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TUHOST A HAMILTONOVSKOST GRAFŮ

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TOUGHNESS AND HAMILTONICITY OF GRAPHS

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**Thesis for the award of the degree
of Doctor of Natural Sciences (RNDr.)**

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Toughness and Hamiltonicity of graphs

THESIS SUBMITTED FOR THE DEGREE OF
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Declaration

I hereby declare that the present thesis is the result of my own work and that all external sources of information have been duly acknowledged.

Adam Kabela

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Abstract

The present thesis is motivated by the study of Chvátal's t_0 -tough conjecture [36]. We recall this conjecture in the context of Hamiltonian Graph Theory. We review partial results to the conjecture, and we analyse constructions which provide lower bounds to this conjecture and to similar problems. We discuss some unifying views on the successful approaches towards this conjecture. Following the ideas presented in the thesis, we suggest possible directions for further research.

We present new results related to the t_0 -tough conjecture. In particular, we apply the hypergraph extension of Hall's theorem [2] and show that every 10-tough chordal graph is Hamilton-connected. Also we show that k -trees of toughness greater than $\frac{k}{3}$ are Hamilton-connected for $k \geq 3$ (and consequently, chordal planar graphs of toughness greater than 1 are Hamilton-connected). These results improve the results of [32, 30, 23]. In addition, we study so-called multi-split graphs and show that toughness at least 2 implies Hamilton-connectedness of these graphs; and studying certain 'cactus-like' graphs, we show that their Hamiltonicity can be decided by using Max-flow min-cut theorem [47].

Furthermore, we present constructions of graphs of relatively high toughness whose longest cycles are relatively short. Namely, we construct maximal planar graphs of toughness $\frac{5}{4}$, $\frac{8}{7}$, greater than 1, respectively; and 1-tough chordal planar graphs, 1-tough planar 3-trees, and k -trees of toughness greater than 1 for $k \geq 4$. These constructions improve the bounds presented in [56, 96, 23, 30].

The Czech and German versions of the abstract follow.

Abstrakt

Hlavním tématem předložené práce je výzkum související s Chvátalovou hypotézou [36] o tuhosti a Hamiltonovskosti grafů. V úvodu práce připomeneme tuto hypotézu v širším kontextu. Dále studujeme vybrané částečné výsledky a analyzujeme konstrukce, které dávají dolní meze vzhledem k této hypotéze a souvisejícím otázkám. Ve snaze o obecnější porozumění hledáme vzájemné souvislosti mezi vybranými přístupy k tomuto problému. V závěrečné kapitole navrhuje možná pokračování ve studiu této problematiky, která vycházejí z myšlenek předložené práce.

Autor práce předkládá nové výsledky související s Chvátalovou hypotézou. S použitím hypergrafové verze Hallovy věty [2] ukážeme, že každý 10-tuhý chordální graf je Hamiltonovsky souvislý. Dále ukážeme, že k -stromy s tuhostí vyšší než $\frac{k}{3}$ jsou Hamiltonovsky souvislé pro $k \geq 3$ (a jako důsledek dostáváme Hamiltonovskou souvislost chordálních rovinných grafů s tuhostí vyšší než 1). Zmíněná tvrzení vylepšují dříve známé výsledky [32, 30, 23]. Další částečný výsledek se týká takzvaných multisplit grafů, pro které ukážeme, že tuhost alespoň 2 zaručuje Hamiltonovskou souvislost. Zabýváme se také grafy, které mají speciální strukturu odvozenou od takzvaných kaktusů. Ukážeme, že otázka Hamiltonovskosti těchto grafů se dá rozhodovat pomocí Fordovy–Fulkersonovy věty [47] o maximálním toku a minimálním řezu.

Studujeme také konstrukce grafů, které mají relativně vysokou tuhost a zároveň relativně krátké nejdelsí kružnice. Konkrétně konstruujeme maximální rovinné grafy, jejichž tuhost je $\frac{5}{4}$ nebo $\frac{8}{7}$ nebo vyšší než 1, a dále 1-tuhé chordální rovinné grafy, 1-tuhé rovinné 3-stromy a k -stromy s tuhostí vyšší než 1 pro $k \geq 4$. Navržené konstrukce vedou k vylepšení dříve známých výsledků [56, 96, 23, 30].

Zusammenfassung

Das zentrale Thema der vorliegenden Dissertation ist die Robustheitsvermutung von Chvátal [36] und die im Zusammenhang stehende Forschung. In der Einleitung betrachten wir die Vermutung in einem breiteren Kontext. Hauptsächlich studieren wir ausgewählte Teilergebnisse zu dieser Vermutung und wir untersuchen Konstruktionsmethoden, die einige untere Schranken für diese Vermutung und für verwandte Probleme geben. Wir betrachten gegenseitige Verbindungen zwischen verschiedenen Ergebnissen und wir geben Anregungen für weitere Forschungsarbeiten.

Der Autor der Dissertation präsentiert neue Ergebnisse zu diesem Thema. Unter Benutzung von der Verallgemeinerung des Heiratssatzes für Hypergraphen [2] beweisen wir, dass jeder 10-robuste triangulierte Graph hamiltonsch zusammenhängend ist. Wir beweisen weiter, dass für $k \geq 3$ jeder k -Baum mit Robustheit größer als $\frac{k}{3}$ hamiltonsch zusammenhängend ist (und als Spezialfall sind die triangulierten plättbaren Graphen mit Robustheit größer als 1 hamiltonsch zusammenhängend). Die vorangehenden Sätze verbessern die Ergebnisse von [32, 30, 23]. Wir studieren ebenfalls sogenannte Multisplitgraphen und wir beweisen, dass jeder 2-robuste Multisplitgraph hamiltonsch zusammenhängend ist. Wir betrachten auch bestimmte „kaktusähnliche“ Graphen und wir zeigen, dass man das Hamiltonkreisproblem dieser Graphen mit Hilfe von max-flow min-cut Satz [47] entscheiden kann.

Außerdem präsentieren wir Konstruktionen Graphen relativ hoher Robustheit mit längsten Kreisen, die eigentlich relativ kurz sind. Wir konstruieren Dreiecksgraphen mit Robustheit $\frac{5}{4}$ oder $\frac{8}{7}$ oder größer als 1, und weiter 1-robuste triangulierte plättbare Graphen, 1-robuste plättbare 3-Bäume und k -Bäume mit Robustheit größer als 1 für $k \geq 4$. Diese Konstruktionen verbessern die Ergebnisse von [56, 96, 23, 30].

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Appendix B An update on non-Hamiltonian $\frac{5}{4}$ -tough maximal planar graphs

Chapter 1

Introduction

The central topic of the present thesis is the study of toughness and Hamiltonicity of graphs and the long-standing open conjecture, stated by Chvátal in 1973, which suggests that there is a constant t_0 such that every t_0 -tough graph (on at least 3 vertices) has a Hamilton cycle.

In the present chapter, the concept of toughness in graphs is recalled and the conjecture is discussed in the context of Hamiltonian Graph Theory. In Chapter 2, we review some of the known partial results on this conjecture in restricted classes of graphs, and in Chapter 3, we analyse constructions of non-Hamiltonian graphs of relatively high toughness. In Chapter 4, the author summarizes his contribution to the topic and suggests several related questions for further research. A more detailed outline of the structure of the thesis can be found in Section 1.2.

Throughout the thesis, we present new results related to Chvátal's t_0 -tough conjecture. In addition, the thesis is concluded with two papers (authored or co-authored by the author of the thesis) presenting new results in relation to this conjecture. These papers are to be found appended to the thesis. The complete list of papers on which the author worked during his Ph.D. studies is included in Section 4.3.

1.1 Hamiltonicity of graphs

For additional background, definitions and concepts in Graph Theory, we refer the reader to [22]. We start by recalling Hamiltonian properties of graphs. We recall that a *spanning* subgraph of a graph G contains all vertices of G . In particular, a spanning path (cycle) is called *Hamilton path* (*Hamilton cycle*). A graph which has a Hamilton cycle is called *Hamiltonian*, and the property 'to have a Hamilton cycle' is referred to as *Hamiltonicity* of a graph. A graph is *Hamilton-connected* if for every pair of its vertices, there is a Hamilton path between them. Clearly, every Hamilton-connected graph (on at least 3 vertices) is Hamiltonian, and every

Hamiltonian graph has a Hamilton path.

The study of Hamiltonian properties of graphs is a classical topic in Graph Theory (for instance, see [22]). Following Chvátal [36], we study relations between these properties and toughness of graphs.

1.2 Chvátal's t_0 -tough conjecture

In [36], Chvátal introduced a graph parameter called toughness, and indicated the importance of toughness for the existence of Hamilton cycles. Simply spoken, this parameter measures the vulnerability of a graph to removing vertices; it reflects the sizes of separating sets of vertices and also the number of components obtained by the removal of these sets. We recall that the *toughness* of a graph G is the minimum of the ratio of $|S|$ to $c(G - S)$, where $c(G - S)$ denotes the number of components of the graph $G - S$, and the minimum is taken over all sets of vertices S such that $c(G - S) \geq 2$. The toughness of a complete graph is defined as infinite. We say a graph is *t-tough* if its toughness is at least t .

Clearly, every Hamiltonian graph is 1-tough. In addition, Chvátal [36] stated a number of conjectures (which suggested that graphs with certain levels of toughness have a Hamilton cycle or cycles of specific properties), including Conjecture 1.1 which is referred to as Chvátal's t_0 -tough conjecture.

Conjecture 1.1. *There exists t_0 such that every t_0 -tough graph (on at least 3 vertices) is Hamiltonian.*

Meanwhile several other conjectures stated in [36] were solved, Conjecture 1.1 is still open. The progress in research on toughness in graphs (in particular, on Conjecture 1.1) is well-documented by a series of surveys, see [14, 15, 16, 7, 8, 25]. An informal introduction to Conjecture 1.1 and additional background can be found in [74]. The present thesis can be viewed as an introduction to some of the concepts and ideas related to the study of Conjecture 1.1. Also the author of the thesis would like to present new results and views related to the topic.

In the present chapter, we view Conjecture 1.1 in a wider context. In particular, in Section 1.3 we compare the concepts of toughness and connectivity of graphs and their relation to Hamiltonicity; and in Section 1.4, we note that these concepts are related more closely when restricted to $K_{1,\ell}$ -free graphs. In Sections 1.5 and 1.6, we mention additional results and problems associated with Conjecture 1.1.

In Chapter 2, we review known and new partial results on Conjecture 1.1 in various restricted classes of graphs. For instance, in interval graphs, split graphs, chordal graphs (see also Appendix A) or circular arc graphs; and in Section 2.6, we discuss the ‘underlying structure’ of these graphs and the possibilities of generalizing these classes and extending the results. In Section 2.4, we unify the view on the partial results in chordal planar graphs and k -trees. In Sections 2.7

and 2.8, we recall known partial results in classes defined by a forbidden subgraph, and we discuss their relation to similar results considering forbidden pairs of graphs. In Section 2.9, we note that certain ‘duality theorems’ are used as the main tools for proving some of the partial results, and we add more such partial results.

In Chapter 3, we analyse constructions of non-Hamiltonian graphs of relatively high toughness. In fact, these constructions provide lower bounds for several related problems. The graphs obtained by these constructions have some stronger property which implies the non-Hamiltonicity (for instance, they have no spanning subgraph whose every vertex has degree 2, or their longest cycles are short). In particular, in Section 3.2 we recall the graphs constructed in [10] which have no Hamilton path and toughness arbitrarily close to $\frac{9}{4}$ (which is the best available lower bound in relation to Conjecture 1.1); and we study this construction in Section 3.3. In Section 3.6, we present constructions of planar graphs of relatively high toughness whose longest cycles are short (see also Appendix B).

In Chapter 4, we summarize the main ideas and new results presented in the thesis. Following these ideas, we suggest possible directions for further research in Section 4.2.

We conclude this section with a note on computational complexity (for an introduction to the theory, see for instance [5]). We recall that Bauer, Hakimi and Schmeichel [11] showed that deciding whether a graph is t -tough is co-NP-complete for every rational number t . Thus, recognizing t -tough graphs is NP-hard. It is well known that deciding whether a graph is Hamiltonian is NP-complete. In fact, both of the problems remain hard even when restricted to bipartite graphs, see [71] and [72], respectively. We note that the computational complexity aspects of the problems shall not be included in the present thesis; we refer an interested reader to [8].

1.3 Toughness viewed as a strengthening of connectivity

In this section, we discuss certain ‘connectivity conditions’ in restricted classes of graphs and their relation to Hamiltonicity and toughness, and we note that these conditions can be viewed as partial results on Conjecture 1.1. We recall that a graph is *connected* if for every pair of its vertices there exists a path between them. Furthermore, a graph (on at least $k + 1$ vertices) is *k -connected* if every graph obtained by a removal of fewer than k vertices is connected. (By convention, the graph K_1 is 1-connected.) The *connectivity* of a graph is the maximum value of k for which the graph is k -connected.

Studying the newly defined parameter, Chvátal [36] observed a number of facts on toughness, for instance, the following relation of toughness and connec-

tivity.

Proposition 1.2. *Every t -tough non-complete graph is k -connected where $k \geq 2t$.*

We recall that, in some restricted classes of graphs, certain level of connectivity implies Hamiltonicity of the graphs. For instance, one of these classes is the class of planar graphs. By the classical theorem of Tutte [99], every 4-connected planar graph is Hamiltonian (in fact, Hamilton-connected by the result of Thomassen [94]). Alternatively, the Hamiltonicity can be viewed as implied by certain level of toughness since toughness is a stronger property by Proposition 1.2. In particular, if the toughness of a non-complete graph is greater than $\frac{3}{2}$, then the graph is 4-connected. Consequently, every planar graph of toughness greater than $\frac{3}{2}$ is Hamilton-connected. On the other hand, there are non-Hamiltonian $\frac{3}{2}$ -tough planar graphs. (A construction of such graphs appeared in [36], although the fact that planar graphs can be obtained was not mentioned. Later, similar constructions were considered by Harant [55] who studied additional properties of the resulting planar graphs.) More results considering non-Hamiltonian planar graphs can be found in Section 3.5.

In addition, the result of Tutte [99] was extended to graphs which can be embedded (without crossing edges) into projective plane [92]. Considering graphs embeddable onto torus, it was shown that every 4-connected such graph has a Hamilton path [93] and every 5-connected such graph is Hamilton-connected [68]. Furthermore, for graphs embeddable on surfaces of higher genus, there exists certain level of connectivity (depending on the genus) ensuring Hamiltonicity [39].

Similarly, for the class of graphs of bounded independence, certain level of connectivity implies Hamiltonian properties. We recall the results of Chvátal and Erdős [37] and we collect them in the following theorem:

Theorem 1.3. *Let G be a k -connected graph and let $\alpha(G)$ be the size of a maximum independent set of vertices of G . Then the following statements are satisfied:*

- (1) *If $k \geq \alpha(G) - 1$, then G is Hamilton-connected.*
- (2) *If $k \geq \alpha(G)$ (and G has at least 3 vertices), then G is Hamiltonian.*
- (3) *If $k \geq \alpha(G) + 1$, then G has a Hamilton path.*

Another example is the class of graphs which contain no copy of $K_{1,3}$ as an induced subgraph. These graphs are discussed in Section 1.4.

Furthermore, considering classes of graphs whose connectivity is bounded, we note that Proposition 1.2 can be restated as follows (and thus when restricted to such class of graphs, Conjecture 1.1 is satisfied trivially).

Proposition 1.4. *Let $k \geq 1$ be an integer and let \mathcal{G} be a class of graphs such that no $k + 1$ -connected graph belongs to \mathcal{G} . Then every graph of \mathcal{G} of toughness greater than $\frac{k}{2}$ (if such exists) is a complete graph (and thus Hamilton-connected).*

Proof. By Proposition 1.2, every graph of toughness greater than $\frac{k}{2}$ is either $k + 1$ -connected or a complete graph. \square

Clearly, Proposition 1.4 applies to the class of all graphs with minimum degree at most k . We remark that some of the classical graph classes are subclasses of this class; for instance, the class of k -regular graphs, or the class of graphs of tree-width at most k (see for instance [88, 21]). Similarly, Proposition 1.4 applies to the class of graphs embeddable on a surface of bounded Euler genus (see for instance [59, 79]), or the class of H -minor-free graphs for any fixed graph H (see for instance [70]).

