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Solution of bending and contact problems for Gao beam using control variational method

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This contribution deals with a nonlinear beam model which was published by Gao in [1] and it is known as the *Gao beam*. Let us denote E the Young's modulus, I the area moment of inertia, q the distributed transverse load, ν the Poissons ratio, L the length of the beam, $2\mathrm{h}$ its height, while its width is b and it is considered as a unit. An axial force P is considered acting at the point $x = \mathrm{L}$. The Gao beam model is described by the fourth-order differential equation

$$EI\,w'''' - E\alpha(w')^2 w'' + P\mu\, w'' = f \qquad \text{in } (0, \mathrm{L}),$$

where w is an unknown deflection of the beam and

$$I = \frac{2}{3}\mathrm{h}^3\mathrm{b}, \quad \alpha = 3\mathrm{hb}(1-\nu^2), \quad \mu = (1+\nu)(1-\nu^2), \quad f = (1-\nu^2)q.$$

The axial load P is constant, $P > 0$ causes compression of the beam and $P < 0$ tension. We will focus on a contact problem for Gao beam and foundation which is situated under the beam. The gap between them is described by the function $\mathrm{g} \leq 0$. For simplicity it will be considered constant. The foundation is assumed to be deformable and governed by the Winkler one-parametric model with a foundation modulus k_F. The Gao beam equation is now modified in the following way

$$EI\,w'''' - E\alpha(w')^2 w'' + P\mu\, w'' = f + T(w) \qquad \text{in } (0, \mathrm{L}),$$

where $T(w)$ represents contact forces between the beam and the foundation and $T(w) = c_F(\mathrm{g} - w)^+$, with $c_F = (1-\nu^2)k_F$ and $v^+(x) = \max\{0, v(x)\}$. The variational formulation of the considered contact problem reads

$$\begin{cases} \text{Find } w^* \in V \text{ such that} \\ \Pi(w) = \min\limits_{v \in V} \Pi(v), \end{cases} \tag{1}$$

where $\Pi(v)$ is the functional of total potential energy in the form

$$\Pi(v) = \frac{1}{2}\int_0^{\mathrm{L}} EI\,(v'')^2 \mathrm{d}x + \frac{1}{12}\int_0^{\mathrm{L}} E\alpha(v')^4 \mathrm{d}x - \frac{1}{2}\int_0^{\mathrm{L}} P\mu(v')^2 \mathrm{d}x - \\ - \int_0^{\mathrm{L}} fv\,\mathrm{d}x + \frac{1}{2}\int_0^{\mathrm{L}} c_F\left((\mathrm{g} - v)^+\right)^2 \mathrm{d}x$$

and V is the space of kinematically admissible displacements.

The minimization problem (1) will be solved by using the control variational method. The main idea is to transform the problem into another one which is easier to solve. It is motivated by [4], where the contact of a cantilever Euler-Bernoulli beam with an obstacle was described. We will extend this conception for nonlinear Gao beam and for four different types of boundary conditions by introducing three different transformations. Each of them consists of several parts, firstly the transformation of variable and the loading function, secondly the definition of the state equation and finally the transformation of the total potential energy functional. For each transformation of variable the Lagrangian can be constructed and the corresponding saddle point equations enable us to define control variable u and state problem. Both of them are used in transformation of functional $\Pi(v)$, so as a result we have a new functional $J(w, u)$. After this process we are able to define a new problem, so called *optimal control problem*

$$\begin{cases} \text{Find } u^* \in U_{ad} \text{ such that} \\ J(w(u^*), u^*) = \min\limits_{u \in U_{ad}} J(w(u), u), \\ \text{where } w(u) \text{ solves the state problem} \\ \text{together with prescribed boundary conditions} \\ \text{and for control value } u \in U_{ad}. \end{cases}$$

The set of admissible controls is defined by

$$U_{ad} = \left\{ u \in L^2((0, L)) : |u(x)| \le C \text{ a.e. in } (0, L) \right\},$$

where constant $C > 0$ is big enough. Under some assumptions the optimal control problem has the unique solution u^* and the function $w^* := w(u^*)$ solves the corresponding variational problem, see [2, 3].

The numerical realization of the optimal control problem consists of evaluation of the state problem and simultaneous minimization of the functional. State problem will be solved by using finite element method and will not make any problems. For minimization process it will be used *conditioned gradient method*. For a given control value u^k and computed state $w^k := w(u^k)$, the next iteration u^{k+1} is found by determining a descent direction and a suitable step size. The descent direction will be chosen as an anti-gradient of $J(w, u)$, which will be evaluated by means of the adjoint problem technique [5].

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