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PROPOSAL OF PROBABILITY MEASURES SUITABLE FOR ACCOUNTING DATA BASED BANKRUPTCY MODELS

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Abstract: Financial distress and bankruptcy modeling represents large and important field of economic research topics. This paper concerns with general structure of score oriented models in form of linear or affine forms built upon data available from standard accounting reports. For illustration purposes, we present well-known models, i.e. Altman, Neumaier, Ohlson, and Zmijewski one. However, the paper is focused mainly to probabilistic framework, which is getting more actual attention in practice for company financial health quantitative analysis, now. At first, the logit and probit transformations used for Ohlson, and Zmijewski model, respectively, are presented, as well as one model for calculation of probability of company default based upon Z-score values. Our contribution to this field is presented in the central part of the paper. We assume the score value to be an outcome of a random variable, which is introduced by its cumulative distribution function. This function is constructed systematically as a piecewise smooth function collected from three branches. Two of them, there are tail functions, belong to exponencial family. Whereas the central branch is specified as a linear function. At each joint, which is related to model grey zone bound, point smoothness of two adjacent branches is maintained by specific boundary conditions applied. Several cumulative distribution functions are presented, together with their probability density functions. Next, we select a SME ranked company, and apply the proposed technique to company survival analysis, as well as an analysis of company default probabilities. All calculations were performed by sw Mathematica.

Keywords: Bankruptcy models, default probability, probability distributions, score oriented models, survival probability.

JEL Classification: C16, D21, G33

INTRODUCTION

It is well-known that bankruptcy is a very actual subject in the financial world. Bankruptcy models and closely related valuation and default prediction models represent very important topics in corporate finance. The main goal of such models is to gauge financial health of company and evaluating its financial distress in order to detect a danger of going bankrupt. There is available a huge number of literature dealing with such topics. In general, there are two main directions thereon. The first and older one is based on traditional processing of accounting data from financial statements, calculating various ratios and computing one or more aggregated objective indicators. However, there is never ending debate about the validity of financial statements and their data with respect of mirroring actual financial health of company and measuring its financial distress. The second approach, which is newer one, is based upon an idea to represent the company by a real option and using market available data and the well-developed methods of financial option pricing to calculate default probability. The present paper is focused on the probabilistic approach and brings new form of cummulative distribution function family suitable for traditional score-oriented models.

1. SOME BANKRUPTCY PREDICTION MODELS AND THEIR SHADOW ZONES

The generally accepted pioneering works in bankruptcy prediction models based on linear discriminant analysis are papers published by Altman (1968), and Beaver (1966), respectively. The results of their studies issues bankruptcy models based on accounting data. Since the sixties of the past century, one can find an intensive utilization of traditional models as well as development of new ones, which use different financial and economic ratios and weights. Well-known consensus says that main ratios have to measure liquidity, leverage and profitability of a firm. In general, when the liquidity is low, profitability is low, and leverage is high, the probability of bankruptcy increases. The differences between these accounting-based models are the explanatory variables, available sets of firm-oriented data, and statistical techniques that are used for processing of corresponding data sets. The main advantage of ratios over the financial statement figures is evident – ratios are dimensionless quantities on the contrary to figures, which are money unit dependent ones. For the number of explanatory variables in the model, a trade-off must be made. If there are too few variables the explanatory power of the model can be low. On the contrary, if there are too many variables, the model can be prone to multi-collinearity problems.

An useful abstract interpretation of financial ratios which can be constructed from firm's financial statements suggests their identification with points in some multi-dimensional space. Let denote $n = C(k,2) = k(k-1)/2$, the number of different ratios built from financial statement containing k data entries in general. Hence, all information contained in financial statement issued for a period (e.g. one year) constitutes, in geometrical sense, a point in R^n. Therefore, we can simply understand any bankruptcy model based upon accounting data as a mapping F, expressed by Eq. (1) in general, from m-dimensional subspace $R^m \subset R^n$, with $m < n$, or more likely $m << n$, into R, which represents a set of real numbers.

$$F : R^m \rightarrow R, \quad \text{as a point mapping } F(\mathbf{x}): \mathbf{x} \rightarrow \rho,$$

$$\mathbf{x} = (x_1, x_2, ..., x_m)^T \in R^m , \ \rho \in R, \qquad (1)$$

Usually, since the multivariate linear discriminant analysis is used for construction of bankruptcy models, the mapping F takes the linear form. It can be expressed by scalar product of two vectors, simply. However, sometimes an additive constant appears too, thus forcing F to take analytically an affine form. Both possibilities are given in Eq. (2).

$$(\mathbf{c}, \mathbf{x}) , \quad \mathbf{c} = (c_1, c_2, ..., c_m)^T, \ \mathbf{x} = (x_1, x_2, ..., x_m)^T$$

$$c_0 + (\mathbf{c}, \mathbf{x}) \qquad (2)$$

where $\mathbf{c}$ is m-dimensional weight vector which is specific for each particular model, c_0 is real constant in affine form specific for each model as well, vectors **c, x** belong to R^m , and symbol ^{T} denotes transposition as usual.

At present, there is a lot of bankruptcy models published which fit into the form (2). The only difference among them is that they assume to use different subspaces R^m from the space of financial ratios R^n in general, and hence the different weight vectors $\mathbf{c}$, too. For the illustration of such approach, we present four well-known models which are also very frequently used in practice – the Altman model known as Z-score model as well, the Neumaiers' model IN05, the Ohlson model denoted by O-score, and finally Zmijewski model denoted, say Q-score. The last two models are defined by affine forms and provide probabilities of company defaults. On the contrary, the first two models, which are calculated by corresponding scalar products, just identify financial position of firm in relation to a shadow zone. The more details about these models can be found in Altman (1968), Beaver (1966), Kislingerová and Hnilica (2008, Chap.7), Kubíčková and Kotěšovcová (2006), Neumaierová and Neumaier (2005), Ohlson (1980), Sedláček (2009), and Vochozka (2011), whereas another Zmijewski model is given in Zmijewski (1984).