We recall that a graph is k -colourable if one of at most k colours can be assigned to every vertex of the graph such that no two adjacent vertices have the same colour. Similarly, we note that Conjecture 1.1 is satisfied when restricted to k -colourable graphs; a similar remark appeared in [17]. (On the other hand, clearly there are k -colourable non-Hamiltonian graphs of arbitrarily connectivity.)

Proposition 1.5. *Every k -colourable graph of toughness greater than $k - 1$ is a complete graph (and thus Hamilton-connected).*

Proof. We let G denote the considered graph and we let X denote the set of all vertices coloured with colour 1 (in a colouring of G). We consider a colouring of vertices with colours $1, 2, \dots, k$ (such that vertices of the same colour are non-adjacent) which maximizes $|X|$. Clearly,

$$\frac{|V(G) \setminus X|}{|X|} \leq k - 1.$$

Thus, the toughness assumption implies that $|X| = 1$. Consequently, G is a complete graph. \square

In Chapter 2, we focus on restricted classes of graphs in which certain level of toughness implies Hamiltonicity of the graphs. In fact, all of these classes contain k -connected non-Hamiltonian graphs for arbitrarily large k .

To conclude this section, we mention that there are other parameters which measure the interconnection of vertices of a graph; for instance, the so-called scattering number (see Section 2.1) or the so-called binding number (see [102]). Simply spoken, the binding number reflects relative sizes of neighbourhoods of sets of vertices; and, in fact, graphs of certain binding number are Hamiltonian. For more details, see [102].

1.4 $K_{1,\ell}$ -free graphs and Matthews-Sumner conjecture

In this section, we discuss the special role of the graph $K_{1,\ell}$ as a forbidden subgraph in relation to Conjecture 1.1. Considering a given graph H , we recall

that a graph is H -free if it contains no copy of H as an induced subgraph (the graph H is referred to as the *forbidden subgraph*). One of the well-studied classes of graphs defined by a forbidden subgraph is the class of $K_{1,3}$ -free graphs (for instance, see [42, 34]).

We recall that for non-complete graphs, toughness at least t implies connectivity at least $\lceil 2t \rceil$ by Proposition 1.2. Considering $K_{1,3}$ -free graphs, Matthews and Sumner [78] showed that the reverse is also true.

Theorem 1.6. *Let G be a non-complete $K_{1,3}$ -free graph. Then the connectivity of G equals twice the toughness of G .*

In the same paper [78], Matthews and Sumner conjectured the following:

Conjecture 1.7. *Every 4-connected $K_{1,3}$ -free graph is Hamiltonian.*

By Theorem 1.6, a non-complete $K_{1,3}$ -free graph is 4-connected if and only if it is 2-tough. In fact, Matthews and Sumner viewed Conjecture 1.7 as related to a stronger version of Conjecture 1.1. (This stronger conjecture, considering general graphs and $t_0 = 2$, was also stated by Chvátal in [36], and it was disproved by Bauer, Broersma and Veldman [10] who constructed graphs with no Hamilton path and toughness arbitrarily close to $\frac{9}{4}$; we recall this construction in Section 3.2.) We view Conjecture 1.7 as a more specific version of Conjecture 1.1, restricted to $K_{1,3}$ -free graphs and the given value of toughness.

Conjecture 1.7 remains open, although partial results and numerous equivalent formulations of the problem are known (see the survey [28]). For instance, as a consequence of a closure concept introduced by Ryjáček [87], Conjecture 1.7 is equivalent to a seemingly weaker conjecture of Thomassen [95] which states that every 4-connected line graph is Hamiltonian. We recall that for a graph G , the *line graph* $L(G)$ is the graph encoding the adjacency of edges of G , that is, vertices of $L(G)$ represent edges of the ‘preimage graph’ G , and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G . (Clearly, every line graph is $K_{1,3}$ -free since in any graph at most two independent edges are adjacent to a common edge. Furthermore, Bermond and Meyer [19] characterized line graphs by a list of forbidden subgraphs, with $K_{1,3}$ being one of the graphs in the list.) In addition, we remark that line graphs play some role in the known constructions of non-Hamiltonian graphs of relatively high toughness, see Section 3.3.

We recall that the best known partial result on Conjecture 1.7 is due to Kaiser and Vrána [67] who showed that every 5-connected line graph of minimum degree at least 6 is Hamiltonian (in fact, Hamilton-connected). By the technique of [87], this result extends to $K_{1,3}$ -free graphs. (Consequently, every 3-tough $K_{1,3}$ -free graph is Hamilton-connected.)

Using a similar argument as in [78], we note that Theorem 1.6 can be extended to $K_{1,\ell}$ -free graphs.

Proposition 1.8. *Let $\ell \geq 3$. Every k -connected $K_{1,\ell}$ -free graph is $\frac{k}{\ell-1}$ -tough.*

Proof. We let G be a k -connected $K_{1,\ell}$ -free graph. Clearly, we can assume that G is connected and that it is a non-complete graph. We consider a set S of vertices of G such that the graph $G - S$ has at least two components, and we let C_1, C_2, \dots, C_m denote these components. Since G is k -connected, at least k vertices of S are adjacent to vertices of C_i for every $i = 1, 2, \dots, m$. On the other hand, every vertex of S has a neighbour in at most $\ell - 1$ of the components since G is $K_{1,\ell}$ -free. Consequently, we note that $|S|(\ell - 1) \geq km$, that is,

$$\frac{|S|}{m} \geq \frac{k}{\ell - 1}.$$

Thus, G is $\frac{k}{\ell-1}$ -tough. □

Interested in possible strengthening of Conjecture 1.7, Jackson and Wormald [60, 61] asked the following:

Question 1.9. *If $k \geq 4$, is every k -connected $K_{1,k}$ -free graph Hamiltonian?*

This question is still open. (We note that Question 1.9 can also be viewed as related to Theorem 1.3 since every graph whose maximum independent set is of size at most $k - 1$ is $K_{1,k}$ -free.) Furthermore, for any $k \geq 4$, it is not known whether there is a certain level of connectivity which implies Hamiltonicity of $K_{1,k}$ -free graphs.

By Proposition 1.8, this problem can be restated as a weaker version of Conjecture 1.1. Namely, for given $\ell \geq 4$, is there t_0 such that every t_0 -tough $K_{1,\ell}$ -free graph (on at least 3 vertices) is Hamiltonian?

We recall that by Proposition 1.8, certain toughness of a $K_{1,\ell}$ -free graph is implied by its connectivity; and we discuss this property of forbidden subgraphs. Clearly, the same applies for ℓK_1 -free graphs (where ℓK_1 is the graph consisting of ℓ vertices and no edges) since every ℓK_1 -free graph is $K_{1,\ell}$ -free. (Recalling Theorem 1.3, we view a graph whose maximum independent set is of size at most $\ell - 1$ as being ℓK_1 -free.) Furthermore, we note that the graphs $K_{1,\ell}$ and ℓK_1 are the only forbidden subgraphs having such property (by Proposition 1.10). Similarly as above, we recall that a graph is \mathcal{H} -free if it contains no copy of a graph from the family \mathcal{H} as an induced subgraph.

Proposition 1.10. *Let H be a graph and let $\alpha(H)$ be the size of a maximum independent set of vertices of H . Then the following statements are equivalent:*

- (1) *There exist k and $t > 0$ such that every k -connected H -free graph is t -tough.*
- (2) *The graph H is either $K_{1,\ell}$ or ℓK_1 where $\ell \geq 1$.*
- (3) *For $\alpha(H) \leq 2$, every H -free graph is a graph whose every component is a complete graph. Furthermore, every non-complete k -connected H -free graph is $\frac{k}{\alpha(H)-1}$ -tough.*

Proof. We show that (1) implies (2). Clearly, $k \geq 1$ since $t > 0$. For every k and t , we consider a graph G taken as a complete bipartite graph $K_{k,m}$ where $m > \max\{k, \frac{k}{t}\}$, and a graph G' obtained from G by adding all edges between vertices of the smaller partity. Clearly, both G and G' are k -connected but not t -tough. Furthermore, G is $\{K_1 \cup K_2, C_3\}$ -free and G' is C_4 -free. Consequently, H is $\{K_1 \cup K_2, C_3, C_4\}$ -free, that is, H is either $K_{1,\ell}$ or ℓK_1 where $\ell \geq 1$.

We show that (2) implies (3). Clearly, every ℓK_1 -free graph is $K_{1,\ell}$ -free. Considering $\ell \leq 2$, we note that every $K_{1,\ell}$ -free graph is a graph whose every component is a complete graph, and thus (3) is satisfied. For $\ell \geq 3$, we have (3) by Proposition 1.8.

Clearly, (3) implies (1) which concludes the proof. \square

Considering \mathcal{H} -free graphs where $|\mathcal{H}| = 2$, we recall that a family \mathcal{H} is referred to as a *forbidden pair*. The forbidden pairs which imply Hamiltonicity (or similar properties) of k -connected graphs are well-studied for $k = 1, 2, 3$, see for instance [43] (and $K_{1,3}$ -freeness plays an important role in this study). In particular, for Hamiltonicity and 2-connected graphs, complete characterizations of such pairs are known, see [18, 43, 76] (depending on the used definitions, the characterisations slightly differ).

Furthermore, considering forbidden pairs and Hamiltonicity of k -connected graphs for $k \geq 2$, the importance of the graph $K_{1,k+1}$ -freeness was indicated in [31] (using a similar argument as in the proof of Proposition 1.10); and it was noted that every k -connected $\{P_4, K_{1,k+1}\}$ -free graph is Hamiltonian for $k \geq 2$ (and a stronger property was shown). We remark that additional such forbidden pairs follow by the combination of Proposition 1.8 and the known results on 1-tough H -free graphs [75] (more details can be found in Section 2.7).

We conclude this section by mentioning two additional results on Hamiltonicity of $K_{1,3}$ -free graphs. Considering chordal graphs (these are graphs which are C_k -free for every $k \geq 4$), Balakrishnan and Paulraja [6] showed that every 2-connected $K_{1,3}$ -free chordal graph is Hamiltonian. In other words, a $K_{1,3}$ -free chordal graph (on at least 3 vertices) is Hamiltonian if and only if it is 1-tough. In Section 2.3, the topic of toughness and Hamiltonicity of chordal graphs is discussed in more detail. In addition, by another result of [78], the square of a connected $K_{1,3}$ -free graph (on at least 3 vertices) is Hamiltonian (and stronger results are known). The squares of graphs are discussed in the following section.

1.5 Hamiltonicity of squares of graphs

In this section, we recall the concept of the square of a graph and its relation to Hamiltonicity and toughness. We let G^2 denote the *square* of a graph G , that is, the graph on the same vertex set as G in which two vertices are adjacent if and only if their distance in G is either 1 or 2.

By the classical result of Fleischner [46], the square of every 2-connected graph is Hamiltonian (in fact, Hamilton-connected [35, 44]). Moreover, several other strengthenings of the result were presented (see for instance [45, 1]). In addition, Fleischner [45] showed that if G^2 is Hamiltonian, then G^2 is pancyclic (that is, for every vertex of G and every $\ell = 3, 4, \dots, |V(G)|$, the graph G^2 has a cycle of length ℓ containing this vertex). Considering pancyclicity and toughness, Bauer, van den Heuvel and Schmeichel [17] constructed C_3 -free graphs of arbitrarily high toughness. In addition, Alon [4] showed that there are such graphs of arbitrary girth (length of a shortest cycle).

Considering squares of graphs, Chvátal [36] showed that the toughness of G^2 is related to the connectivity of G . (Actually, some of the conjectures stated in [36] were motivated by properties of squares of graphs.)

Theorem 1.11. *The square of a k -connected graph is k -tough.*

In particular, the square of a connected graph is 1-tough (but not necessarily Hamiltonian). Studying squares of trees, Neuman [81] presented necessary and sufficient conditions for the existence of a Hamilton path between a given pair of vertices (Neuman viewed the path as an ordering of vertices in which consecutive vertices have distance at most 2 in the tree). Consequently, the characterization of trees whose square has a Hamilton cycle (Hamilton path) follows by considering a pair of vertices which are adjacent in the square of the tree (by considering an arbitrary pair of vertices). Later, these results were also proven separately. In particular, Harary and Schwenk [58] showed that the square of a tree (on at least 3 vertices) is Hamiltonian if and only if the tree is $S(K_{1,3})$ -free (where $S(K_{1,3})$ is the graph obtained by adding one pendant edge to every leaf of $K_{1,3}$). Similarly, Gould [52] characterized trees whose square has a Hamilton path.

In Section 3.6, we recall the results of [58, 52], and we use these characterizations and Theorem 1.11 in the study of non-Hamiltonian 1-tough chordal planar graphs.

1.6 Factors, walks and trestles

In this section, we consider properties somewhat similar to the Hamiltonicity of a graph; namely, the existence of k -factors, k -walks and k -trestles, and we discuss these properties in relation to toughness of a graph.

We recall that a k -factor of a graph is a spanning subgraph whose every vertex has degree k . In particular, a 1-factor is a perfect matching, and a 2-factor is a spanning subgraph whose every component is a cycle (a connected 2-factor is a Hamilton cycle). Indeed, one of the facts observed by Chvátal [36] is that (as a consequence of Tutte's matching theorem [98]) every 1-tough graph (on even number of vertices) has a perfect matching. This was generalized by Enomoto et al. [41] who showed the following:

Theorem 1.12. *For $k \geq 1$, every k -tough graph (on n vertices such that $n \geq k+1$ and kn is even) has a k -factor.*

(In particular, every 2-tough graph (on at least 3 vertices) has a 2-factor.) In fact, this result was also stated as a conjecture in [36]. In addition, Enomoto et al. [41] presented a construction of graphs which show that the result is sharp (we recall the case $k = 2$ of this construction in Section 3.1). For more details, see [41]. A reader interested in factors of graphs is referred to [3].

We recall the concept of k -walks. A *walk* in a graph is an alternating vertex-edge sequence (starting and ending with a vertex) in which every two consecutive elements are incident. A *closed walk* is a walk starting and ending with the same vertex. A k -walk is a closed walk in which every vertex of the graph is used at least once and at most k times. In particular, a 1-walk is a Hamilton cycle.

Motivated by Conjecture 1.1, Win [101] showed that for $k \geq 3$, every $\frac{1}{k-2}$ -tough graph has a spanning tree whose maximum degree is at most k (and remarked that for $k = 2$, such tree is a Hamilton path). Jackson and Wormald [60] observed that for every k , traversing (a symmetric orientation of) such tree gives a k -walk; and furthermore, that a simple modification of a k -walk yields a spanning tree whose maximum degree is at most $k + 1$. In addition, they conjectured the following strengthening of the result of [101].

Conjecture 1.13. *For $k \geq 2$, every $\frac{1}{k-1}$ -tough graph has a k -walk.*

In particular, the case $k = 2$ of Conjecture 1.13 suggests that every 1-tough graph has a 2-walk. This case was studied by Ellingham and Zha [40], and as a consequence of their result it follows that every 4-tough graph has a 2-walk. Clearly, asking about a certain level of toughness for the remaining case $k = 1$ corresponds to Conjecture 1.1.

We recall that a k -trestle is a 2-connected spanning subgraph of maximum degree at most k . Influenced by Conjecture 1.1, Tkáč and Voss [97] conjectured the following:

Conjecture 1.14. *For every integer $k \geq 2$, there exists t_k such that every t_k -tough graph (on at least 3 vertices) has a k -trestle.*

In particular, a 2-trestle is a Hamilton cycle. Thus, the case $k = 2$ of Conjecture 1.14 is precisely Conjecture 1.1. Furthermore, Conjecture 1.14 is not solved for any value of k .

In relation to the ‘connectivity conditions’ discussed in Section 1.3, we remark that for every k , there are graphs of arbitrary connectivity which have no k -factor, no k -walk and no k -trestle. For instance, we consider complete bipartite graphs $K_{n,m}$ for $m \geq n$. Clearly, these graphs are n -connected. For $m \geq n + 1$, they have no k -factor; for $m \geq kn + 1$, no k -walk; and for $2m \geq kn + 1$ no k -trestle.

Constructions of graphs of relatively high toughness without a k -walk or a k -trestle are discussed in Section 3.4. More details on walks and trestles (in particular, in relation to toughness) can be found in [91].

Chapter 2

Partial results on the t_0 -tough conjecture in restricted classes of graphs

For an overview of known results related to Conjecture 1.1, we refer the reader to the survey of Bauer, Broersma and Schmeichel [8]. The survey collects partial results on Conjecture 1.1 and also results considering toughness in relation to circumference of a graph, graph factors, and computational complexity.

