COMPUTATIONAL MECHANICS 2019
35th conference with international participation
Srní
November 4 - 6, 2019

Sensitivity of the generalized van der Pol equation to sub- or super-harmonic resonance

C. Fischer[a], J. Náprstek[a]

[a]Institute of Theoretical and Applied Mechanics of the Czech Academy of Sciences, Prosecká 76, 190 00 Prague, Czech Republic

1. Introduction

The flow-induced vibration causes very interesting effects namely in the case of slender engineering structures. The airflow around the structure induces a wide spectrum of the non-linear aero-elastic processes. The beating effects which emerge due to vortex shedding may represent (almost) periodic load. This type of excitation is dangerous to the functionality and safety of structures. When the vortex frequency becomes close to the structure eigenfrequency, i.e., when the lock-in regime occurs, the quasiperiodic beatings constitute regular load and possibly cause significant fatigue in the material of the structure. Unfortunately, the non-linear character of slender or soft structures is often neglected in the engineering practice. This could lead to a significant underestimation of the overall response properties because the non-linear physical systems of such type are prone to the effect of the sub- or super-harmonic synchronization.

The sub- or super-harmonic synchronization effect represent cases when the driving frequency is close to integer multiples or fractions of the eigenfrequency of the structure. Due to non-linear effects, such load can induce significant vibration of the structure in the dominant eigenmode and can cause undesired or even dangerous effects.

The quasiperiodic phenomena of the resonance which occurs in the basic aero-elastic model and its stability properties were theoretically investigated by the authors in the past [3]. The recent study of the authors [4] concentrates on the stability assessments of the sub- or super-harmonic synchronization and its effect on the free component of the system response. The both works use the harmonic balance method for analytical investigation and their results depend on the fulfilment of the relevant assumptions. The authors also tried to illustrate some properties of the system using the numerical study in [1]. On the analysis of numerically obtained resonance curves, the authors shown dependence of the sub- and super-harmonic synchronization effect on the value of the excitation amplitude. It has been shown that due to the synchronization effect loses the response its beating character in a vicinity of integer multiples of the eigenfrequency of the structure and exhibits stationary response. The effect was better visible for lower excitation amplitudes, where the width of affected frequency interval was wider than it was for higher amplitudes. The super-harmonic synchronization effect (for driving frequency close to integer fractions of the dominant eigenfrequency) was much lower than the sub-harmonic one.

The present contribution studies numerically effect of the sub-harmonic, resonant and super-harmonic excitation on the frequency content of the response. This way it tries to measure effects of individual excitation modes on the character of the response, namely the influence of the sub- and super-harmonic excitation on the dominant eigenmode vibration.

The commonly used Single-Degree-of-Freedom (SDOF) or the more complicated Two-Degree-of-Freedom (TDOF) section models of a structure in the air stream represent a reasonable compromise between complexity and ability to characterise the dynamic processes. Such type of models is used often in the aerodynamic wind tunnel experiments and well serve their purpose. However, it appears that in many cases when the TDOF model is used, one of the components is dominant and, thus, the second one can be neglected. It reveals that majority of the resulting SDOF systems can be modelled by the van der Pol-Duffing or generalized van der Pol type equations or their combination adjusting degree of individual non-linear terms or their coefficients. This hypothesis is generally accepted, see, e.g., [2].

The paper is organized as follows. First, the generalized van der Pol model is described and modified to separate the forced and induced parts of the response. Then the resonance properties of the model are discussed. In Section 3, the individual sub- and super-harmonic cases are briefly mentioned. Results are summarized in the last section.

2. Mathematical model

Vibration of a slender structure in an airflow is usually modelled using the generalized van der Pol equation with a harmonic right hand side. The inclusion of the fourth order term in the description of the damping allows to better describe the lock-in regime and the corresponding most important limit cycles, from whose one is stable (attractive) and the other is unstable (repulsive). Consequently, the governing equation reads

$$\ddot{u} - (\eta - \nu u^2 + \vartheta u^4)\dot{u} + \omega_0^2 u = \omega_0^2 P \cos \omega t, \tag{1}$$

where u is the response of the system, $\omega_0^2 = K/m$ is the eigenfrequency of the associated linear system with stiffness K and concentrated mass m, η, ν, ϑ are positive coefficients of linear viscous and non-linear damping, $\omega_0^2 P$ is the amplitude of the harmonic excitation (excitation force per unit mass, frequency ω). P can be interpreted as an amplitude of the air pressure variation during vortex shedding.

Following [4], the solution in the sub- or super-harmonic cases can be written in the form

$$u = v + F_n \cos \omega t\,, \qquad F_n = P/(1 - n^2), \tag{2}$$

where $n = 2, 3, \dots$ for sub-harmonic cases and $n = 1/2, 1/3, \dots$ for super-harmonic cases. The solution u consists from a harmonic forced term ($F_n \cos \omega t$) and an auto-oscillation component v which represents the (possible) sub- or super-harmonic effects induced by the equation. The auto-oscillation part does not occur in a linear case. This approach neglects the damping, however, the introduced inaccuracy is acceptable, see the discussion in [4].