Let denote the particular models as follows: Z-score (${}_{A}\mathbf{c}$,${}_{A}\mathbf{x}$), index IN05 (${}_{N}\mathbf{c}$,${}_{N}\mathbf{x}$), O-score (${}_{O}\mathbf{c}$,${}_{O}\mathbf{x}$), and Q-score (${}_{Z}\mathbf{c}$,${}_{Z}\mathbf{x}$), and their corresponding subspaces as ${}_{s}R^{m}$, where s = A, N, O, Z, denotes the model.
Application of bankruptcy model in practice follows two steps in general. The first one is calculation of the indicator value, say $\gamma(\mathbf{x}) = (\mathbf{c},\mathbf{x})$, for the given values of vector $\mathbf{x} \in R^{m}$. The second one is the test if the calculated value belongs to a grey zone or not, which reflects the financial health of the company and can indicate its position against a prospective bankruptcy. The grey zone of model is given as a finite interval [r, s] on R, which is specific for particular model as well. Denoting the grey zone D one can summarize these two steps from mathematical point of view as follows:

- Calculation of value of linear mapping $\gamma : R^{m} \rightarrow R$ at given vector $\mathbf{x}$,
- Test if $\gamma(\mathbf{x}) \in D$ then company heads serious threats of bankruptcy.

Prof. Altman published several models since his seminal paper from 1968, which differ slightly each other either in selection of ${}_{A}R^{m}$ and appropriate weights or grey zones ${}_{A}D$ or both. We have chosen two from the family of his models. The similar situation holds for Neumaiers' models. At present, there exist a family of such models, and we have selected just one for illustration. The models are summarized in the scheme bellow:

Altman model (version 1983): $Z = \gamma({}_{A}\mathbf{x}) = ({}_{A}\mathbf{c},{}_{A}\mathbf{x})$,

${}_{A}\mathbf{c} = (0.717, 0.847, 3.107, 0.420, 0.998)^{T}$, ${}_{A}\mathbf{x} = ({}_{A}x_1, {}_{A}x_2, ..., {}_{A}x_5)^{T} \in {}_{A}R^{5}$,

${}_{A}x_1$ = WC/TA, ${}_{A}x_2$ = EAR/TA, ${}_{A}x_3$ = EBIT/TA, ${}_{A}x_4$ = VE/TL, ${}_{A}x_5$ = S/TA,

WC – working capital, TA – total assets, EAR – retained earnings, EBIT – earnings before interest and taxes, VE – market value of equity, S – sales, TL – total liabilities,

${}_{A}D$ = [1.20, 2.90] – grey zone for Altman model (version 1983).

Altman model (version 1995): ${}_{95}Z = \gamma({}_{A95}\mathbf{x}) = ({}_{A95}\mathbf{c},{}_{A95}\mathbf{x})$,

${}_{A95}\mathbf{c} = (6.56, 3.26, 6.72, 1.05)^{T}$, ${}_{A95}\mathbf{x} = ({}_{A95}x_1, {}_{A95}x_2, ..., {}_{A95}x_4)^{T} \in {}_{A95}R^{4}$,

${}_{A95}x_1 = {}_{A}x_1$, ${}_{A95}x_2 = {}_{A}x_2$, ${}_{A95}x_3 = {}_{A}x_3$, ${}_{A95}x_4 = {}_{A}x_4$,

${}_{A95}D$ = [1.20, 2.60] – grey zone for Altman model (version 1995).

Neumaiers' model (version 2005): $N = \gamma({}_{N}\mathbf{x}) = ({}_{N}\mathbf{c}, {}_{N}\mathbf{x})$,

${}_{N}\mathbf{c} = (0.13, 0.04, 3.97, 0.21, 0.09)^{T}$, ${}_{N}\mathbf{x} = ({}_{A}x_1, {}_{A}x_2, ..., {}_{A}x_5)^{T} \in {}_{N}R^{5}$,

${}_{N}x_1$ = TA/FC, ${}_{N}x_2$ = EBIT/CI, ${}_{N}x_3$ = EBIT/TA, ${}_{N}x_4$ = S/TL, ${}_{N}x_5$ = CA/CL,

FC – foreign capital, CI – interest of costs,

${}_{N}D$ = [0.90, 1.60] – grey zone for Neumaiers' model (IN05).

Ohlson model: $O = \gamma({}_{O}\mathbf{x}) = {}_{O}c_0 + ({}_{O}\mathbf{c}, {}_{O}\mathbf{x})$,

${}_{O}\mathbf{c} = (-0.41, 6.03, -1.43, 0.08, -2.37, -1.83, 0.285,-1.72,-0.52)^{T}$,

${}_{O}c_0$ = -1.32 is additive constant in affine form, ${}_{O}\mathbf{x} = ({}_{O}x_1, {}_{O}x_2, ..., {}_{O}x_9)^{T} \in {}_{O}R^{9}$,

${}_{O}x_1$ = ln(TA/GNP_PI), ${}_{O}x_2$ = TL/TA, ${}_{O}x_3$ = WC/TA, ${}_{O}x_4$ = CL/CA,

${}_{O}x_5$ = NI/TA, ${}_{O}x_6$ = FFO/TL, ${}_{O}x_7$ = if (EAR in the last 2 years < 0) then 1 else 0,

${}_{O}x_8$ = if (TL > TA) then 1 else 0, ${}_{O}x_9 = (EAR_t – EAR_{t-1})/(|EAR_t| + |EAR_{t-1}|)$,

GNP_PI – gross national product price index, CL – current liabilities, CA – current assets, FFO – funds from operations, NI – net income.