In this chapter, we focus on partial results on Conjecture 1.1 in several restricted classes of graphs. We discuss known and new results, and we outline some of the ideas and proof techniques. Many of the discussed classes of graphs admit some sort of natural intersection representation (namely, interval graphs, split graphs, spider graphs, chordal graphs and circular arc graphs). In Section 2.6, we consider similar intersection representations given by an ‘underlying graph’ and a family of its connected subgraphs, and we discuss the resulting classes of graphs in relation to Conjecture 1.1. In particular, we recall the concept of H -graphs and we suggest possible generalizations of chordal graphs and circular arc graphs. In Section 2.4, we note the relation between chordal planar graphs and k -trees of relatively high toughness, and we unify the view on the partial results in these classes of graphs. In Sections 2.7 and 2.8, we recall known results in classes given by a particular forbidden subgraph, and we discuss these results in relation to forbidden pairs and Hamiltonicity of k -connected graphs (mentioned in Section 1.4). Finally, in Section 2.9 we discuss a similar nature of some of the tools which are used for proving partial results on Conjecture 1.1.

We present new partial results on Conjecture 1.1, see for instance Theorems 2.5, 2.11, 2.26, 2.27 and Corollary 2.8. In addition, we present Propositions 2.6 and 2.21 which we view as an outline to the technique used for proving Theorem 2.5.

2.1 Interval graphs and scattering number

Partial results on Conjecture 1.1 were shown in several classes of graphs, many of which admit simple intersection representations of some sort. One of these classes is the class of interval graphs.

We recall that an *interval graph* is a graph whose vertex set can be represented by a family of intervals on a straight line so that two vertices of the graph are adjacent if and only if the corresponding intervals intersect.

It is known that every 1-tough interval graph (on at least 3 vertices) is Hamiltonian. (As remarked in [32], the idea first appeared implicitly in the paper of Keil [69].) We recall that every Hamiltonian graph is 1-tough. Thus, toughness equal to 1 is the lowest bound that could possibly ensure Hamiltonicity of a graph. Moreover, there are interval graphs which are not 1-tough, but their toughness can be arbitrarily close to 1 (a construction of such graphs is shown in Figure 2.1).

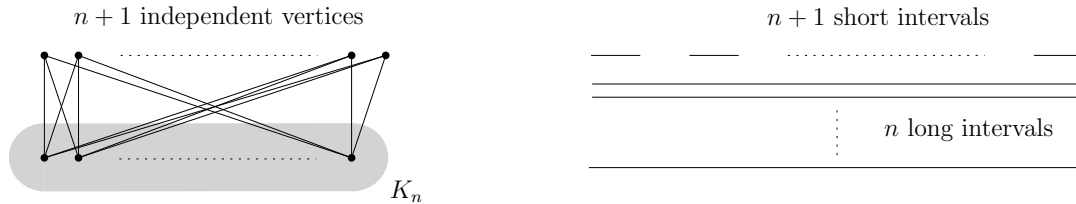


Figure 2.1: An example of a construction of interval graphs (left) and their intersection representations (right). The lines depict intervals (representing vertices of the interval graph); and two vertices of the graph are adjacent if and only if the corresponding intervals intersect (intersecting intervals are depicted as overlapping). Clearly, the interval graphs obtained by this construction have no 2-factor (and thus no Hamilton cycle), and furthermore their scattering number is 1, and with increasing n their toughness goes to 1. We note that by removing one universal vertex (that is, a vertex adjacent to every vertex of the graph) we obtain graphs which have no Hamilton path.

Considering graphs whose toughness is close to 1, we recall the definition of the scattering number. The *scattering number* of a graph G is the maximum of $c(G - S) - |S|$, where $c(G - S)$ denotes the number of components of the graph $G - S$, and the maximum is taken over all sets of vertices S such that $c(G - S) \geq 2$. The scattering number of a complete graph is defined as negative and infinite.

We note that scattering number equal to 0 corresponds to toughness equal to 1, and graphs of greater toughness are precisely graphs which have negative scattering number. So an interval graph (on at least 3 vertices) is Hamiltonian if and only if its scattering number is at most 0. The construction depicted in Figure 2.1 gives non-Hamiltonian interval graphs with scattering number 1. (We

remark that in general there are non-Hamiltonian graphs with arbitrarily small negative scattering number, for instance, see Figures 2.2, 3.4 or 3.5.) In fact, the interval graphs depicted in Figure 2.1 have a Hamilton path. This follows, for instance, from a result of Deogun et al. [38] who showed that for $k \geq 1$, every co-comparability graph of scattering number k has a spanning subgraph consisting of k disjoint paths (the class of co-comparability graphs is mentioned in Section 2.5 as a superclass of interval graphs).

The Hamiltonicity of interval graphs was studied in more detail by Broersma et al. [26] who showed that negative values of the scattering number are related to generalized Hamiltonian properties of interval graphs; namely, to the existence of a spanning p -stave and to k -Hamilton-connectivity. We recall that a p -stave is a graph consisting of p paths all of which have the same ends and (apart from the ends) are pairwise disjoint. (In particular, a spanning 1-stave is a Hamilton path, and a spanning 2-stave is a Hamilton cycle.) The results of [26] are recalled in Theorems 2.1 and 2.2.

Theorem 2.1. *An interval graph G (distinct from K_k where $k \leq p$) contains a spanning p -stave between u_1 and u_n if and only if the scattering number of G is at most $2 - p$ (where u_1 and u_n are vertices of G chosen so that the distance between the corresponding intervals is maximal).*

We recall that a graph is k -Hamilton-connected if for every set of at most k of its vertices, the removal of this set results in a graph which is Hamilton-connected. (Clearly, 0-Hamilton-connected means Hamilton-connected.)

Theorem 2.2. *For $k \geq 0$, an interval graph is k -Hamilton-connected if and only if its scattering number is at most $-(k + 1)$.*

In particular, Theorems 2.1 and 2.2 yield the following:

Corollary 2.3. *Every 1-tough interval graph (on at least 3 vertices) is Hamiltonian. Furthermore, every interval graph of toughness greater than 1 is Hamilton-connected.*

As an important ingredient of the proof of Theorems 2.1 and 2.2, Broersma et al. present an algorithm (influenced by [69]) which finds a spanning p -stave (between u_1 and u_n chosen as in Theorem 2.1) of an interval graph (for maximum p for which such stave exists). Furthermore, backtracking the algorithm leads to a cut which determines the scattering number of the graph. (In the proof of Theorem 2.2, a Hamilton path between any two given vertices is obtained by modifying a spanning 3-stave between u_1 and u_n . We remark that a similar idea is used for constructing Hamilton paths in k -trees in Section 2.4.)

To outline the idea of this algorithm, we discuss its simplified version which tries to find a spanning p -stave for given p . We view the intersection representation as a family of intervals on a line, and we view this line as going from left

to right. Considering some family of intervals, we say an interval of this family is *right-shortest* if the right-end of this interval is most to the left (that is, every interval in the family contains this right-end or a point which is more to the right).

We let I_1 denote a right-shortest of all intervals, and I_n denote an interval whose left end is most to the right. The algorithm tries to construct p disjoint sequences of intervals (without using I_1 and I_n) so that in each sequence every two consecutive intervals intersect. The construction starts with p empty sequences and proceeds iteratively. At each step, the family of last intervals of all sequences is considered and one sequence with the right-shortest last interval is chosen (the last interval of an empty sequence is viewed as a copy of I_1 , and an empty sequence is chosen with higher priority). The algorithm tries to extend this chosen sequence by adding an interval which is not yet used in any of the sequences. In particular, by adding a right-shortest interval of the family of all unused intervals which intersect the last interval of the chosen sequence. If no such interval is available, then the algorithm stops.

The obtained sequences translate to p disjoint paths (possibly empty) in the interval graph. After the algorithm stops, if every interval (except for I_1 and I_n) is used and every sequence is non-empty (one of the sequences can be empty in case I_1 intersects I_n) and the last interval of every sequence intersects I_n , then we obtain a desired spanning p -stave. For more details, we refer the reader to [26].

2.2 Split graphs and spider graphs

We recall that a graph is a *split graph* if its vertex set admits a division into two disjoint sets: one inducing a complete graph and the other inducing a graph with no edges. Among other results, Kratsch et al. [71] showed that every $\frac{3}{2}$ -tough split graph (on at least 3 vertices) is Hamiltonian. In [71], this division of vertices of a split graph is emphasized by a black-and-white colouring, by which all vertices of the set inducing a complete graph are coloured black and the remaining vertices are coloured white. The idea of the proof is to consider paths whose both ends are black, and show that there is a system of disjoint such paths covering all white vertices (this is done with an involved minimality argument). Clearly, such system of paths can be extended to a Hamilton cycle since the set of black vertices induces a complete graph.

In the same paper [71], Kratsch et al. presented split graphs with no 2-factor and toughness arbitrarily close to $\frac{3}{2}$. In Figure 2.2, we outline the construction and we present simple intersection representations of these graphs. (For a formal description of this construction, we refer the reader to [71].)

We note that split graphs are precisely intersection graphs of subtrees of a star (with the additional property that for every leaf of the star, there is at most one subtree consisting of this leaf as its only vertex).

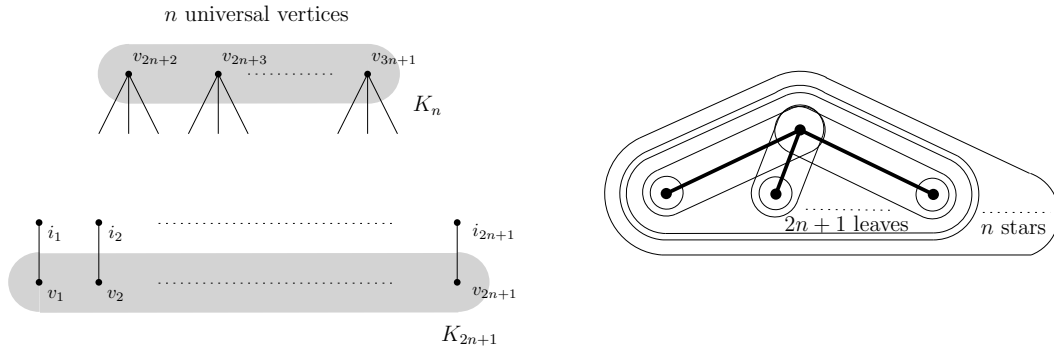


Figure 2.2: The construction of split graphs presented in [71] (left) and their intersection representations (right). The underlying graph of the representation is the star $K_{1,2n+1}$ (depicted in bold), and the ovals depict subgraphs of the star. For every leaf of the star, there is one subgraph consisting of this leaf as its only vertex and one subgraph whose only vertices are this leaf and the centre of the star; and additionally, there are n subgraphs containing all vertices of the star. Each subgraph represents a vertex of the split graph, and two vertices are adjacent if and only if the two corresponding subgraphs have a vertex in common. We note that the split graphs obtained by this construction have no 2-factor, and with increasing n their toughness goes to $\frac{3}{2}$.

In [66], split graphs are generalized by considering intersection graphs of subtrees of a tree which has at most one vertex of degree greater than 2; such trees are called *spiders* and the intersection graphs are called *spider graphs*. Clearly, the class of spider graphs is a superclass of split graphs and interval graphs.

Kaiser, Král' and Stacho [66] showed that every $\frac{3}{2}$ -tough spider graph (on at least 3 vertices) is Hamiltonian. In addition, they presented a simplified version of the argument which provides an alternative proof of the Hamiltonicity of $\frac{3}{2}$ -tough split graphs (the proof technique is quite different from [71]). In both versions of the argument, Matroid intersection theorem and Hall's marriage theorem are used as the main tools (for more details, see [66]).

Additional generalizations of split graphs are discussed in Sections 2.3, 2.8 and 2.9.

2.3 Chordal graphs

We recall that a graph is *chordal* if each of its cycles of length at least 4 has a *chord*, that is, an edge joining two non-consecutive vertices of this cycle. In other words, chordal graphs are C_k -free for every $k \geq 4$. An equivalent definition is due to Gavril [49] who showed that chordal graphs are precisely intersection graphs of subtrees of a tree. Considering this equivalent definition, we note that interval graphs, split graphs and spider graphs are chordal.

2.3. Chordal graphs

In contrast to the former classes of graphs, the tight bound of toughness ensuring Hamiltonicity is not known for chordal graphs. In [32], Chen et al. proved that every 18-tough chordal graph (on at least 3 vertices) is Hamiltonian. On the other hand, Bauer, Broersma and Veldman [10] constructed chordal graphs with no Hamilton path and toughness arbitrarily close to $\frac{7}{4}$. (We recall this construction in Section 3.2; and in addition, we present intersection representations of these graphs in Figure 3.5).

In [12], Bauer, Katona, Kratsch and Veldman studied 2-factors in a superclass of chordal graphs (in graphs whose every cycle of length at least 5 has a chord), and proved that every $\frac{3}{2}$ -tough such graph (on at least 3 vertices) has a 2-factor. The construction depicted in Figure 2.2 shows that this bound is tight. In addition, they conjectured that the tight value of toughness ensuring Hamiltonicity for chordal graphs is 2.

Conjecture 2.4. *Every 2-tough chordal graph (on at least 3 vertices) is Hamiltonian, and for every $\epsilon > 0$, there exists a $(2 - \epsilon)$ -tough non-Hamiltonian chordal graph.*

The above mentioned result of Chen et al. was improved by the author and the adviser of the present thesis who showed the following:

Theorem 2.5. *Every 10-tough chordal graph is Hamilton-connected.*

The corresponding paper [65] is enclosed in Appendix A. In the paper, we use the characterization of Gavril [49] and we represent a chordal graph by an underlying tree T and a family \mathcal{F} of its subtrees. The idea of the proof is to choose a particular intersection representation (T, \mathcal{F}) of the chordal graph, and to break the tree T into paths and assign two subfamilies of \mathcal{F} to each of these paths. The existence of a desired assignment is guaranteed by using the hypergraph extension of Hall's theorem (proved by Aharoni and Haxell [2]). To outline this idea, we present a simplified version of the argument and prove Proposition 2.6. In this simplified version, we shall use the classical Hall's marriage theorem for bipartite graphs.

Proposition 2.6. *Let G be a 4-tough chordal graph (on at least 3 vertices) which has an intersection representation (T, \mathcal{F}) such that the underlying tree T contains no vertex of degree 2, and for every leaf of T , there is a subtree in \mathcal{F} consisting of this leaf as its only vertex. Then G is Hamiltonian.*

Proof. We construct an auxiliary bipartite graph (A, B) which encodes the structure of (T, \mathcal{F}) as follows. In the partity A , there are two vertices for every edge of T ; and there is one vertex in B , for every subtree of \mathcal{F} . Two vertices are adjacent if and only if the corresponding edge of T is contained in the corresponding subtree of \mathcal{F} . We prove the following claim:

Claim 2.1. *There is a matching covering all vertices of A .*

Proof of Claim 2.1. To the contrary, we suppose that there is no such matching. By Hall's marriage theorem, there is a subset A' of A such that $|A'| > |N(A')|$ where $N(A')$ denotes the set of all vertices adjacent to a vertex of A' . We consider the corresponding subset E' of edges of T and the subfamily \mathcal{F}' of \mathcal{F} . Clearly, $|E'| > 2|\mathcal{F}'|$. The removal of all edges of E' disconnects T to $|E'| + 1$ components; and since T contains no vertex of degree 2, we observe that more than $\frac{1}{2}|E'|$ of these components contain a leaf of T . We remove from G all vertices corresponding to \mathcal{F}' , and we conclude that the resulting graph has more than $\frac{1}{4}|\mathcal{F}'|$ components, contradicting the toughness assumption of G . ■

We note that a matching covering A gives an assignment of subtrees of \mathcal{F} to the edges of the symmetric orientation of T . We consider an Euler tour in the symmetric orientation of T and the corresponding sequence of assigned subtrees of \mathcal{F} . In this sequence, every two consecutive subtrees have a vertex in common (the first and last subtrees are also considered consecutive), and we observe that this property can be preserved while adding the remaining subtrees of \mathcal{F} to the sequence. The resulting sequence of subtrees translates to a Hamilton cycle of G . □

We remark that the bound shown in Theorem 2.5 is unlikely to be tight. A better bound is known when restricted to a subclass of chordal graphs, for instance, to $K_{1,3}$ -free chordal graphs [6], spider graphs [66], or chordal planar graphs [23] (see Theorem 2.10).