Introduction of Eq. (2) into (1) results in the following differential relation for the complementing auto-oscillation component v:

$$\ddot{v} + \omega_0^2 v(t) = \left(\eta - \nu(v + F\cos\omega t)^2 + \vartheta(v + F\cos\omega t)^4\right)(\dot{v} - F\omega\sin\omega t) + \left(\omega^2 - n^2\omega_0^2\right) F \cos\omega t\,. \tag{3}$$

3. Numerical analysis

A thorough analysis was conducted to illustrate the general properties of the theoretical system. Results obtained for particular setting of system parameters ($\eta = 1$, $\nu = 0.5$, $\gamma = 0.1$, $\vartheta = 0.025$) are described in this paragraph. The natural frequency of the oscillator is set as $\omega_0 = 1$ and thus the integer fractions and multiples of the natural frequency are easy to follow.

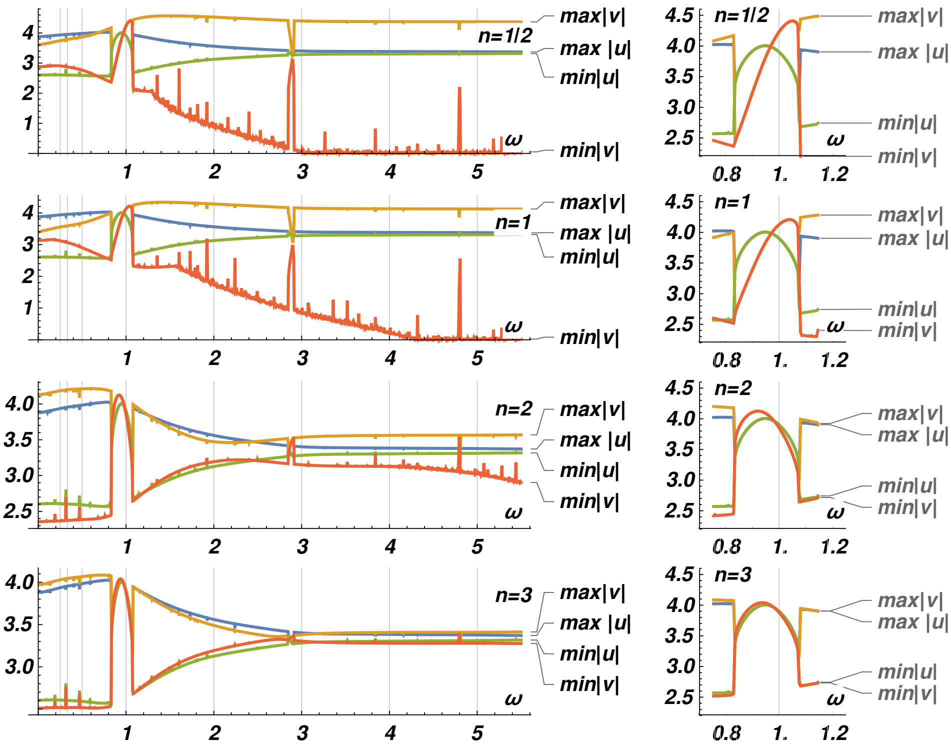

Fig. 1. Numerically obtained resonance curves of the generalized van der Pol equation response u, Eq. (1), and the corresponding auto-oscillation component v, Eq. (3), for the excitation amplitude $P = 0.75$ and $n = 1/2, 1, 2, 3$; $\eta = 1$, $\nu = 0.5$, $\gamma = 0.1$, $\vartheta = 0.025$, $\omega_0 = 1$

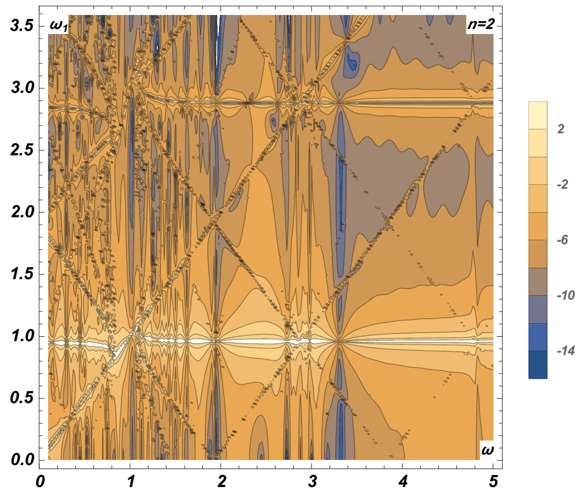

Fig. 2. Frequency properties of the auto-oscillatory part v for $n = 2$.