Zmijewski model: $Q = \gamma({}_Z\mathbf{x}) = {}_Zc_0 + ({}_Z\mathbf{c}, {}_Z\mathbf{x})$,

${}_Z\mathbf{c} = (-4.5, 5.7, 0.004)^T$,

${}_Zc_0 = -4.3$ is additive constant in affine form, ${}_Z\mathbf{x} = ({}_Zx_1, {}_Zx_2, {}_Zx_3)^T \in {}_ZR^3$,

${}_Zx_1$ = NI/TA, ${}_Zx_2$ = TL/TA, ${}_Zx_3$ = CA/CL

Ranges of indicator function $\gamma(\mathbf{x})$ of models belong to R. On the R are also defined grey zones of different models in the form of closed intervals, each of which is given by its two bounds – lower bound and the upper one. Let denote such interval D = [r,s] $\in$ R, where *r* is the lower bound, and *s* the upper one. Usual interpretation of calculated value $\gamma(\mathbf{x})$ for given **x** in correspondence with company financial health is following –

- if $\gamma(\mathbf{x}) < r$ then the company suffers serious financial troubles and its bankruptcy is almost to come,
- if if $\gamma(\mathbf{x}) > s$ then the company stands in good financial condition enabling it to cover all its liabilities smoothly, so the company survives without any danger to default,
- if $\gamma(\mathbf{x}) \in [r,s]$ then the company is financially constrained in various measure and could be endangered by prospective bankruptcy.

An idea to adjoint and compare values from different bankruptcy models by mapping their grey zones each other was submitted already in Lukáš (2012). Having different grey zones one may construct linear mappings between two of them in quite natural way thus providing useful possibilities to compare corresponding issues from different bankruptcy models for one company. For example, let choose a couple ${}_A$D and ${}_N$D, and construct a mapping M: ${}_N$D $\rightarrow$ ${}_A$D, which is linear one given by Eq. (3), and using that we may map values calculated by Neumaiers' model into the grey zone of Altman model. And vice versa, we may also construct the inverse mapping M^{-1}: ${}_A$D $\rightarrow$ ${}_N$D given by Eq. (4), which maps the ${}_A$D on ${}_N$D.

$M: \xi \rightarrow \eta = \alpha_0 + \alpha_1\xi, \quad \xi \in {}_ND, \ \eta \in {}_AD,$ (3)

α_0, α_1 are determined by interpolation conditions M: ${}_Nr \rightarrow {}_Ar$, M: ${}_Ns \rightarrow {}_As$ expressing the correspondence of bounds of both grey zones.

$M^{-1}: \eta \rightarrow \xi = \beta_0 + \beta_1\eta, \quad \xi \in {}_ND, \ \eta \in {}_AD$ (4)

β_0, β_1 are determined by similar interpolation conditions $M^{-1}: {}_Ar \rightarrow {}_Nr$, $M^{-1}: {}_As \rightarrow {}_Ns$.

Using these mappings one can obtain both the values ${}_N\gamma = \gamma({}_N\mathbf{x})$ mapped into ${}_A$D and ${}_A\gamma = \gamma({}_A\mathbf{x})$ mapped into ${}_N$D. Usually, we process not a single financial statement but a sequence of them reporting time development of company during a period of time, say {1, 2, …, T} years.

Let assume, we have got time series $\{{}_A\gamma_t\}$ and $\{{}_N\gamma_t\}$, t = 1, 2, .., T, by Altman model and Neumaiers' one, respectively. Applying the mapping (3), we get $M:\{{}_N\gamma_t\} \rightarrow \{{}_{NA}\gamma_t\}$ which gives the values of IN05 index mapped into ${}_A$D. Similarly, the mapping (4) yields the sequence $\{{}_{AN}\gamma_t\}$, i.e. the values of Z-score mapped into ${}_N$D. As usual, we may construct the simplest interpolation function in form of continuous piecewise linear interpolants of the time series over the domain [1,T], which are given by expression (5).

${}_A\gamma(t)$, ${}_{NA}\gamma_t(t)$, $t \in [1,T]$, with images in ${}_A$D ,

${}_N\gamma(t)$, ${}_{AN}\gamma_t(t)$, $t \in [1,T]$, with images in ${}_N$D. (5)

These functions provide useful and interesting information relating the calculated indicators of financial health of company, especially when taking differences, the absolute differences in aprticular, and scaled differences, as defined by Eq. (6).

${}_N\Delta_A(t) = |{}_{NA}\gamma_t(t) - {}_A\gamma(t)|$, ${}_Nd_A(t) = {}_N\Delta_A(t)/({}_As - {}_Ar)$, $t \in [1,T]$,

${}_A\Delta_N(t) = |{}_{AN}\gamma_t(t) - {}_N\gamma(t)|$, ${}_Ad_N(t) = {}_A\Delta_N(t)/({}_Ns - {}_Nr)$, $t \in [1,T]$. (6)

The functions ${}_N\Delta_A(t)$ and ${}_A\Delta_N(t)$, defined as absolute values of differences, may express a variability of calculated values of two financial health indicators, here $\gamma({}_A\mathbf{x})$ and $\gamma({}_N\mathbf{x})$, compared properly each other over the whole time period [1,T].