We observe that a better bound can also be obtained for chordal graphs with bounded size of maximum clique. For such graphs, certain level of toughness implies that these graphs are, in fact, interval graphs (and thus Hamilton-connected).

Proposition 2.7. *Let $n \geq 3$ be an integer and let G be a K_n -free chordal graph. If the toughness of G is greater than $\frac{n-1}{3}$, then G is an interval graph.*

Proof. We choose an intersection representation (T, \mathcal{F}) of G minimizing the number of vertices of the underlying tree T . By the choice of the representation, for every leaf of T , there is a subtree in \mathcal{F} consisting of this leaf as its only vertex.

For a vertex v of T , we let \mathcal{F}_v denote the family of all subtrees of \mathcal{F} containing v . We consider a graph obtained from G by removing all vertices corresponding to the subtrees of \mathcal{F}_v , and we note that the number of components of this graph is at least the degree of v .

Since G is K_n -free, every vertex of T belongs to at most $n - 1$ subtrees. Consequently, the toughness of G implies that T contains no vertex of degree greater than 2, that is, T is a path. Thus, the members of \mathcal{F} are paths. We view T as a line and the paths of \mathcal{F} as intervals, and we conclude that G is an interval graph. □

We remark that the existence of a vertex of degree greater than 2 in the underlying tree of a particular intersection representation is similar to the concept of the asteroidal triple, which is used in the known characterization of interval graphs as a subclass of chordal graphs [73].

As a consequence of Proposition 2.7 and Corollary 2.3 (that is, every interval graph of toughness greater than 1 is Hamilton-connected), we obtain the following:

Corollary 2.8. *Let $n \geq 5$ be an integer. Every K_n -free chordal graph of toughness greater than $\frac{n-1}{3}$ is Hamilton-connected.*

In the following section, we discuss similar results in the class of k -trees which is a subclass of K_n -free chordal graphs for $k = n + 2$.

2.4 k -trees and chordal planar graphs

We present new results on long paths and toughness of k -trees and chordal planar graphs, and we note that these results are also available in the manuscript [64] (written by the author of the present thesis). This section and Section 3.6 are based on the recently submitted version of this manuscript. In the present section, we include the results considering toughness and Hamilton-connectedness of k -trees and chordal planar graphs. In Section 3.6, we include the constructions of 1-tough 3-trees and 1-tough chordal planar graphs whose longest paths are relatively short (compared to the number of vertices of the graph).

We recall that a k -tree is either the graph K_k or a graph obtained from a k -tree by choosing its complete subgraph of size k and adding a new vertex adjacent to the vertices of the chosen subgraph. In particular, every k -tree is chordal and K_{k+2} -free. Thus, by Corollary 2.8 every k -tree of toughness greater than $\frac{k+1}{3}$ is Hamilton-connected for $k \geq 3$.

Using a different technique, Broersma, Xiong and Yoshimoto [30] proved a similar result.

Theorem 2.9. *Let $k \geq 2$. Every $\frac{k+1}{3}$ -tough k -tree (except for K_2) is Hamiltonian.*

An older result considering toughness and Hamiltonicity in another subclass of chordal graphs is due to Böhme et al. [23] who showed the following:

Theorem 2.10. *Every chordal planar graph (on at least 3 vertices) of toughness greater than 1 is Hamiltonian.*

In [50], Gerlach generalized Theorem 2.10 for planar graphs whose separating cycles of length at least four have chords. We present a different generalization of Theorem 2.10 which also improves the result of Theorem 2.9.

Theorem 2.11. *Let $k \geq 3$. Every k -tree of toughness greater than $\frac{k}{3}$ is Hamilton-connected. Furthermore, every 1-tough 2-tree (except for K_2) is Hamiltonian.*

The proof of Theorem 2.11 is included in this section. We remark that under this toughness restriction a graph is chordal planar if and only if it is a 3-tree or K_1 or K_2 (see Proposition 2.18). Hence, Theorem 2.11 is a generalization of Theorem 2.10. In particular, chordal planar graphs of toughness greater than 1 are Hamilton-connected.

In addition Böhme et al. [23], constructed 1-tough chordal planar graphs whose longest cycles are short. (We remark that the constructed graphs are, in fact, 3-trees. Thus, the bound of Theorem 2.11 is tight for $k = 3$. We present similar constructions in Section 3.6.)

In order to prove Theorem 2.11, we shall need several technical statements. Simply spoken, the proof is inductive; we choose a vertex on a path and we extend the path using particular neighbours of this vertex.

For a vertex v , we let $N_G(v)$, or simply $N(v)$, denote its *neighbourhood*, that is, the set of all vertices adjacent to v in a graph G . We say a set $S \subseteq N(v)$ is a *squeeze* by v if $2 \geq |S| \geq 1$ and $|R| \geq 2$ and every vertex of S is adjacent to at least $|R| - 1$ vertices of R and every vertex of R is adjacent to at least $|S| - 1$ vertices of S where $R = N(v) \setminus S$.

Proposition 2.12. *Let P be some set of vertices of a graph G and let x_1, x_2 and v be distinct vertices of P and let S be a squeeze by v . If $G - S$ has a path between x_1 and x_2 whose vertex set is P , then G has such path whose vertex set is $P \cup S$.*

Proof. We let uv and vw be the edges (incident with v) of the considered path in $G - S$. We note that the graph induced by $\{u, v, w\} \cup S$ has a Hamilton path between u and w . Thus, we can extend the considered path into a path between x_1 and x_2 whose vertex set is $P \cup S$. \square

We recall that a vertex whose neighbourhood induces a complete graph is called *simplicial*. For further reference, we state the following fact (shown, for instance, in [63]).

Proposition 2.13. *Adding a simplicial vertex to a graph does not increase its toughness.*

By definition, k -trees can be viewed as graphs constructed iteratively from K_k by adding one new simplicial vertex of degree k in each step. We recall that a vertex adjacent to all vertices of a graph is called *universal*. Considering a non-universal vertex v of a k -tree and a set S of all its neighbours of degree k , we say v is a *twig* if $N(v) \setminus S$ induces K_k and $|S| \geq 1$; and we say S is the *bud* of this twig. We note the following:

Proposition 2.14. *Let $k \geq 1$ and let G be a k -tree (on at least $k+3$ vertices) of toughness greater than $\frac{k}{3}$. Then G has a twig. Furthermore, if v is a twig of G and S is its bud, then $G - S$ is a k -tree of toughness greater than $\frac{k}{3}$. In addition, if $k \geq 2$, then S is a squeeze by v .*

Proof. For every $k \geq 1$, we consider iterative constructions of a k -tree, and we note that a non-universal and non-simplicial vertex v is a twig if and only if there exists an iterative construction such that all neighbours of v added later by this construction have degree k in the whole k -tree. We note that G has such vertex. We consider a twig v and its bud S , and we let $R = N(v) \setminus S$. Clearly, $G - S$ is a k -tree. Furthermore, Proposition 2.13 implies that the toughness of $G - S$ is at least the toughness of G .

In addition, every vertex of S is adjacent to $|R| - 1$ vertices of R . Since v is non-universal, the toughness of G implies that no two vertices of S have the same neighbourhood. In particular, for $k = 2$, we have $|S| \leq 2$. For $k \geq 3$, the same follows from the fact that $G - R - v$ has at least $|S| + 1$ components and $|R| = k$. Clearly, if $k \geq 2$, then $|R| \geq 2$; and we conclude that S is a squeeze by v . \square

We note that, with Propositions 2.12 and 2.14 on hand, we get an elementary proof of the Hamiltonicity of k -trees of toughness greater than $\frac{k}{3}$. (We remark that 2-trees of toughness greater than $\frac{2}{3}$ are, in fact, 1-tough.)

Lemma 2.15. *Let $k \geq 2$. Every k -tree (except for K_2) of toughness greater than $\frac{k}{3}$ is Hamiltonian.*

Proof. We let G be the considered k -tree and we let n denote the number of its vertices. Clearly, if $n \leq k + 2$, then G is Hamiltonian. We can assume that $n \geq k + 3$. We suppose that the statement is satisfied for graphs on at most $n - 1$ vertices, and we show it for G .

By Proposition 2.14, G has a twig v ; and we let S be the bud of v . Furthermore, $G - S$ is a k -tree of toughness greater than $\frac{k}{3}$. (Clearly, $G - S$ is distinct from K_2 .) By the hypothesis, $G - S$ has a Hamilton cycle, and we view it as a Hamilton path containing v as an interior vertex. By Propositions 2.12 and 2.14, we can extend this path and obtain a Hamilton path in G whose ends are adjacent, that is, a Hamilton cycle. \square

Aiming for the Hamilton-connectedness, we shall need two additional ingredients which are given by Propositions 2.16 and 2.17. For $k \geq 2$, a *basic 3-twig* is the graph obtained from K_{k+1} by choosing its three different subgraphs K_k and by adding one new simplicial vertex to each of them. For instance, the graph B depicted in Figure 3.12 is the basic 3-twig for $k = 3$.

Proposition 2.16. *Let $k \geq 1$ and let G be a k -tree (on at least $k + 4$ vertices) of toughness greater than $\frac{k}{3}$. Then either G has two non-adjacent twigs (whose buds are disjoint) or G is the basic 3-twig.*

Proof. We observe that if a k -tree has two non-adjacent vertices of degree greater than k , then it has two non-adjacent twigs. Clearly, their buds are disjoint. We note that the condition is satisfied if G is not the basic 3-twig. \square

We shall use Propositions 2.17 to address the setting in which the ends of the desired Hamilton path are the only twigs of a k -tree. We note that a similar idea is used in [26]. In a graph G , we say a Θ -spanner between vertices x_1 and x_2 is a spanning subgraph of G consisting of 3 paths with the same ends x_1, x_2 such that (except for the ends) these paths are mutually disjoint and each of them has an interior vertex.

Lemma 2.17. *Let $k \geq 3$ and let G be a k -tree (distinct from K_4) of toughness greater than $\frac{k}{3}$ and let x_1 and x_2 be distinct vertices of degree k . Then G has a Θ -spanner between x_1 and x_2 .*

Proof. Clearly, for every $k \geq 1$, K_k has no vertex of degree k . Furthermore, there exists exactly one k -tree on $k + 1, k + 2$ vertices, respectively; and exactly one k -tree on $k + 3$ vertices has the required toughness.

Considering these k -trees, we note that the statement is satisfied for graphs on at most $k + 3$ vertices. We let n denote the number of vertices of G , and we assume that $n \geq k + 4$. We suppose that the statement is satisfied for graphs on at most $n - 1$ vertices, and we show it for G .

Let us suppose that there is a twig v and its bud S such that neither x_1 nor x_2 belongs to S . By Proposition 2.14 and by the hypothesis, we can consider a Θ -spanner between x_1 and x_2 in $G - S$; and we let P be the set of vertices of one of the three paths between x_1 and x_2 of this Θ -spanner such that v belongs to P . By Propositions 2.12 and 2.14, there is a path with the same ends whose vertex set is $P \cup S$. Thus, G has a Θ -spanner between x_1 and x_2 .

We assume that every twig is adjacent to x_1 or x_2 . By Proposition 2.16, we can assume that there is a twig x'_1 and its bud S' such that x_1 belongs to S' and x_2 does not. Clearly, x'_1 has degree k in $G - S'$. We consider a Θ -spanner Y between x'_1 and x_2 in $G - S'$; and we let N denote the set of all vertices adjacent to x'_1 in Y . We choose a vertex y of N such that y is adjacent to x_1 in G . Clearly, the subgraph of Y induced by $N \cup \{x'_1\} \setminus \{y\}$ is a path, and we apply Propositions 2.12 and 2.14 and extend this path by adding vertices of S' . We consider the resulting path and the edge x_1y and we extend the graph $Y - x'_1$ into a Θ -spanner between x_1 and x_2 in G . \square

Finally, we use the tools introduced in this section and prove Theorem 2.11.

Proof of Theorem 2.11. For $k = 2$, the statement is satisfied by Lemma 2.15. We assume that $k \geq 3$. We let G be a k -tree of toughness greater than $\frac{k}{3}$ and we let n denote the number of its vertices. We note that if $n \leq k + 3$, then G is Hamilton-connected; so we can assume that $n \geq k + 4$. We suppose that the statement is satisfied for graphs on at most $n - 1$ vertices, and we show it for G ,

that is, we show that for an arbitrary pair of vertices x_1 and x_2 , G has a Hamilton path between x_1 and x_2 .

Let us suppose that G has a twig v distinct from x_1 and x_2 , and we let S be the bud of v . If neither x_1 nor x_2 belongs to S , then we consider a Hamilton path between x_1 and x_2 in $G - S$; and we note that it can be extended into a desired path in G by Propositions 2.12 and 2.14. In addition, let us suppose that x_1 belongs to S and x_2 does not. We consider a Hamilton path between v and x_2 in $G - x_1$, and we extend it by adding the edge x_1v .

We assume that no such v exists. By Proposition 2.16, we can assume that x_1 and x_2 are non-adjacent twigs and the corresponding buds S_1 and S_2 are disjoint. We consider the graph $G' = G - S_1 - S_2$. Since x_1 and x_2 are non-adjacent, the number of vertices of G' is at least $k+2$ and x_1 and x_2 have degree k in G' . Using Proposition 2.14, we note that G' is a k -tree of toughness greater than $\frac{k}{3}$.

We consider a Θ -spanner Z between x_1 and x_2 in G' given by Lemma 2.17. Clearly, Z forms 3 paths in $G' - x_1 - x_2$. We note that we can join these paths (using the adjacency of their ends and using the vertices of $S_1 \cup S_2$) and obtain a Hamilton path from S_1 to S_2 in $G - x_1 - x_2$. Thus, we get a Hamilton path between x_1 and x_2 in G . \square

In relation to Theorems 2.10 and 2.11, we remark the following:

Proposition 2.18. *Let G be a graph of toughness greater than 1. Then the following statements are equivalent:*

- (1) G is chordal planar,
- (2) G is chordal and K_5 -free,
- (3) G is either a 3-tree or K_1 or K_2 .

We shall use the facts stated in Propositions 2.19 and 2.20 (shown by Patil [85] and by Markenzon et al. [77, Lemma 24], respectively).

Proposition 2.19. *Let $k \geq 1$. A graph (distinct from K_k) is a k -tree if and only if it is chordal k -connected and K_{k+2} -free.*

Proposition 2.20. *Let G be a 3-tree. Then G is planar if and only if $G - C$ consists of at most two components for every set of vertices C inducing K_3 .*

The combination of Propositions 2.19 and 2.20 gives the desired equivalence.

Proof of Proposition 2.18. Since planar graphs are K_5 -free, (1) implies (2). Clearly, every graph of toughness greater than 1 is either 3-connected or K_1 or K_2 or K_3 ; and we apply Proposition 2.19 to 3-trees, and we note that (2) implies (3). For every graph of toughness greater than 1, a removal of three vertices creates at most two components. Thus, the application of Proposition 2.20 concludes the proof. \square

2.5 On superclasses of interval graphs

As discussed in the previous sections, Conjecture 1.1 is satisfied when restricted to interval graphs and also to spider graphs and chordal graphs; and we recall that these are superclasses of interval graphs. In this section, we consider other superclasses of interval graphs; namely, co-comparability graphs and circular arc graphs.

We recall that a *comparability graph* is a graph whose vertices can be represented by elements of a partially ordered set, so that two vertices of the graph are adjacent if and only if the corresponding elements are comparable; and a *co-comparability graph* is the complement of a comparability graph.

Considering a partial ordering of intervals in which two intervals are comparable if and only if they are non-intersecting, and considering the corresponding comparability graph, we note that every interval graph is a co-comparability graph. (In fact, Gilmore and Hoffman [51] showed that the class of interval graphs is precisely the intersection of the classes of co-comparability and chordal graphs.)

We recall that by Corollary 2.3, every 1-tough interval graph (on at least 3 vertices) is Hamiltonian. This fact extends to co-comparability graphs by the result of Deogun et al. [38] (in fact, they studied spanning subgraphs whose every component is a path in relation to the scattering number of the graph, and showed that the obtained result implies Hamiltonicity of 1-tough co-comparability graphs).