The resonance curves for the selected example are depicted in Fig 1. Each row shows the complete plot on the left and a detailed view of interval $(0.7, 1.3)$ on the right hand side. Four curves are shown in individual graphs. For each excitation frequency ω they represent the maximal and minimal values of the solution envelope curves for u and v separately. When both minimal and maximal envelope curves coincide, the response is stationary. The response u does not change within individual plots, as it is apparent from Eq. (1), however, character of the part v varies significantly with increasing value of n. Fig. 2 shows dependence of the frequency content of the response (vertical axis, ω_1) on the excitation frequency (horizontal axis, ω). The heat map in the logarithmic scale shows amplified responses for $\omega_1 \approx 1$ and 3 for almost all excitation frequencies (horizontal), and also narrow peaks corresponding to selected excitation frequencies (vertical lines).

The most important results of the paper measure the influence of the sub- or super-harmonic excitation to individual frequency components of the response. These characteristics are shown in Fig. 3. The excitation intervals surrounding $2\omega_0$ and $3\omega_0$, $\omega_0 = 1\ \mathrm{s}^{-1}$, are considered in the

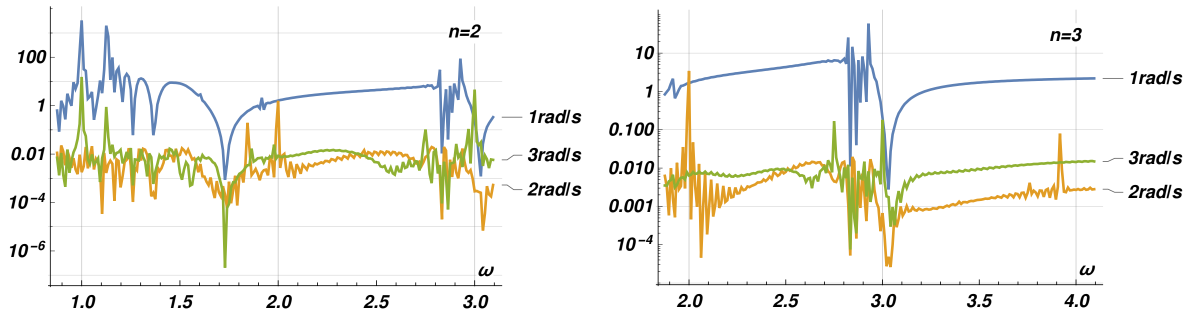

Fig. 3. Contributions of the individual frequency components 1, 2, 3 rad.s^{-1} in to the total auto-oscillation part of the response to the sub-harmonic excitation

left and right hand plots, respectively. The curves show amplitudes of the frequency components $\omega = 1, 2, 3$ rad^{-1} of the auto-oscillation response in dependence on excitation frequencies. In both plots is the component of the response in the eigenfrequency $\omega_0 = 1$ dominant (blue curves). In the left plot for $n = 2$, the response component "2rad/s" (brown curve) exhibits resonance for excitation $\omega = 2$, however, the system is apparently not very sensitive. Similar results are predicted also in [4]. More interesting is the case $n = 3$. The effect of the sub-harmonic excitation is clearly visible in all three component curves in the right hand plot. Note that the curves in Fig. 3 are approximative only because the actual resonance frequency is generally shifted down from the nominal values due to non-linear effects.

Results for $\omega = {}^1\!/_2, {}^1\!/_3, \ldots$ rad^{-1} are less apparent and are omitted due to space limitation.

4. Conclusions

The generalized van der Pol equation describes the state when the linear damping component becomes negative and the stability of the system is maintained due to non-linear effects only. Its selected resonance properties are identified numerically in the present contribution as a supplement to the approximate analytical results by the authors published in the past. The quantitative results support those theoretical, however, only a single value set was used and thus they serve for illustrative purposes only.

Acknowledgements

The kind support of the Czech Science Foundation project No. 19-21817S and of the RVO 68378297 institutional support are gratefully acknowledged.

References

[1] Fischer, C., Náprstek, J., Local stabilization of the quasiperiodic response of the generalized van der Pol oscillator, Proceedings of the conference Engineering Mechanics 2019, Institute of Thermomechanics of the Czech Academy of Sciences, Svratka, 2019, pp. 105-108.

[2] Koloušek, V., Pirner, M., Fischer, O., Náprstek, J., Wind effects on civil engineering structures, Elsevier, Amsterdam, 1984.

[3] Náprstek, J., Fischer, C., Analysis of the quasiperiodic response of a generalized van der Pol nonlinear system in the resonance zone, Computers and Structures 207 (2018) 59-74.

[4] Náprstek, J., Fischer, C., Super and sub-harmonic synchronization in generalized van der Pol oscillator, Computers and Structures 224 (2019) No. 106103, doi: 10.1016/j.compstruc.2019.106103.