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We may suggest that the narrower these functions each other are the better accordance of indicator values of both models is to expect. Hence, the financial health state of company may be assessed with elevated reliability prospectively. The scaled functions, i.e. ${}_{N}d_{A}(t)$ and ${}_{A}d_{N}(t)$, do not depend upon the measures of corresponding grey zones ${}_{A}D$ and ${}_{N}D$ as they are relative differences, hereby expressible in percents.

2. CONSTRUCTIONS OF PROBABILITY MEASURES

In general, when speaking about probability measure one needs to introduce probability space, first. It is also called probability triple, denoted (Ω, F, P), and it consists from three mathematical objects. A sample space is denoted Ω, F represents a set of all events, and function P assigns a probability, a real number between zero and one, to an event. In our case, the event is defined as a set of real numbers less or equal to value $\gamma(\mathbf{x})$, calculated by corresponding bankruptcy model with given $\mathbf{x}$. The well-known models of Ohlson (1980), and Zmijewski (1984) use logistic, and probit transformations to calculate company probability of default, respectively.

In logistic models, it is assumed that the errors are standard logistically distributed, which is different from probit models, where it is assumed that the errors are normally distributed. Such specifics have to be considered when calculating company probability of bankruptcy correctly.

Probability of company bankruptcy using Ohlson model is given by Eq.(7)

$$\varphi(\gamma({}_{O}\mathbf{x})) = \frac{a}{1+a} = \frac{1}{1+b} = \frac{1}{2} + \frac{1}{2}\tanh(\gamma({}_{O}\mathbf{x})/2), \quad a = \exp(\gamma({}_{O}\mathbf{x})),\ b = \frac{1}{a} = \exp(-\gamma({}_{O}\mathbf{x})) \tag{7}$$

where standard logistic function, called also standard sigmoid or S-shaped function, is given by following expression

$$\varphi(x) = \frac{\alpha}{1+\alpha} = \frac{1}{1+\beta} = \frac{1}{2} + \frac{1}{2}\tanh(x/2), \quad \alpha = \exp(x),\ \beta = \frac{1}{\alpha} = \exp(-x). \tag{8}$$

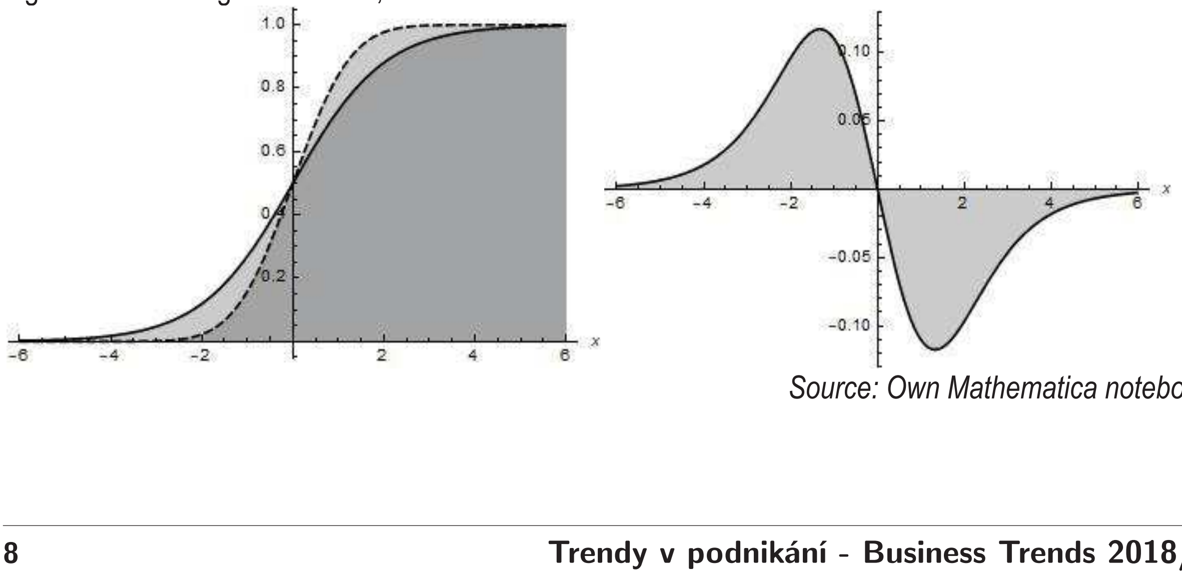
As usual, for the O-score, any value $\varphi(\gamma({}_{O}\mathbf{x})) > 0.5$ suggests the company will go default within two years.

Zmijewski model is used to predict a company bankruptcy within two years, too. The ratio used in Zmijewski Q-score was determined by probit analysis. Company with a Q-score giving $\Phi(\gamma({}_{Z}\mathbf{x})) > 0.5$ is classified as going bankrupt, otherwise if $\Phi(\gamma({}_{Z}\mathbf{x})) \leq 0.5$ then it is classified as non-bankrupt one, where $\Phi(x)$ denotes cumulative distribution function of random variable N(0,1) and is given by expression (9)

$$\Phi(x) = \int_{-\infty}^{x} \varphi(\xi)d\xi, \quad \varphi(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\xi^2}. \tag{9}$$

The difference between logistic function $\varphi(x)$ and probit one $\Phi(x)$ is that the curve of the probit regression approaches the axes more quickly than the curve of the logit regression, in other words, the function $\varphi(x)$ has heavier tails than $\Phi(x)$.

Fig. 1: Standard logistic function, and cumulative distribution function of standard normal distribution.



Source: Own Mathematica notebook

In Fig. 1, left panel, there are displayed both standard logistic function $\varphi(x)$ (full-line) and cumulative distribution function $\Phi(x)$ (dashed-line) of standard normal distribution N(0,1), whereas in the right panel, the difference function of both ones, i.e. $\Phi(x) - \varphi(x)$, $x \in [-6,6]$, is presented.