Another natural generalization of interval graphs are *circular arc graphs*, these are, intersection graphs of arcs of a circle. We note that the results considering Hamiltonian properties of interval graphs (discussed in Section 2.1) do not extend to circular arc graphs (for example, see Figure 2.3). In particular, not every 1-tough circular arc graph is Hamiltonian. On the other hand, Deogun et al. [38] mentioned that toughness greater than 1 ensures Hamiltonicity for circular arc graphs by the result of Shih et al. [90].

We present an alternative approach to toughness and Hamiltonicity of circular arc graphs, and we show a weaker result. In particular, by applying the technique of [65] (see Appendix A), we can obtain the following:

Proposition 2.21. *Every 4-tough circular arc graph (on at least 3 vertices) is Hamiltonian.*

Proof. We let G be the considered graph and \mathcal{F} be a corresponding family of arcs of a circle, that is, for every vertex u , there is an arc F_u representing it. We say two arcs F and F' *intersect* if there is a point of the circle belonging to both. We say F' is a *proper subarc* of F if every point of F' belongs to F and there is a point of F not belonging to F' . Furthermore, we say an arc F is *good* if there is no proper subarc of F in \mathcal{F} .

2.5. On superclasses of interval graphs

We let \mathcal{I} be a maximum set of pairwise non-intersecting good arcs of \mathcal{F} . Clearly, we can assume that G is a non-complete graph, and thus $|\mathcal{I}| \geq 2$. We consider a clockwise ordering of the arcs of \mathcal{I} along the circle, and we let I_0, I_1, \dots, I_k denote the arcs of \mathcal{I} (in accordance with the ordering). A point of the circle is called *substantial* if it is an endpoint of an arc of \mathcal{I} . For every $i = 0, 1, \dots, k$, we let s_i denote the clockwise endpoint of I_i , and t_i denote the counterclockwise endpoint of I_{i+1} (we view the calculations as done modulo $k + 1$, in particular, t_k is the counterclockwise endpoint of I_0).

For every $i = 0, 1, \dots, k$, we construct a so-called ‘overspan graph’ A_i . The vertex set of A_i consists of the vertices of G corresponding to the arcs of $\mathcal{F} \setminus \mathcal{I}$. The edges of A_i (simple edges and loops) encode possible ways of connecting s_i to t_i by at most two arcs. In particular, a vertex u is incident with a loop if and only if s_i and t_i belong to F_u ; and there is an edge vw if and only if s_i belongs to F_v and t_i belongs to F_w and F_v and F_w intersect.

Let us suppose that for every $i = 0, 1, \dots, k$, we can choose one edge (a simple edge or a loop) e_i from A_i such that every vertex is incident with at most one of the edges e_0, e_1, \dots, e_k . We let F_i, F'_i be the pair of arcs corresponding to vertices incident with e_i such that s_i belongs to F_i and t_i belongs to F'_i (in case e_i is a loop, the arc F'_i is an auxiliary copy of F_i). We consider the sequence $I_1, F_1, F'_1, I_2, F_2, F'_2, \dots, I_k, F_k, F'_k$, and we note that every two consecutive arcs intersect (the last and the first element of the sequence are also considered consecutive); and we shall preserve this property as we further modify the sequence. By the choice of \mathcal{I} , every arc contains a substantial point. For every arc of \mathcal{F} which is not in the sequence, we choose one of its substantial points arbitrarily, and we extend the sequence as follows. If the chosen point is s_i (t_i), then we add the arc as an immediate successor of I_i (F'_i). We consider the extended sequence, and we remove all auxiliary copies of arcs (the copies which are used to deal with loops). Clearly, the resulting sequence contains every element of \mathcal{F} exactly once. Thus, the sequence of the corresponding vertices of G defines a Hamilton cycle.

To the contrary, we suppose that there is no such choice of edges from the overspan graphs. In other words, (viewing edges as sets of vertices and viewing the overspan graphs as sets of edges) we suppose that there is no system of disjoint representatives. We view the overspan graphs as hypergraphs of rank at most 2, and we apply the hypergraph extension of Hall’s theorem [2]. By a corollary of this theorem (see Corollary 6 in [65]), there is a family \mathcal{B} of overspan graphs and a set E of edges with the following properties:

- every edge of every graph of \mathcal{B} is incident with a vertex of X where X is the set of all vertices incident with an edge of E ,
- and $|E| \leq 2(|\mathcal{B}| - 1)$.

We note that if $|\mathcal{B}| = 1$, then G is an interval graph, and thus it is Hamiltonian by Corollary 2.3. Consequently, we can assume that $|\mathcal{B}| \geq 2$. Clearly, $|X| \leq$

$2|E| < 4|\mathcal{B}|$. We consider the graph $G - X$, and we observe that it has at least $|\mathcal{B}|$ components, a contradiction. \square

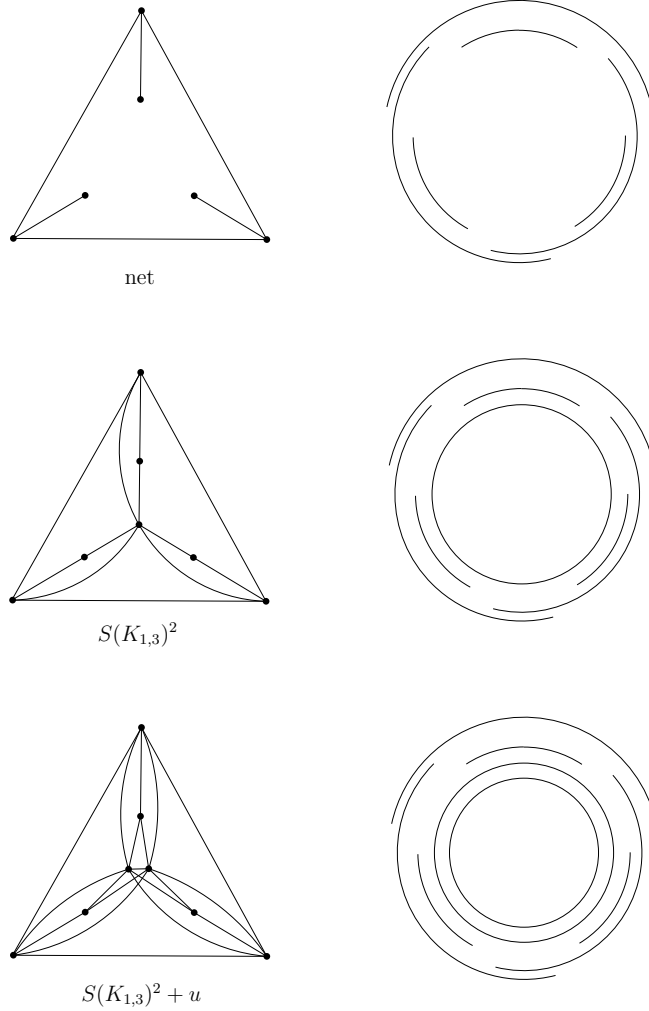


Figure 2.3: Circular arc graphs (left) and their intersection representations (right). The arcs of a circle represent vertices of the graph; and two vertices of the graph are adjacent if and only if the corresponding arcs intersect (intersecting arcs are depicted as overlapping). We note that the net has no Hamilton path, $S(K_{1,3})^2$ has no Hamilton cycle, $S(K_{1,3})^2 + u$ is not Hamilton-connected, and their scattering numbers are 1, 0 and -1 , respectively. In particular, the graph $S(K_{1,3})^2$ is 1-tough and non-Hamiltonian. We remark that there are infinite families of circular arc graphs which have these properties. (For instance, we can extend the present representations by adding an arbitrary number of copies of the ‘shortest’ arc.)

2.6 H-graphs (topological intersection graphs)

Motivated by the partial results reviewed in this chapter and by a discussion with Tomáš Kaiser and Peter Zeman, we consider the concepts of underlying graphs and H -graphs as a possible approach to restrict and study Conjecture 1.1.

For instance, we consider the intersection representations of interval graphs. Clearly, for every interval graph, the ‘underlying line’ can be viewed as an ‘underlying path’ (of sufficient length), and the intervals can be viewed as subpaths of this path (that is, two vertices of the graph are adjacent if and only if the corresponding subpaths have a vertex in common). Similarly, every circular arc graph can be viewed as an intersection graph of a family of connected subgraphs of a cycle. We recall that every split graph, spider graph, chordal graph is an intersection graph of a family of connected subgraphs of a star, of a tree with at most one vertex of degree greater than 2, of a tree, respectively. Furthermore, we recall that Conjecture 1.1 is satisfied when restricted to these classes of graphs. So it seems natural to study Conjecture 1.1 under similar restrictions. In this section, we discuss these restrictions.

Clearly, considering a fixed underlying graph H , there is certain level of connectivity which implies Hamiltonicity of the intersection graphs (as noted by Stéphan Thomassé). In particular, for a graph H on k vertices, every intersection graph has a maximum independent set of size at most k . Thus, every k -connected such intersection graph (on at least 3 vertices) is Hamiltonian by Theorem 1.3.

Therefore, we should consider a class of intersection graphs which is defined by an infinite class of underlying graphs (instead of one fixed underlying graph). Following Biró et al. [20], we recall the concept of H -graphs (topological intersection graphs). A *subdivision* H' of a graph H is a graph obtained by replacing every edge of H by a path of arbitrary length (such that the ends of the path are the vertices incident with the replaced edge, and the paths are internally disjoint). A graph G is an H -graph if there is a subdivision H' of H such that the vertices of G can be represented by some connected subgraphs of H' in such a way that two vertices u, v are adjacent in G if and only if the corresponding subgraphs H_u, H_v share a vertex of H' .

As noted in [20], every graph H is an H -graph, so the class of all H -graphs such that H is arbitrary is simply the class of all graphs. Clearly, interval graphs are precisely H -graphs where H is K_2 (the underlying path can be viewed as a subdivision of K_2). Similarly, circular arc graphs are precisely H -graphs where H is C_3 (or any cycle). Furthermore, every chordal graph (spider graph) is an H -graph where H is a tree (a star), and vice versa.

Similarly as above, we can consider H -graphs where H is a fixed graph. The question is: can we show that for every H , there exists t_0 such that every t_0 -tough H -graph (on at least 3 vertices) is Hamiltonian? In relation to the technique which we used for chordal graphs and circular arc graphs (see Appendix A and

the proof of Proposition 2.21, respectively), there is a more general question: what about considering H -graphs where H can be any graph with a bounded number of cycles? Similar restrictions on H are suggested in Section 4.2 as questions for further research. We view the arising classes of graphs as natural generalizations of chordal graphs and circular arc graphs.

Another approach would be to consider an arbitrary underlying graph and some restrictions on the subgraphs. For instance, considering all (distinct) P_2 subgraphs of a graph H , the intersection graph is precisely the line graph of H . (Similarly, considering some family of P_2 subgraphs of H , the intersection graph is the line graph of the multigraph whose edges are given by the considered family.) Considering all P_3 subgraphs (not necessarily induced), we note the following:

Proposition 2.22. *Let H be a graph. Then the intersection graph of all P_3 subgraphs of H is precisely $L(L(H))^2$, that is, the square of the line graph of the line graph of H . Furthermore, if H is connected (and contains at least three P_3 subgraphs), then $L(L(H))^2$ is Hamiltonian.*

Proof. Clearly, every P_3 subgraph of H corresponds to an edge of $L(H)$ which corresponds to a vertex of $L(L(H))$. Furthermore, two P_3 subgraphs intersect in H if and only if the corresponding vertices are at distance at most 2 in $L(L(H))$.

Furthermore, $L(L(H))$ is connected and $K_{1,3}$ -free (and it has at least 3 vertices), and we recall that the square of a connected $K_{1,3}$ -free graph is pancyclic by [78]. \square

We remark that a similar argument applies to intersection graphs of all P_ℓ subgraphs of H where ℓ is fixed.

In Section 3.3 (in a different context), we consider intersection graphs such that every subgraph is a copy of P_2 or a copy of the graph H itself (that is, we consider line graphs with additional universal vertices).

We conclude this section with a remark on the proof technique of [65] (see Appendix A and Proposition 2.21). We extend the concept of H -graphs as follows. We say a graph G is a *relaxed H -graph* if there is a subdivision H' of the graph H such that every vertex of G can be represented by a connected subgraph of H' in such a way that

- two vertices of G are adjacent if the corresponding subgraphs of H' have a vertex in common, and
- two vertices of G are non-adjacent if the corresponding subgraphs of H' are at distance at least 2.

Clearly, every H -graph is a relaxed H -graph. (We note that a relaxed H -graph can be viewed as an M -graph where M is obtained from H' by replacing every edge by a number of subdivided edges, and every subgraph of M considered in the representation contains a vertex of H' .)

In other words, the difference is that in a relaxed H -graph two vertices may or may not be adjacent if the corresponding subgraphs of H' are at distance 1. We note that a similar situation occurs in [65]; after suppressing vertices of degree 2 in the underlying graph, the corresponding adjacencies are encoded by the overspan graphs. Considering the proof, we remark that Theorem 2.5 extends to the class of relaxed H -graphs such that H is a tree, which is a superclass of chordal graphs. Similarly, Proposition 2.21 can be extended to a superclass of circular arc graphs.

2.7 Question of $K_1 \cup P_4$ -free graphs

We recalled that every Hamiltonian graph is 1-tough, and we mentioned some restricted classes of graphs in which toughness at least 1 implies Hamiltonicity. Li et al. [75] studied such classes of H -free graphs, and they obtained an almost complete characterisation of the forbidden subgraphs H , leaving the subgraph $K_1 \cup P_4$ as the only open case. We collect their main results in the following theorem:

Theorem 2.23. *Let H be a graph distinct from $K_1 \cup P_4$. Then every 1-tough H -free graph (on at least 3 vertices) is Hamiltonian if and only if H is an induced subgraph of $K_1 \cup P_4$.*

(In addition to the Hamiltonicity, Li et al. [75] studied spanning subgraphs whose every component is a path in relation to the scattering number of the graph. Also, they noted that the case of $K_1 \cup P_3$ -free graphs was shown independently by Nikoghosyan [82], and the case of P_4 -free graphs was solved formerly by Jung [62].) Interested in the remaining case, Li et al. [75] asked the following: is every 1-tough $K_1 \cup P_4$ -free graph (on at least 3 vertices) Hamiltonian? (In addition, they noted that this question was stated as a conjecture in [82].) Furthermore, they asked a weaker version of the question: is Conjecture 1.1 satisfied when restricted to $K_1 \cup P_4$ -free graphs?

We recall the topic of forbidden pairs and Hamiltonicity of k -connected graphs (mentioned in Section 1.4), and we remark that Theorem 2.23 can be viewed as related to this topic. In particular, it seems that not much is known for $k \geq 4$, apart from the fact that every k -connected $\{P_4, K_{1,k+1}\}$ -free graph is Hamiltonian for $k \geq 2$, see [31]. We remark that a similar result follows as a consequence of Proposition 1.8 and Theorem 2.23.

Corollary 2.24. *Let $k \geq 2$, and let H belong to $\{P_4, K_1 \cup P_3, 2K_1 \cup K_2\}$. Then every k -connected $\{H, K_{1,k+1}\}$ -free graph is Hamiltonian.*

Similarly, the results discussed in Section 2.8 imply that certain level of connectivity ensures Hamiltonicity for $\{2K_2, K_{1,k}\}$ -free graphs.

We recall that the subgraph $K_1 \cup P_4$ presents the open case in relation to Theorem 2.23. In order to exclude other forbidden subgraphs (other than the induced subgraphs of $K_1 \cup P_4$), Li et al. [75] presented several infinite families of non-Hamiltonian 1-tough graphs. We note that some of these graphs are $K_{1,3}$ -free (or $K_{1,4}$ -free). It seems that the ideas of [75] in combination with [31] might be viewed as an introduction to a more detailed study of forbidden pairs and Hamiltonicity of k -connected graphs for $k \geq 4$. (However, this may be a challenging topic since, for instance, it is related to Conjecture 1.7 and Question 1.9.)

Motivated by [75] and [31] (and Proposition 1.8), we note that it might be of interest to ask about Hamiltonicity of k -connected $\{K_1 \cup P_4, K_{1,k+1}\}$ -free graphs.

2.8 $2K_2$ -free graphs

In addition to spider graphs and chordal graphs, we discuss toughness and Hamiltonicity in another superclass of split graphs; namely, in $2K_2$ -free graphs.

