} u_i & v_i & \varphi_i & u_j & v_j & \varphi_j & \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{bmatrix}^T \tag{24}$$

and vector of nodal loads is defined as

$$\begin{bmatrix} \mathbf{F}^e \\ \mathbf{Q}^e \end{bmatrix} = \begin{bmatrix} F_{xi} & F_{yi} & M_i & F_{xj} & F_{yj} & M_j & Q_1 & Q_2 & Q_3 & Q_4 \end{bmatrix}^T \tag{25}$$

where Q_1, Q_2, Q_3 and Q_4 are electric charge on electrodes 1, 2, 3 and 4, respectively.

5 Conclusions

The paper presents new beam finite element with piezoelectric layers, where core of the beam can be made of FGM materials. Such combination of materials is very attractive for mechatronic applications, because material composition of FGM core can be optimized for design stress state and deformation can be controlled by voltages on electrodes. The beam finite element can be used for analysis of such systems very effectively and accurately.

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References

[1] MOHEIMANI, S.O.R. and FLEMING, A.J. *Piezoelectric Transducers for Vibration Control and Damping*. Springer, 2006. ISBN 978-1-84628-332-1.

[2] BALLAS, R.G. *Piezoelectric Multilayer Beam Bending Actuators*. Springer, 2007. ISBN 978-3-540-32642-7.

[3] BARNETT, A.R., PEELAMEDU, S.M., DUKKIPATI, R.V. and NAGANATHAN, N.G. Finite Element Approach to Model and Analyze Piezoelectric Actuators. *JSME International Journal*. 2001, **44**, 476–485.

[4] KUTIŠ, V., MURÍN, J., BELÁK, R., PAULECH, J. Beam element with spatial variation of material properties for multiphysics analysis of functionally graded materials. *Computers & Structures*. 2011, **89**, 1192–1205.

[5] BRUANT, I., COFFIGNAL, G., LENE, F., VERGE, M. Active control of beam structures with piezoelectric actuators and sensors: modeling and simulation. *Smart Material Structures*. 2001, **10**, 404–408.

[6] LUO, Q., TONG, L. An accurate laminated element for piezoelectric smart beam including peel stress. *Computational Mechanics*. 2004, **33**, 108–120.

[7] PREUMONT, A. *Mechatronics: Dynamics of Electromechanical and Piezoelectric Systems*. Springer, 2006. ISBN 978-1-4020-4696-4.

[8] PREUMONT, A. *Vibration Control of Active Structures*. Kluwer, 2002. ISBN 978-94-007-2033-6.