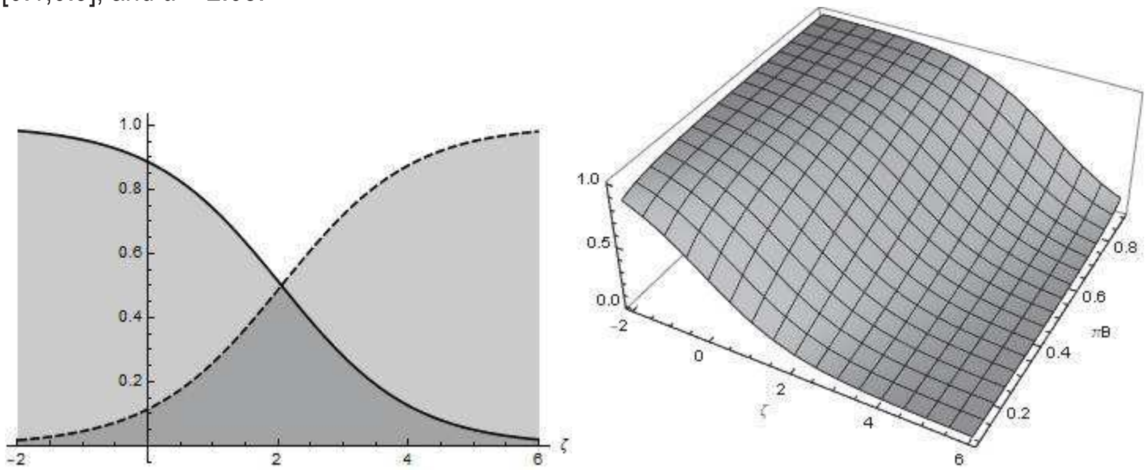
Gurný and Gurný (2013) gives also formula for calculation of probability of company bankruptcy from Altman Z-score value in following form

$$\varphi_A(\gamma({}_A\mathbf{x})) = \frac{1}{1+\frac{1-\pi_B}{\pi_B}\exp(\zeta-\alpha)} \tag{10}$$

where ζ is ether $Z = \gamma({}_A\mathbf{x}) = ({}_A\mathbf{c},{}_A\mathbf{x})$, or ${}_{95}Z = \gamma({}_{A95}\mathbf{x}) = ({}_{A95}\mathbf{c},{}_{A95}\mathbf{x})$, alternatively, α is equal to the mid-point of corresponding model grey zone, and π_B represents the prior probability of default, e.g. 0.5.

In Fig. 2, left panel, there are displayed probability functions of company bankruptcy $\varphi_A(\zeta)$ (full-line) and company survival $1 - \varphi_A(\zeta)$ (dashed-line) for Altman Z-score $\zeta=Z=\gamma({}_A\mathbf{x})=({}_A\mathbf{c},{}_A\mathbf{x}) \in [-2,6]$, and $\alpha = 2.05$, $\pi_B = 0.5$, whereas in the right panel, the surface of function $\varphi_A(\zeta;\pi_B)$ is depicted, which was calculated by formula (10), setting $\zeta \in [-2,6]$, $\pi_B \in [0.1,0.9]$,and $\alpha = 2.05$. It shows a high dependence of $\varphi_A(\zeta;\pi_B)$ upon the parameter π_B, thus showing a weak point of the formula (10), precisely.

Fig. 2: Left panel: Probability functions of company bankruptcy $\varphi_A(\zeta)$ (full-line) and company survival $1–\varphi_A(\zeta)$ (dashed-line), with $\zeta \in [–2,6]$; Righ panel: Surface of function $\varphi_A(\zeta;\pi_B)$, with $\zeta \in [–2,6]$, $\pi_B \in [0.1,0.9]$, and $\alpha = 2.05$.



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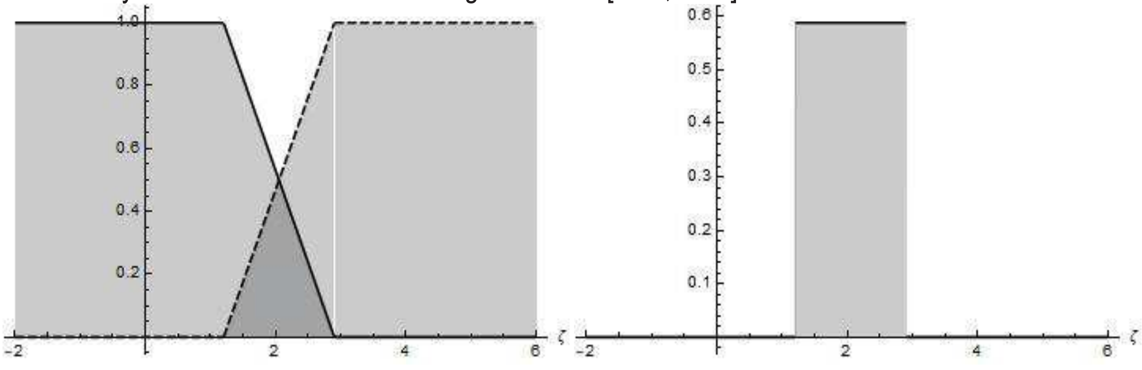
Our approach for constructing probability distribution function family suitable for traditional score-oriented bankruptcy models is based upon general idea to interpret corresponding grey zone as a domain of continuous random variable, say *g*, with uniform distribution. In case of Altman-83 model with grey zone ${}_AD = [1.20, 2.90]$, we get naïve cumulative distribution functions of company bankruptcy $\psi_A(\zeta)$ (full-line) and company survival $\Psi_A(\zeta) = 1 - \psi_A(\zeta)$ (dashed-line), which are depicted in Fig. 3, together with corresponding probability density function of a random variable *g* with uniform distribution over [1.20, 2.90].