We recall that a $2K_2$ -free graph contains no copy of the graph $2K_2$ as an induced subgraph. In other words, a graph is $2K_2$ -free if and only if its complement is C_4 -free. In particular, a complement of a chordal graph is a $2K_2$ -free graph (this was remarked in [27] regarding the richness of the class of $2K_2$ -free graphs).

In [27], Broersma, Patel and Pyatkin showed that every 25-tough $2K_2$ -free graph (on at least 3 vertices) is Hamiltonian. The key idea of the proof is to find a so-called *PT-factor*, that is, a spanning subgraph whose every component is either a triangle (a copy of C_3) or a graph obtained by connecting two disjoint triangles with a path (such that each end of the path is identified with one vertex of one of the triangles). The assumption on toughness is only used to ensure the existence of a PT-factor; and it is shown that if a $2K_2$ -free graph has a PT-factor, then it has a connected PT-factor, and this yields a Hamilton cycle.

As noted in [27], the known lower bound for split graphs [71] (see Figure 2.2) applies to $2K_2$ -free graphs (since every split graph is $2K_2$ -free). The authors [27] mentioned that there is a large gap between the presented upper bound of 25 and the lower bound of ‘almost’ $\frac{3}{2}$, and that they are not sure whether the lower bound is extremal, but they are almost certain that the upper bound is not.

It is also shown that, with the additional restriction of being C_3 -free or $K_{1,3}$ -free, every 1-tough $2K_2$ -free graph (on at least 3 vertices) is Hamiltonian. In particular, the C_3 -freeness implies that the $2K_2$ -free graph is either a bipartite $2K_2$ -free graph or a so-called C_5^* -graph, and the Hamiltonicity is proven separately in each case. (We discuss a generalization of the C_5^* -graphs in Section 2.9.) Considering the $K_{1,3}$ -free property, it is argued that the Hamiltonicity follows from the fact that every 2-connected $2K_2$ -free graph has a dominating cycle [100]. (We remark that, alternatively, the Hamiltonicity of 1-tough $\{K_{1,3}, 2K_2\}$ -free graphs can be viewed as following from some of the results considering forbidden pairs, see [53, 29, 18, 76].)

Recently, an improvement of the known upper bound from 25 to 3 was announced by Shan [89]. The idea of the proof is to use the case $k = 2$ of Theorem 1.12, that is, every 2-tough graph (on at least 3 vertices) has a 2-factor; and to consider a 2-factor with a minimum number of components. Shan distinguishes different types of vertices and edges of the 2-factor, and (using the distinctions and edges which connect different components of the 2-factor) argues that there is a 2-factor which is connected (that is, a Hamilton cycle). In addition, Shan notes that the property of being 3-tough is used just once in the argument; for proving one of the claims [89, Claim 2.5].

We conclude this section with a technical remark considering this part of the proof. In a discussion with Hajo Broersma, we noted that it seems like in some cases this claim is satisfied trivially. In particular, if a cycle of the considered 2-factor contains a so-called B -type edge, then all other cycles are of even lengths by [89, Claim 2.3 (2)]. Thus, in case the 2-factor contains a cycle of odd length, the desired claim follows trivially from [89, Claims 2.3 (2) and 2.2]. For instance, this is satisfied for graphs whose number of vertices is odd.

2.9 Approaching the t_0 -tough conjecture with duality theorems

In this section, we include results of a joint work with Hajo Broersma, Hao Qi and Elkin Vumar (the results shall also be available in the manuscript [86] which we are preparing for submission).

In addition, we make a remark regarding the tools which are used in [86]. First, we discuss the main tools which can be used to show partial results on Conjecture 1.1 for split graphs, spider graphs, chordal graphs or circular arc graphs. We recall that the Hamiltonicity of $\frac{3}{2}$ -tough split graph was shown in [71]. For the sake of the discussion, we note that it is easy to show a weaker version of this result.

Proposition 2.25. *Every 2-tough split graph (on at least 3 vertices) is Hamiltonian.*

Proof. We let G be the considered split graph and S be an independent set of vertices of G such that the set $T = V(G) \setminus S$ induces a complete graph. Clearly, if we can find a subgraph of G consisting of vertex-disjoint paths such that every vertex of S has degree 2 in this subgraph, then we easily obtain a Hamilton cycle of G .

We find such subgraph consisting of copies of P_3 . We consider the bipartite graph (S, T) obtained from G by ignoring all edges which are incident with two vertices of T . Since G is 2-tough, we have $|N(S')| \geq 2|S'|$ for every subset S' of S . The existence of the desired graph follows, for instance, from the Generalized

marriage theorem of Akiyama and Kano [3]. (Alternatively, it can be observed using the classical Hall's marriage theorem by considering the bipartite graph (S, T) , cloning every vertex of S once, finding a matching which saturates all vertices of S and their clones, and identifying every vertex of S with its clone.) \square

We view the finding of a Hamilton cycle in a split graph as an assignment problem, and we choose a suitable tool to approach this problem. We either find a desired assignment, and we use it to obtain a Hamilton cycle; or the conditions for applying the tool are not satisfied, and we use this fact to obtain a contradiction with the toughness assumption on the graph.

We view the usage of the Matroid intersection theorem (and Hall's marriage theorem) in [66], and the usage of the hypergraph extension of Hall's theorem in [65] (and in Proposition 2.21) as examples of a similar 'duality reasoning'.

In this section, we apply a similar reasoning. We use basic 'duality theorems'; namely, the Max-flow min-cut theorem and Hall's marriage theorem, and we present additional partial results on Conjecture 1.1. The author of the present thesis views the fact that these theorems are used as having some interest on its own (separately from considering the results themselves). In particular, it seems natural to ask the following: what other suitable theorems (of similar nature) should we use in the study of Conjecture 1.1?

We proceed with the results of [86]. We recall the concept of C_5^* -graphs [27] (see also Section 2.8), and we generalize it as follows. For a graph H , we say an H^* -graph is either the graph H or a graph obtained from an H^* -graph by choosing its arbitrary vertex and by adding a new vertex to the graph such that the two vertices have the same neighbourhood (in particular, the new vertex is not adjacent to the chosen vertex). For instance, taking H as a copy of C_5 , the resulting class is the class of C_5^* -graphs.

We study the Hamiltonicity of particular 1-tough 'cactus-like' graphs (originally, we were interested in C_p^* -graphs). The present technique can be viewed as an alternative approach to the method used for C_5^* -graphs in [27]. We use the concept of a flow network (see for example [22]) and the classical Max-flow min-cut theorem of Ford and Fulkerson [47] to decide the Hamiltonicity of the considered graphs. The existence of a Hamilton cycle is reduced to the existence of a particular flow in an auxiliary flow network. Furthermore, if no such flow exists, then (using the Max-flow min-cut theorem and the structure of the flow network) we obtain a contradiction with the toughness assumption on the graph. In fact, the proof of Theorem 2.26 translates to an algorithm which provides a certificate for the decision. Namely, we either construct a Hamilton cycle, or provide a set of vertices showing that the graph is not 1-tough. We recall that a graph is a cactus if every edge is in at most one cycle.

Theorem 2.26. *Let H be either a bipartite cactus or an odd cycle and let G be an H^* -graph (on at least 3 vertices). Then G is Hamiltonian if and only if G is 1-tough.*

The proof of Theorem 2.26 is included in this section.

We say a graph is a *multisplit graph* if its vertex set admits a division into two disjoint sets: one inducing a complete k -partite graph and the other inducing a graph with no edges. (In case every class of the k -partition consists of a single vertex, the graph is a split graph. In case $k = 2$, the graph is a so-called bisplit graph; more on bisplit graphs can be found in [24].) We use the basic idea of the proof of Proposition 2.25 and we show the following:

Theorem 2.27. *Every 2-tough multisplit graph is Hamilton-connected.*

We recall that there are non-Hamiltonian split graphs of toughness ‘almost’ $\frac{3}{2}$ (see Figure 2.2), and we note that the bound of 2 is probably not optimal. In the remainder of this section, we prove Theorems 2.26 and 2.27.

Proof of Theorem 2.26. Clearly, every Hamiltonian graph is 1-tough.

For the reverse direction, we note that H is connected (since G is 1-tough). We let v_1, v_2, \dots, v_k denote the vertices of H . For every vertex v_i of H , we let A_i denote the corresponding set of vertices of G . The basic idea of the proof is the following:

Claim 2.2. *Suppose that there is an assignment w of non-negative integer weights to edges of H such that for every v_i , the sum of weights of all edges incident with v_i equals $2|A_i|$. Then G is Hamiltonian.*

Proof of Claim 2.2. First of all, we show the following properties of such assignments.

- (1) The weight of every cut-edge of H is even. Furthermore, for every cycle of H , either all of its edges have odd weights or all have even.
- (2) The weight of every cut-edge of H is positive. Furthermore, there exists such assignment w^+ with the additional feature that for every cycle of H , at most one of its edges has weight 0.

We consider an edge-cut E of H such that there is a component Y of $H - E$ with the property that every edge of E is incident with exactly one vertex of Y . Clearly, the sum of weights of all edges of E plus twice the sum of weights of all edges of Y is even. Thus, by parity the sum of weights of the edges of E is even. In particular, property (1) follows by considering E as a set which consists of one cut-edge or of two consecutive edges of a cycle.

In addition, we note that every proper subgraph of H is bipartite. Hence Y is bipartite, and so is the corresponding subgraph Z of G . Let us suppose that all edges of E have weight 0. Then Z is balanced (that is, its parties are of the same size). Moreover, if all vertices of Y incident with an edge of E belong to the same party, then we consider all vertices of Y belonging to this party and we remove from G all corresponding vertices of Z ; and we note that the resulting

graph has at least $|Z| + 1$ components, a contradiction. In particular, we have that every cut-edge of H has a positive weight. Furthermore, we observe that if a cycle of H has at least two edges of weight 0, then the cycle is even and the weights of its edges can be adjusted by alternating $+1$ and -1 along the cycle. We apply this adjustment for every such cycle, and we note that the resulting assignment has the additional feature stated in property (2).

We use the properties (1) and (2) and show that G is Hamiltonian. We consider an assignment w^+ given by property (2), and a maximal subgraph H' of H whose every edge has positive weight. By (2), H' is a connected spanning subgraph of H . Since H' is a cactus, we use (1) and we observe that we can start in a vertex of H' and traverse through H' and return to the same vertex so that every edge of H' with odd weight is used exactly once and every edge of H' with even weight is used exactly twice.

We consider this traversal through H' and we extend it as follows. For every edge e_{ij} (incident with v_i and v_j) of H' , we let ℓ_{ij} be the value equal to $w^+(e_{ij})$ minus the number of appearances of e_{ij} in the traversal; and we note that ℓ_{ij} is non-negative and even. For every positive ℓ_{ij} in sequence, we replace one of the appearances of e_{ij} with $\ell_{ij} + 1$ consecutive traversals through e_{ij} . We note that in the resulting closed walk every vertex v_i is visited exactly $|A_i|$ times.

We consider a corresponding traversal through G (that is, visiting a vertex v_i corresponds to visiting a vertex of A_i) such that it starts and ends in the same vertex and no vertex of G is used more than once. We note that this traversal defines a Hamilton cycle of G . ■

To find the desired assignment w , we construct a flow network N as follows. We let $s, u_1, u_2, \dots, u_k, u'_1, u'_2, \dots, u'_k, t$ denote the nodes of N . The set of arcs of N consists of arcs su_i and $u'_i t$ for every $i = 1, 2, \dots, k$; and of arcs $u_i u'_j$ and $u_j u'_i$ for every edge $v_i v_j$ of H . The node s is the source and the node t is the sink of N . For every arc incident with neither s nor t , we define its capacity as infinite (or sufficiently large); and for every $i = 1, 2, \dots, k$, the capacities of arcs su_i and $u'_i t$ equal $|A_i|$.

Let us suppose that there is a flow of value $|V(G)|$, that is, every arc with a finite capacity is saturated. Then there exists such flow f whose all values are integers. To every edge $v_i v_j$ of H , we assign weight equal to $f(u_i u'_j) + f(u_j u'_i)$; and we note that G is Hamiltonian by Claim 2.2.

We proceed with the following claim.

Claim 2.3. *If there is no saturating flow in N , then there is a symmetrical min-cut (that is, for every $i = 1, 2, \dots, k$, the arc su_i belongs to this min-cut if and only if $u'_i t$ does).*

Proof of Claim 2.3. We let \mathcal{X} be a min-cut of N . Clearly, \mathcal{X} contains no arc with infinite capacity. We consider the set O of all vertices v_i of H such that exactly one of the arcs $su_i, u'_i t$ belongs to \mathcal{X} . By the Max-flow min-cut theorem,

the capacity of \mathcal{X} is less than $|V(G)|$. In particular, not all vertices of H belong to O .

Thus, the graph induced by O is bipartite. We fix a bipartition of the graph, and we emphasize it by a black-and-white colouring of the vertices of O , and we consider the black-and-white colouring of the corresponding arcs of N (we note that for every vertex of O , there are two coloured arcs, one incident with s and one with t); and we let \mathcal{B} , \mathcal{W} denote the set of all black, white arcs, respectively. Similarly, we consider the set of all vertices v_i of H such that both su_i and $u_i't$ belong to \mathcal{X} ; and we let \mathcal{R} denote the set of all these arcs of N .

We consider an arc $u_iu'_j$ (with infinite capacity) such that one of its nodes is incident with an arc of \mathcal{B} ; clearly, the other node is not incident with an arc of \mathcal{B} . Furthermore, considering the arcs $u_iu'_j$ and $u_ju'_i$, we note that the other node is incident with an arc of $\mathcal{R} \cup \mathcal{W}$ (since \mathcal{X} is a cut). Similarly, for every arc with infinite capacity, if one of its nodes is incident with an arc of \mathcal{W} , then the other node is incident with an arc of $\mathcal{R} \cup \mathcal{B}$.

Consequently, the set $\mathcal{R} \cup \mathcal{W}$ is a cut in N , and so is $\mathcal{R} \cup \mathcal{B}$. Clearly, both of these cuts are symmetrical. We conclude that both of these sets are min-cuts (since the capacity of $\mathcal{R} \cup \mathcal{W}$ plus the capacity of $\mathcal{R} \cup \mathcal{B}$ equals twice the capacity of \mathcal{X}). \blacksquare

To conclude the proof, we suppose that there is no saturating flow in N . We recall that by the Max-flow min-cut theorem, the capacity of a min-cut is less than $|V(G)|$. We consider a symmetrical min-cut given by Claim 2.3. In sequence, for every i such that su_i belongs to this min-cut, we remove all vertices of A_i from G ; and we note that the resulting graph contains no edges. So after removing less than $\frac{1}{2}|V(G)|$ vertices, the resulting graph has more than $\frac{1}{2}|V(G)|$ components, a contradiction. \square

In order to show Theorem 2.27, we shall use a technical lemma considering complete k -partite graphs. We recall that a *linear forest* is a graph whose every component is a path (we view isolated vertices as trivial paths).

Lemma 2.28. *Let $k \geq 2$ and let G_0 be a complete k -partite graph (distinct from K_2) with classes Q_1, Q_2, \dots, Q_k . Let \overline{F} be a linear forest in the complement of G_0 and let G be a graph obtained from G_0 by adding all edges of \overline{F} . Let F be a spanning linear forest in G such that \overline{F} is a subgraph of F . Then the following statements are equivalent:*

- (1) G has a Hamilton cycle containing F .
- (2) For every $j = 1, 2, \dots, k$, we have $2f_j \leq \sum_{i=1}^k f_i$ where $f_i = \sum_{u \in Q_i} (2 - d_F(u))$.

Clearly, for every $k \geq 2$, every 1-tough complete k -partite graph (distinct from K_2) is Hamiltonian. We note that Lemma 2.28 is a stronger statement.

In particular, considering forests F which have at most one edge, we note the following.

Corollary 2.29. *Let G be a complete k -partite graph. Then the following statements are satisfied:*

- (1) G is Hamilton-connected if and only if its toughness is greater than 1.
- (2) G is Hamiltonian if and only if it is 1-tough (and it has at least 3 vertices).
- (3) G has a Hamilton path if and only if G^+ is 1-tough where G^+ is the graph obtained from G by adding one universal vertex.

(We remark that for $k \geq 3$, a complete k -partite graph can be viewed as an H^* -graph where H is a complement of C_k .) We apply a greedy argument and prove Lemma 2.28.