Fig. 3: Left panel: Naive probability functions of company bankruptcy $\psi_A(\zeta)$ (full-line), and company survival $\Psi_A(\zeta) = 1 - \psi_A(\zeta)$ (dashed-line), with $\zeta \in [-2,6]$; Right panel:Probability density function of uniformly distributed random variable g over ${}_AD = [1.20, 2.90]$.



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The piece-wise defined function $\Psi_A(\zeta)$, depicted in Fig. 3, is given by Eq. (11), assuming $r = 1.20$, and $s = 2.90$, and it enables to map any value $\gamma({}_A\mathbf{x})$ into [0,1], thus providing probabilistic interpretation of Z-score, uniquely. However, it has a serious weak point. The tails are absolutely insensitive to $\gamma({}_A\mathbf{x})$ values outside the ${}_AD$.

$\Psi_A(\zeta) = \omega, \quad \omega = 0, \text{ if } \zeta \le r, \quad \omega = (\zeta - r)/(s - r), \text{ if } r < \zeta \le s, \quad \omega = 1, \text{ if } \zeta > s$. (11)

In general, for score-oriented bankruptcy models hold following convention regarding usual interpretation of company score value $\gamma(\mathbf{x})$ in grey zone $D = [r,s]$. As the higher value it gets the better is a financial health of company, which leads naturally to company survival conclusions. Hence, we shall concern with probability survival functions denoted $\Psi(x)$, primarily.

Now, we release the conditions in vicinities of grey zone bounds r, and s, introducing tail functions from exponential family, which will be defined by suitable interpolation conditions. Let r_b, s_b, are given so that $r_b < s_b$, and $r_b, s_b \in [r,s]$, in general, and at these points we assume p_b, q_b, to be given as survival probabilities, correspondingly, so that $p_b < q_b$, and $p_b, q_b \in]0,1[$.

Survival cumulative distribution function $\Psi(\xi)$ is defined as piece-wise smooth function with continuous first derivative in R, by Eq. (12), in order to provide continuous probability density function $\theta(\xi) = \Psi'(\xi)$ in R, too

$\Psi(\xi) = \omega, \quad \omega = f_1(\xi), \text{ if } \xi \le r_b, \quad \omega = p_b + (\xi - r_b)(q_b - p_b)/(s_b - r_b), \text{ if } r_b < \xi \le s_b, \quad \omega = f_2(\xi), \text{ if } \xi > s_b,$ (12)

where tail functions $f_1(\xi)$, $f_2(\xi)$, are definied as follows

$f_1(\xi) = \alpha \exp(\beta\,\xi), \quad \alpha = p_b/\exp(r_b\lambda_b/p_b), \quad \beta = \lambda_b/p_b, \quad \xi \in]-\infty, r_b]$

$f_2(\xi) = 1 - \gamma \exp(-\delta\,\xi), \quad \gamma = (1 - q_b)/\exp(s_b\lambda_b/(q_b - 1)), \quad \delta = \lambda_b/(1 - q_b), \quad \xi \in]s_b, +\infty[$

with $\lambda_b = (q_b - p_b)/(s_b - r_b)$, thus giving a slope of linear function

$$p_b + (\xi - r_b)(q_b - p_b)/(s_b - r_b) = p_b + \lambda_b(\xi - r_b), \text{ if } r_b < \xi \le s_b.$$

Derivation of the coefficients α, β, γ, δ, stems from interpolation conditions (13), applied upon tail functions $f_1(\xi)$, $f_2(\xi)$, after taking several technical steps

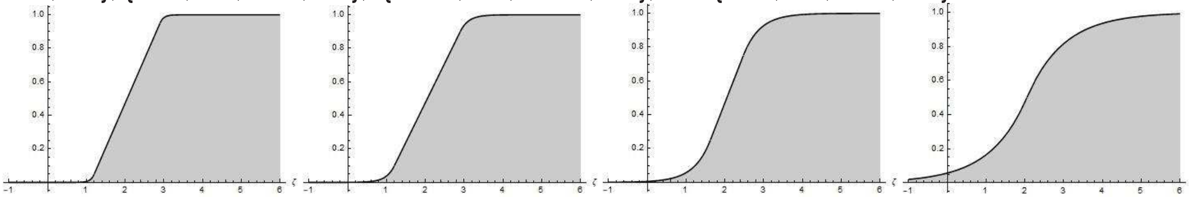
$f_1(rb) = pb$, $f_1'(rb) = df_1(rb)/d\xi = \lambda b$, $\quad f_2(sb) = qb$, $f_2'(sb) = df_s(sb)/d\xi = \lambda b$ (13)

The theoretical framework for that construction is following. The calculated value γ(x) of a score-oriented bankruptcy model with grey zone $D = [r,s]$, is assumed to be an outcome of a random variable, say η, which is defined by its cumulative distribution function $\Psi(\xi)$

$\Pr(\eta < \xi) = f_1(\xi), \text{ if } \xi \le r_b, \quad \Pr(\eta < \xi) = p_b + \lambda_b(\xi - r_b), \text{ if } r_b < \xi \le s_b, \quad \Pr(\eta < \xi) = f_2(\xi), \text{ if } \xi > s_b.$ (14)