Proof of Lemma 2.28. We recall that F is a spanning linear forest. We shall extend F by adding edges of G one by one, and obtain a Hamilton cycle of G . After the addition of every edge, we re-calculate all values $f_i = \sum_{u \in Q_i} (2 - d_F(u))$ for the extended F . We view f_i as counting the ends of the considered paths (we view a trivial path as having two identical ends); and the inequality $f_j \leq -f_j + \sum_{i=1}^k f_i$ guarantees that not too many end vertices belong to the same class of G_0 .

In each step, we choose two distinct paths in the extended F (in F in the first step) such that an end vertex of one of the paths is in the class Q_i which currently maximizes f_i , and an end vertex of the other path is in a different class of G_0 ; and we extend F by adding the edge incident with these two vertices.

We show that the inequality $2f_j \leq \sum_{i=1}^k f_i$ is satisfied in each step for every $j = 1, 2, \dots, k$. Clearly, after each addition of an edge, the value of $\sum_{i=1}^k f_i$ decreases by 2. We note that in each step if $2f_j < \sum_{i=1}^k f_i$, then $2f_j + 2 \leq \sum_{i=1}^k f_i$ (since $\sum_{i=1}^k f_i$ is even). Furthermore, if $2f_j = \sum_{i=1}^k f_i$ for some j , then there are at most two classes of G_0 which maximize f_i ; and in this case there are two such classes, all other values of f_i are equal to 0. Consequently, the inequality is satisfied in each step, and thus we can choose suitable paths (as mentioned above) in each step until $\sum_{i=1}^k f_i = 2$. At this point, F is a Hamilton path with end vertices in different classes of G_0 , and we join these two end vertices and obtain a Hamilton cycle in G . \square

To prove Theorem 2.27, we shall find a particular linear forest in the multisplit graph and suppress all vertices of degree 2 in this forest, and apply Lemma 2.28 to the resulting graph. The linear forest is found with a basic argument using alternating paths and matchings in bipartite graphs.

Proof of Theorem 2.27. We let G be a 2-tough multisplit graph and let S be an independent set of vertices of G such that $T = V(G) \setminus S$ induces a complete k -partite graph with classes Q_1, Q_2, \dots, Q_k .

Using a similar argument as in the proof of Proposition 2.25, we observe that there is a spanning subgraph P of G consisting of vertex-disjoint copies of P_3 and K_1 , with the property that every vertex of S has degree 2 and every vertex of T has degree at most 1 in P .

For such subgraph P , we consider $f_i = \sum_{u \in Q_i} (2 - d_P(u))$ for every $i = 1, 2, \dots, k$, and we let f be the maximum of f_i over all i ; and we choose P in such a way that it minimizes f . We show the following.

Claim 2.4. *We have that $2f \leq \sum_{i=1}^k f_i$.*

Proof of Claim 2.4. To the contrary, we suppose that $2f > \sum_{i=1}^k f_i$, that is, $2f \geq 2 + \sum_{i=1}^k f_i$ (since $\sum_{i=1}^k f_i$ is even). We note that there is exactly one class of T for which the maximum f is obtained, and we let X denote this class. We call the edges of P *red*, and the remaining edges which are incident with a vertex of S are called *black*. We let A denote the set of all vertices of X not incident with any red edge, and we divide the remaining vertices of X over two disjoint sets B and C . In particular, we let B denote the set of all vertices b of $X \setminus A$ for which there is no alternating black-red path starting in b (with a black edge) and ending in a vertex of $T \setminus X$, and we let $C = X \setminus (A \cup B)$. Clearly, $f = 2|A| + |B| + |C|$ and $-f + \sum_{i=1}^k f_i \geq |T \setminus X|$. Thus, the assumption $f > -f + \sum_{i=1}^k f_i$ implies the following:

$$2|A| + |B| + |C| > |T \setminus X|. \quad (2.1)$$

In addition, we let β denote the set of all vertices of S adjacent to a vertex of B by a red edge, and we let γ denote the set of all vertices of $S \setminus \beta$ adjacent to a vertex of X in G . By the definition of β , and since no vertex of T is incident with two red edges, we have:

$$|B| \geq |\beta|. \quad (2.2)$$

By definition, no vertex of X is adjacent to a vertex of $S \setminus (\beta \cup \gamma)$. We show that no vertex of $A \cup B$ is adjacent to a vertex of γ . We recall that vertices of A are not incident with any red edge, and all vertices adjacent to B by a red edge are in β , so no vertex of γ is adjacent to $A \cup B$ by a red edge. For the black edges, we first observe that for every vertex of γ , there is an alternating red-black path starting in this vertex and ending in a vertex of $T \setminus X$. If a vertex a of A is adjacent to a vertex of γ by a black edge, then we modify P by swapping the colors of the edges along an alternating black-red path which starts in a and ends in a vertex of $T \setminus X$, a contradiction with the minimality of f (since we assumed that $2f \geq 2 + \sum_{i=1}^k f_i$). If a vertex of B is adjacent to γ by a black edge, then we observe that there is an alternating path contradicting the definition of B . This confirms the claim that no vertex of $A \cup B$ is adjacent to a vertex of γ .

We now distinguish two cases, and reach a contradiction in both of them. First, we suppose that $|C| \geq |\gamma|$. We delete all vertices of $T \setminus X$ and β and γ from G , and we note that the resulting graph has at least $|A| + |B| + |C|$ components.

By the assumption on C and by (2.1) and (2.2), we obtain $2(|A| + |B| + |C|) > |T \setminus X| + |\beta| + |\gamma|$, contradicting that G is 2-tough.

Next, we suppose that $|C| < |\gamma|$. After deleting all vertices of $T \setminus X$ and β and C from G , the resulting graph has at least $|A| + |B| + |\gamma|$ components. Similarly as above, we obtain $2(|A| + |B| + |\gamma|) > |T \setminus X| + |\beta| + |C|$, a contradiction. We conclude that $2f \leq \sum_{i=1}^k f_i$. \blacksquare

With Claim 2.4 on hand, we can easily apply Lemma 2.28 and conclude that G is Hamiltonian (if it has at least 3 vertices). To show the Hamilton-connectedness, we shall use the following technical claim.

Claim 2.5. *If $2f_j = \sum_{i=1}^k f_i$ and $|Q_j| \geq 2$ for some $j = 1, 2, \dots, k$, then there are vertices s, t, q of $S, T \setminus Q_j, Q_j$, respectively; such that the edge st belongs to P and the edge sq belongs to G but not to P .*

Proof of Claim 2.5. We consider the graph obtained from G by removing all vertices of $S \cup T \setminus Q_j$, and we note that this graph has $|Q_j|$ components. Since G is 2-tough, we have $|S \cup T \setminus Q_j| \geq 2|Q_j|$; and considering $2f_j = \sum_{i=1}^k f_i$, we observe that P contains an edge incident with a vertex of $T \setminus Q_j$ (and with a vertex of S).

To the contrary, we suppose that for every vertex s of S adjacent to a vertex of $T \setminus Q_j$ in P , we have that $N_G(s) \cap Q_j = N_P(s) \cap Q_j$. Considering an arbitrary vertex of S , we note that if it is adjacent to exactly one vertex (at least two vertices) of Q_j in G , then it is adjacent to exactly one vertex (exactly two vertices) of Q_j in P ; and we let S_1 (S_2) denote the set of all such vertices of S . We observe that $f_j = 2|Q_j| - |S_1| - 2|S_2|$ and $-f_j + \sum_{i=1}^k f_i = 2|T \setminus Q_j| - |S_1| - 2|S \setminus (S_1 \cup S_2)|$. Consequently, we have $|Q_j| + |S \setminus (S_1 \cup S_2)| = |T \setminus Q_j| + |S_2|$. Since $|Q_j| \geq 2$, we have $|Q_j| + |S \setminus (S_1 \cup S_2)| \geq 2$. We consider the graph obtained from G by removing all vertices of $(T \setminus Q_j) \cup S_2$, and we note that this graph has $|Q_j| + |S \setminus (S_1 \cup S_2)|$ components, a contradiction. \blacksquare

We note that if $2f_j = \sum_{i=1}^k f_i$ and $|Q_j| = 1$, then the toughness of G implies that G is K_3 (which is Hamilton-connected). Thus, if $2f_j = \sum_{i=1}^k f_i$, then we can assume that $|Q_j| \geq 2$.

We let u and v be vertices of G , and we show that there is a Hamilton path from u to v . We add the edge uv to G and to P (if not already present), and we let G_{uv} and P_{uv} denote the resulting graphs. If the edge uv is not in P , then we adjust P_{uv} as follows:

- If u and v belong to S , then we choose a vertex u' of $N_P(u)$ and v' of $N_P(v)$ (if possible such that u' and v' belong to different classes of T) and we remove edges uu' and vv' from P_{uv} . Furthermore, if u' and v' belong to the same class, say Q_j , (that is, all vertices of $N_P(u) \cup N_P(v)$ belong to Q_j) and $2f_j = \sum_{i=1}^k f_i$ (calculated in P), then we apply Claim 2.5 and we remove the edge st from P_{uv} and we add the edge sq .

- Let us suppose that u belongs to S and v belongs to T . If both vertices of $N_P(u)$ belong to Q_j and v belongs to $T \setminus Q_j$ and $2f_j = \sum_{i=1}^k f_i$, then we apply Claim 2.5 and we remove st from P_{uv} and we add sq . In any case, we remove from P_{uv} an edge e incident with u and with a vertex of $N_P(u)$ (in particular, if precisely one vertex of $N_P(u)$ belongs to Q_j and v belongs to $T \setminus Q_j$ and $2f_j = \sum_{i=1}^k f_i$, then we choose e as incident with the other vertex of $N_P(u)$; or if adding sq creates a cycle in P_{uv} , then we choose e such that it belongs to this cycle).
- Let us suppose that u and v belong to T and they have no common neighbour in P . If u and v belong to $T \setminus Q_j$ and $2f_j = \sum_{i=1}^k f_i$, then we apply Claim 2.5 and we add sq to P_{uv} , and we choose an edge incident with s and with a vertex of $T \setminus Q_j$ and remove this edge (in particular, if adding sq creates a cycle, then we choose such edge of this cycle).
- If u and v belong to T and they have a common neighbour c in P , then we choose a vertex of $\{u, v\}$, say u , and a vertex of $N_G(c) \setminus \{u, v\}$, say c' , (if possible such that one of the chosen vertices belongs to a class maximizing the value of f_i and the other belongs to a different classes of T) and we remove cv from P_{uv} and we add cc' . Furthermore, if all vertices of $N_G(c)$ belong to $T \setminus Q_j$ and $2f_j = \sum_{i=1}^k f_i$, then we apply Claim 2.5 and we add sq to P_{uv} and we remove st .

We note that the resulting P_{uv} is a linear forest such that every vertex of S has degree 2, and that $2f_j = \sum_{i=1}^k f_i$ is satisfied for the resulting P_{uv} . We suppress every vertex of S in P_{uv} , and we let F denote the resulting graph. We consider the complete k -partite graph induced by T , and we view this graph as playing the role of G_0 and we apply Lemma 2.28. In the graph $G - S$ extended by the edges of F , we get a Hamilton cycle containing F . Thus, in G_{uv} we have a Hamilton cycle containing the edge uv , that is, a Hamilton path from u to v in G . \square

Chapter 3

Constructions of relatively tough non-Hamiltonian graphs

In the previous chapter, we mentioned some examples of non-Hamiltonian graphs of relatively high toughness. Namely, the interval graphs, split graphs and circular arc graphs depicted in Figures 2.1, 2.2 and 2.3, respectively. In this chapter, we review additional constructions of such graphs. In fact, each of the constructed graphs has some stronger property which implies the non-Hamiltonicity (for instance, it has no 2-factor, no k -walk, no k -trestle, or its longest cycles are short). In particular, in Section 3.2 we recall the graphs constructed in [10] which provide the best available lower bound to Conjecture 1.1; and in Section 3.3, we study these graphs and we present alternative views on the construction. In addition, we present new results considering the shortness of longest cycles of planar graphs with relatively high toughness, see Theorems 3.2, 3.4 and 3.5.

3.1 No 2-factor

In addition to the result of Theorem 1.12, Enomoto et al. [41] presented a construction of graphs (satisfying the necessary condition on the number of vertices) with no k -factor and toughness arbitrarily close to k for every $k \geq 1$. (It seems that for $k = 1$, there is a technical problem in the construction. Anyway, we can simply consider graphs $K_{n,n+2}$ instead.)

We recall the case $k = 2$ of this construction. For $n \geq 1$, we consider the graph consisting of eight disjoint copies of K_{2n+2} ; with $x_{i,1}, x_{i,2}, \dots, x_{i,n}, y_{i,1}, y_{i,2}, \dots, y_{i,n+1}, z_i$ denoting the vertices for $i = 1, 2, \dots, 8$. We extend the graph by adding edges as follows. For every $i = 1, 3, 5, 7$ and $j = 1, 2, \dots, n$, we add $x_{i,j}x_{i+1,j}$; and for every $i = 2, 4, 6$ and $j = 1, 2, \dots, n+1$, we add $y_{i,j}y_{i+1,j}$; and for every $j = 1, 2, \dots, n+1$, we add $y_{1,j}y_{8,j}$. We subdivide each of these additional edges by adding one new vertex. Finally, we add two universal vertices. The construction is depicted in Figure 3.1.

3.1. No 2-factor

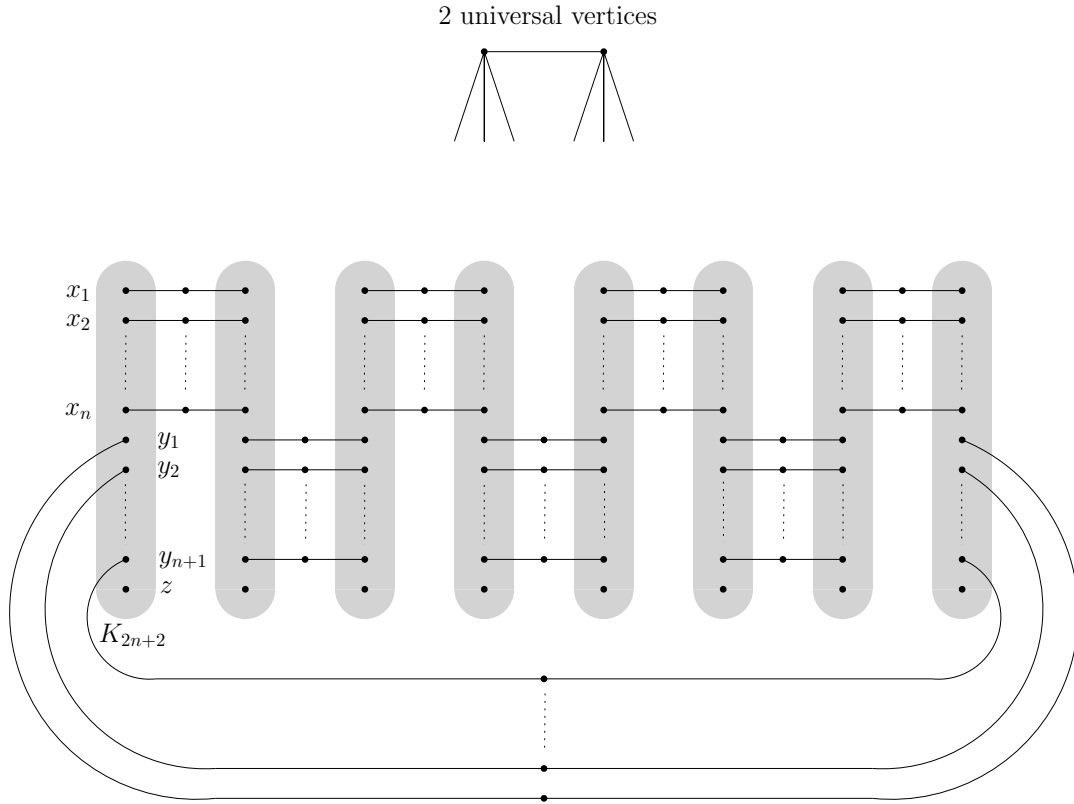


Figure 3.1: Enomoto et al. [41] constructed graphs with no k -factor and toughness arbitrarily close to k . This figure depicts the case $k = 2$ of the construction.

We note that the resulting graphs have no 2-factor, and with increasing n their toughness goes to 2; for more details, see [41]. (Clearly, a graph with no 2-factor is non-Hamiltonian.)