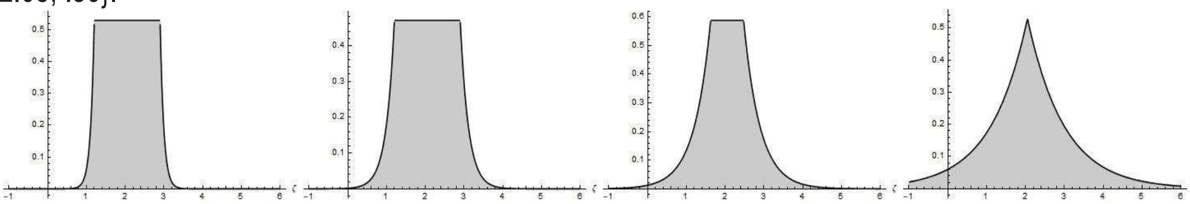
In Fig. 4, and Fig. 5, there are depicted various survival cumulative distribution functions $\Psi(\xi)$ and their probability density functions $\theta(\xi) = \Psi'(\xi)$, with different adjusted values of the interpolation data $\{r_b, p_b, s_b, q_b\}$. We emphasize that general advantage of that setting is its possibility to some extent to mitigate the errors, which are inherent ones in linear discriminant analysis (LDA) standing in background of score-oriented bankruptcy prediction models and producing bounds of corresponding grey zones with such errors. However, these ones aren't accessible by the LDA directly itself, but as usual, they are detected in practice a posteriori, prospectively.

Fig. 4: Panels from left to right: Functions $\Psi(\xi)$ with $\{r_b, p_b, s_b, q_b\}$, taking the following values {1.20, .05, 2.90, .95}, {1.20, .10, 2.90, .90}, {1.625, .25, 2.475, .75}, and {2.05, .50, 2.05, .50}.



Source: Own Mathematica notebook

Fig. 5: Panels from left to right: Probability density functions $\theta(\xi) = \Psi'(\xi)$ with $\{r_b, p_b, s_b, q_b\}$, taking the following values {1.20, .05, 2.90, .95}, {1.20, .10, 2.90, .90}, {1.625, .25, 2.475, .75}, and {2.05, .50, 2.05, .50}.



Source: Own Mathematica notebook

Sure, we checked numerically the general property of the probability density functions $\theta(\xi)$

$\int_{-\infty}^{+\infty} \theta(\xi)\mathrm{d}\xi = 1$

using Mathematica function Integrate[expr,{ξ,*a*,*b*}], where expr gives the integrand, i.e. $\theta(\xi)$, and *a*, and *b* are lower and upper bounds of the definite integral. We choose *a* = -2, and *b* = 6, and we have got the following results: 1., 1., 0.999888, 0.985504, for PDF-s $\theta(\xi)$, depicted in Fig. 5, from left to right. The numerical error of the last result was caused evidently by too short interval [*a*,*b*], and we have got the result 0.999975 when taking [*a*,*b*] = [-8,12], which is already quite acceptable.

3. APPLICATION OF PROPOSED PROBABILITY MEASURES TO SME COMPANY

We have selected one company from West Bohemian region ranked into SME range with accounting data contained in financial statement time series available for nine consecutive years.

In Tab. 1: Calculated values of Z-score, IN05 index, and η(IN05) index mapped on $_{A}$D, are listed. All calculations and graphs were performed in Mathematica, Wolfram Research Inc, with own Mathematica notebook developed.

Tab. 1: Company values of Z-score, IN05 and η(IN05) index mapped on $_{A}$D, within a period of nine years.

year	1	2	3	4	5	6	7	8	9
Z-score	1.24044	1.63358	1.85194	1.4351	1.79291	1.55467	2.19040	2.40995	1.67931
IN05	1.13656	1.45369	157586	1.34576	1.56543	1.26013	1.66072	1.70960	1.15466
η(IN05)	1.77450	2.54468	2.84138	2.28256	2.81604	2.07460	3.04746	3.16617	1.81845

Source: Own Mathematica notebook

The values η(IN05) are simply calculated from IN05 values $\{_{N}\gamma_t\}$, t = 1, 2, .., 9, using Eq. (3). In particular, this mapping takes the following form

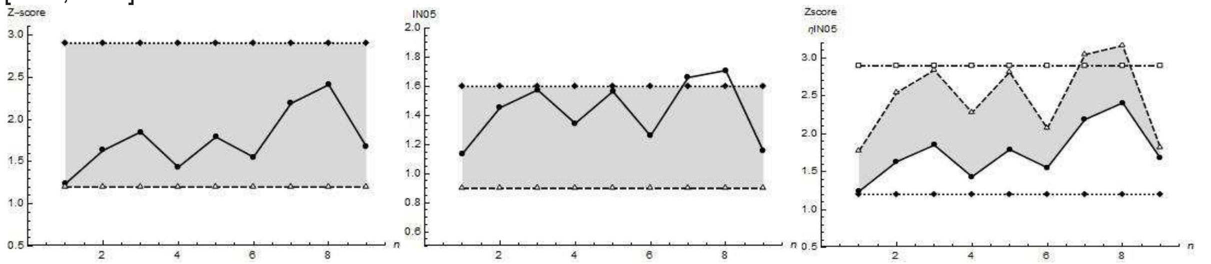
$$\eta = 1.20 + 2.42857(v - 0.9), \quad (15)$$

which maps $_{N}$D [0.90, 1.60] on $_{A}$D [1.20, 2.90], uniquely, where v substitutes any value $_{N}\gamma_t$, as an argument of linear function.

In Fig. 6, all values from Tab. 1 are plotted. The left panel presents Z-score values $\{_{A}\gamma_t\}$, t = 1, 2, .., 9, with shadowed grey zone $_{A}$D [1.20, 2.90], whereas the central panel shows all values of IN05 model, i.e. $\{_{N}\gamma_t\}$, with shadowed grey zone $_{N}$D [0.90, 1.60], respectively. The right panel depicts values of $\{_{A}\gamma_t\}$ (full-line), and η(IN05) values, i.e. $\{_{NA}\gamma_t\}$ (dashed-line), mapped into $_{A}$D, which is given by its bounds $_{A}r$ =

1.20, $_As$ = 2.90, and finally, the shadow band captures variable dispersions between Z-score values (full-line), and *η*(IN05) ones (dashed-line), for whole period of nine years.

Fig. 6: Left panel: Z-score values with $_AD$ [1.20, 2.90]; Central panel: IN05 values with $_ND$ [0.90, 1.60]; Right panel: dispersions between Z-score values (full-line), and *η*(IN05) ones (dashed-line), with $_AD$ [1.20, 2.90].



Source: Own Mathematica notebook

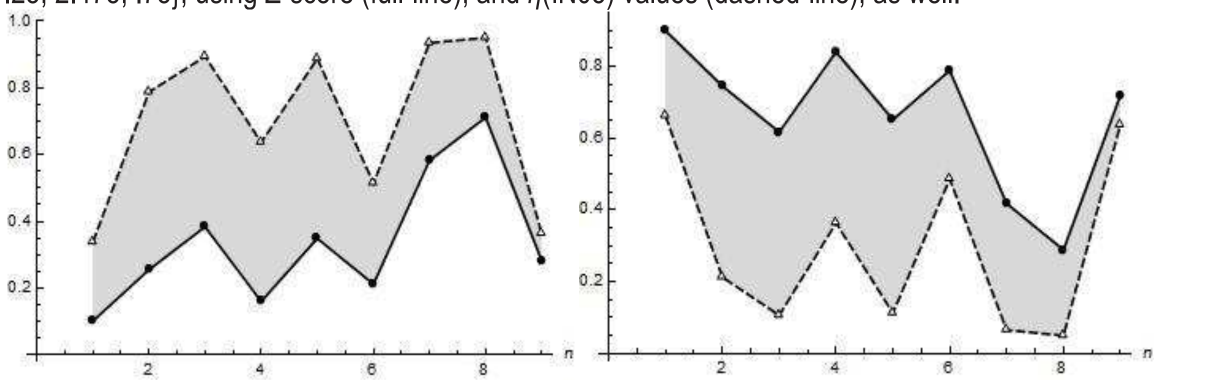
Now, we use our technique proposed above to convert these results directly into the probabilistic framework. First, we need to set data for the parameters $\{r_b, p_b, s_b, q_b\}$. Let choose $\{r_b, p_b, s_b, q_b\}$ = {1.625, .25, 2.475, .75}, so we will apply the function *Ψ*(*ξ*), displayed in Fig. 4, as the third one from the left. It means, we preserve linear conversion of the score values exactly at a half of the $_AD$, in particular at the central part [1.625, 2.475]. The score values lying outside that part will be converted by corresponding tail functions.

In figure 7, we present our results. Left panel shows company survival probability band calculated by function *Ψ*(*ξ*) with $\{r_b, p_b, s_b, q_b\}$ = {1.625, .25, 2.475, .75}, using Z-score (full-line) values, and *η*(IN05) ones (dashed-line), respectively. This pattern looks alike that one depicted in Fig. 5, the right panel, which is correct. However, we would like to promote the probabilistic interpretation of values calculated by score models over their classical one, definitively, albeit the last one being more popular in practice, yet.

In figure 7: Right panel, we plotted the company default probabilities, depicted as a band again. These probabilities are calculated simply as probabilities of complement events to company survival chances. The default probability is usually denoted PD, in literature, and we preserve such notation. Hence, we can write

$$PD(t) = 1 - \Psi(\xi). \tag{16}$$

Fig. 7: Left panel: company survival probability band calculated by function *Ψ*(*ξ*), with $\{r_b, p_b, s_b, q_b\}$ = {1.625, .25, 2.475, .75}, using Z-score (full-line), and *η*(IN05) values (dashed-line); Right panel: company default probability band calculated by function *PD*(*t*) = 1 – *Ψ*(*ξ*), with $\{r_b, p_b, s_b, q_b\}$ = {1.625, .25, 2.475, .75}, using Z-score (full-line), and *η*(IN05) values (dashed-line), as well.



Source: Own Mathematica notebook

Concluding our presentation, we would like to emphasize that both functions, either survival cumulative distribution function *Ψ*(*ξ*), and its probability density function *θ*(*ξ*) = *Ψ'*(*ξ*), or default cumulative distribution function *PD*(*t*) = 1 – *Ψ*(*ξ*), with the same probability density function *θ*(*ξ*), can be used as versatile probability measures for converting score values obtained from score-oriented bankruptcy models into probabilistic framework.

CONCLUSION

In the paper, we sketched general structure of bankruptcy models briefly, highlighting the role of linear mappings between grey zones of particular models each other. Such mappings enable as better insight into values of different score-oriented models, as they provide a good platform for their probabilistic comparisons. The main contribution of the paper, there is the development of new family of survival cumulative distribution functions containing four adjusting parameters, with quite transparent interpretations. We hope, proper setting of these parameters would enable to mitigate, prospectively, the errors which are inherent to score-oriented models, in general. Finally, the proposed technique is illustrated by bankruptcy analysis of a SME ranked company from West Bohemian region. Classical results based upon Altman Z-score and IN05 index are presented, as well as their probabilistic counterparts, survival probabilities, and default probabilities, too. Future research will be focused on a role of common ratios in indicator functions of different bankruptcy models, and on an accumulation of experience with proper parameter setting when using this probabilistic measures in practice.

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