Simply spoken, the role of the universal vertices used in this construction is to eliminate small separating sets. Considering a graph obtained by this construction, we remark that by removing one universal vertex we obtain a graph of toughness $\frac{3}{2}$ (since the resulting graph has a vertex of degree 3). On the other hand, when adding (up to 13) extra universal vertices, the resulting graph does not become 2-tough (but when adding only two such vertices, the resulting graph is Hamiltonian).

We note that two similar constructions of graphs with no 2-factor and toughness arbitrarily close to 2 were outlined in [13], see for instance Figure 3.2. Although the constructions of [13] provide graphs with arbitrarily many universal vertices, the role of these vertices is very similar as we discussed for the construction of [41]. (Furthermore, it seems that only two such vertices are needed in the constructions of [13].)

In the following section, we discuss a different construction of graphs in which many universal vertices are used and, in fact, these vertices increase the bound

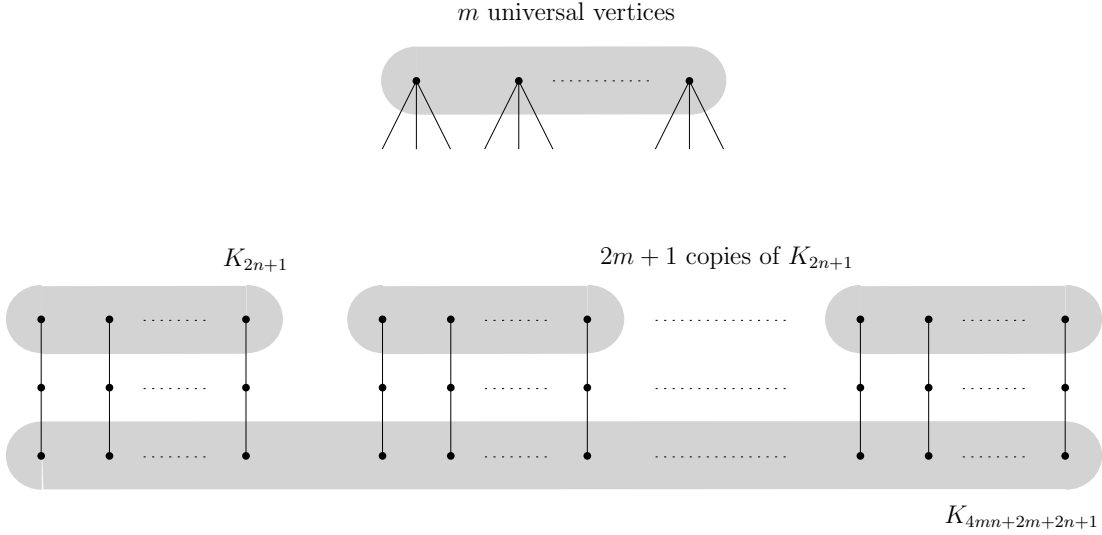


Figure 3.2: A construction of graphs with no 2-factor which was outlined in [13]. We note that for $m \geq 2$, the toughness of these graphs goes to 2 with increasing n .

on toughness attained by the construction. The constructed graphs have no Hamilton path (on the other hand, they have a 2-factor).

3.2 No Hamilton path

In this section, we recall the construction of Bauer, Broersma and Veldman [10] which provides the best available lower bound to Conjecture 1.1. The constructed graphs contain no Hamilton path and their toughness can be arbitrarily close to $\frac{9}{4}$ (which disproved a stronger version of Conjecture 1.1 which was also stated in [36]). We also recall that chordal graphs with no Hamilton path and relatively high toughness were obtained by a similar construction [10].

The graphs are obtained by interconnecting copies of a small graph which is referred to as a ‘building block’. The building block L with two distinguished vertices u and v is depicted in Figure 3.3. The key observation is that the graph L has no Hamilton path joining its two distinguished vertices. (We note that a similar construction using a ‘hypothetical’ building block appeared in [9].) For $n \geq 1$, we consider a graph obtained by taking $2n + 3$ disjoint copies of L and by adding edges such that the set of all $4n + 6$ distinguished vertices induces a complete graph. Clearly, this graph has no Hamilton path; and furthermore, this property is preserved when adding n universal vertices to the graph. (If a long path contains all vertices of a building block, then it contains an edge incident with a vertex of this block and with a universal vertex.) Furthermore, considering this construction with increasing n , the toughness of the resulting graphs goes to $\frac{9}{4}$. The construction is outlined in Figure 3.4. The complete argument can be

3.2. No Hamilton path

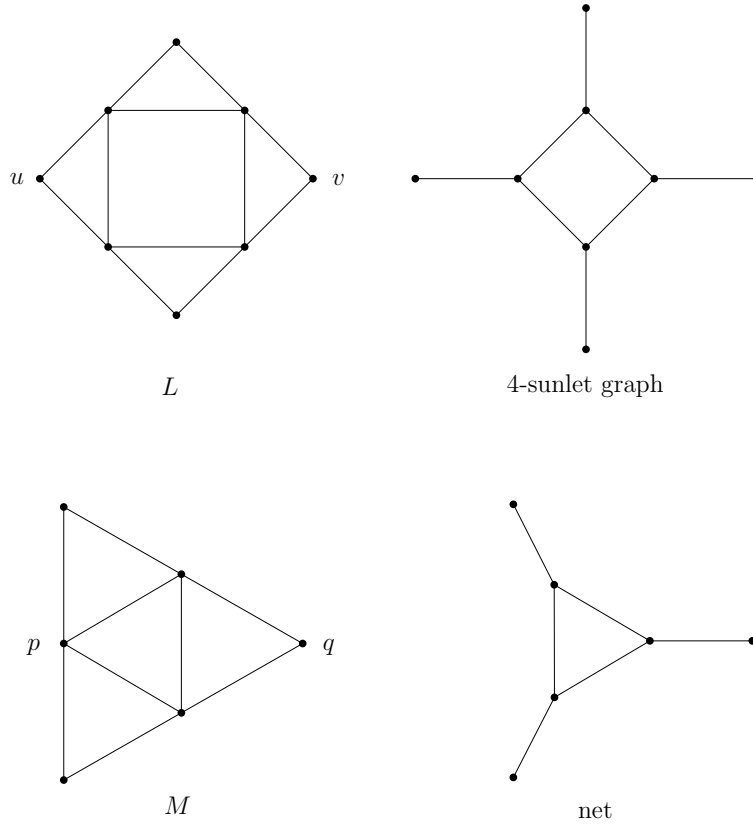


Figure 3.3: The building blocks L and M used in the construction of [10]. Clearly, the graph L has no Hamilton path between u and v , and M has no Hamilton path between p and q . (We remark that the blocks can be viewed as line graphs of the 4-sunlet graph and the net, respectively.)

found in [10].

The same construction with a different building block (the graph M depicted in Figure 3.3 with two distinguished vertices p and q) provides chordal graphs of toughness arbitrarily close to $\frac{7}{4}$. This construction is outlined in Figure 3.5 (for more details, see [10]); in addition, we present intersection representations of these graphs (since the graphs are chordal, the underlying graph is a tree, see also Sections 2.3 and 2.6). In the following section, we discuss the intersection representations of the graphs depicted in Figure 3.4.

We conclude this section by remarking that the split graphs presented in [71] (see Figure 2.2) can be viewed as obtained by a similar construction using the graph K_2 (with only one distinguished vertex) as a building block. In principle, the interval graphs depicted in Figure 2.2 can be viewed as a trivial version of this construction.

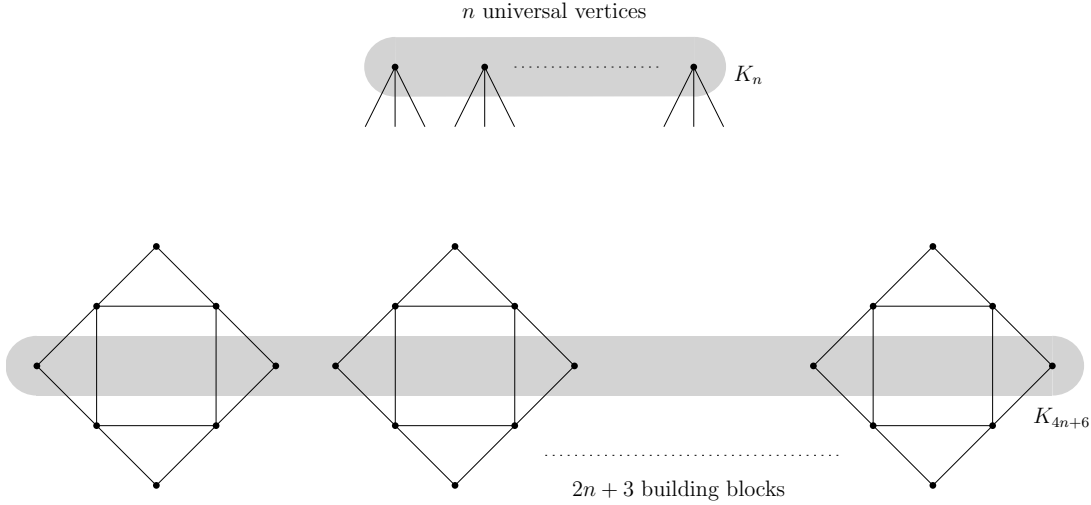


Figure 3.4: The construction of graphs presented in [10]. The constructed graphs have no Hamilton path, and with increasing n the toughness of these graphs goes to $\frac{9}{4}$.

3.3 Representing known non-Hamiltonian graphs of high toughness

We recall the concept of H -graphs discussed in Section 2.6. We view the graphs constructed in [10] (see Figure 3.4) as H -graphs; and in Figure 3.6, we present intersection representations of these graphs. (We note a similarity with the intersection representations depicted in Figure 3.5.)

In addition, we recall the graphs constructed in [71, 41, 13, 10] (see Figures 2.2, 3.1, 3.2, 3.4, respectively), and we remark that there is a similarity in the structure of these graphs. Namely, each of the graphs can be viewed as obtained from a line graph by adding universal vertices. For instance, we consider the graphs depicted in Figure 3.4 and the corresponding representations presented in Figure 3.6, and we note that most of the subgraphs in the intersection representations are copies of K_2 and K_1 . Considering a K_1 subgraph, we can add a pendant edge to the corresponding vertex of the underlying graph and extend the K_1 subgraph to the K_2 subgraph (by adding the new vertex); clearly, this modification preserves the intersections (that is, the extended representation gives the same graph). We apply such extending on all K_1 subgraphs, and we note that every subgraph of the resulting representation is either a copy of K_2 or a subgraph intersecting all other subgraphs. Furthermore, every edge of the extended underlying graph corresponds to precisely one K_2 subgraph. Thus, the graphs can be viewed as line graphs (of the extended underlying graphs) with additional universal vertices. We note that similar reasoning applies to the graphs constructed in [71, 41, 13], see Figures 2.2, 3.7, 3.8, respectively.

3.3. Representing known non-Hamiltonian graphs of high toughness

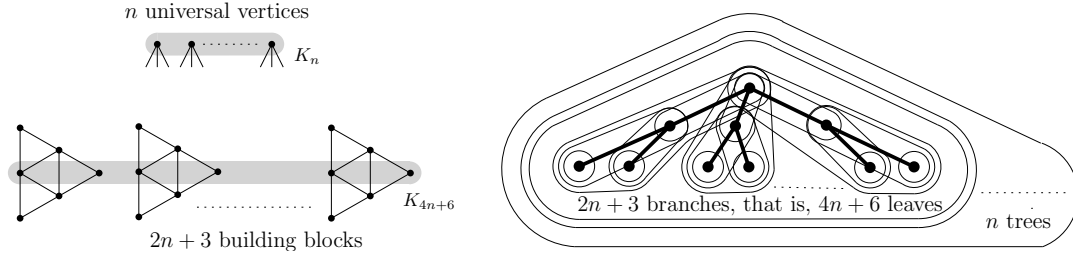


Figure 3.5: The construction of chordal graphs presented in [10] (left) and their intersection representations (right). Every oval depicts a subtree of an underlying tree (the tree is depicted in bold). Every subtree represents a vertex of the chordal graph, and two vertices are adjacent if and only if the corresponding subtrees share a vertex of the underlying tree. The graphs obtained by this construction have no Hamilton path, and with increasing n their toughness goes to $\frac{7}{4}$.

We view the intersection representations as a way of demonstrating that all these graphs have a similar and relatively simple structure. It seems natural to ask the following: Is this the best approach, or is it just the easiest approach to constructing such graphs?

We recall that 3-tough line graphs are Hamilton-connected [67]. Discussing the class of line graphs with additional universal vertices with Tomáš Kaiser and Petr Vrána, we observed that Conjecture 1.1 is also satisfied when restricted to this class. (In other words, if we try to improve the lower bound to Conjecture 1.1 by constructing line graphs with additional universal vertices, then the potential improvement is limited.) In fact, a more general statement can be observed, see Proposition 3.1. We recall that a class \mathcal{G} of graphs is *hereditary* if for every graph of \mathcal{G} , all its induced subgraphs belong to \mathcal{G} .

Proposition 3.1. *Let \mathcal{G} and \mathcal{G}' be classes of graphs such that \mathcal{G} is hereditary and every graph of \mathcal{G}' can be obtained from a graph of \mathcal{G} by adding a number of universal vertices. If Conjecture 1.1 is satisfied when restricted to \mathcal{G} , then it is satisfied when restricted to \mathcal{G}' .*

Proof. By the assumption, there exists t_0 such that every t_0 -tough graph (on at least 3 vertices) of \mathcal{G} is Hamiltonian. We say a graph H is *good* if either it is a complete graph or

$$\frac{|S|}{c(H - S) - 1} \geq t_0$$

for every set S of vertices such that $c(H - S) \geq 2$ (where $c(H - S)$ denotes the number of components of the graph $H - S$).

We show that every $2t_0 + 1$ -tough graph (on at least 3 vertices) of \mathcal{G}' is Hamiltonian. We consider such graph G' and a corresponding graph G of \mathcal{G} . Clearly, if G is good, then it is Hamiltonian (or it is a copy of K_1 or K_2), and

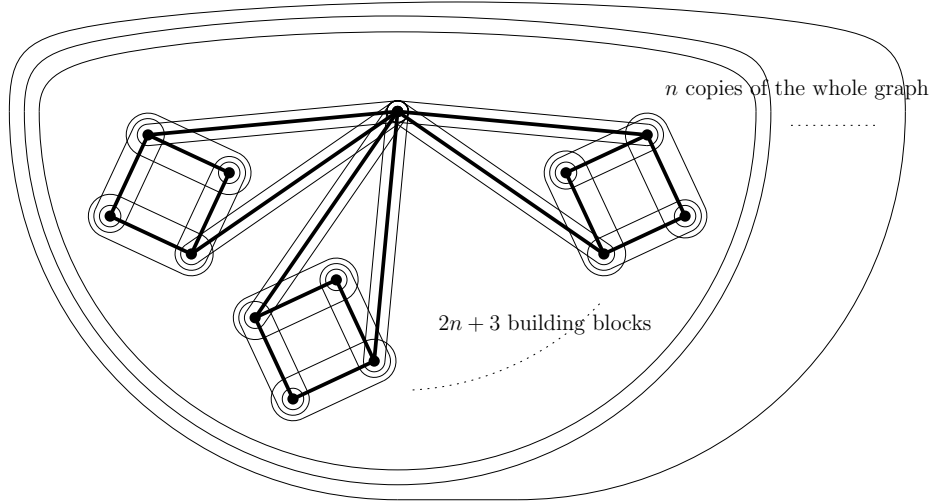


Figure 3.6: The intersection representations of the graphs depicted in Figure 3.4. We consider $2n + 3$ copies of $K_{2,3}$ with one distinguished vertex of degree 2 in each copy; the underlying graph (depicted in bold) is obtained by identifying the distinguished vertices. Every oval depicts a subgraph of the underlying graph. The subgraphs represent vertices of the intersection graph, and two vertices are adjacent if and only if the corresponding subgraphs have a common vertex.

thus G' is Hamiltonian. We suppose that G is not good, and we consider a set X of vertices of G maximizing $c(G - X)$ such that

$$\frac{|X|}{c(G - X) - 1} < t_0.$$

For the sake of simplicity, we let $u = |V(G') \setminus V(G)|$ and $x = |X|$ and $c = c(G - X)$; that is, we have

$$t_0(c - 1) > x.$$

Clearly, the toughness of G' implies that $x + u \geq c(2t_0 + 1)$. Consequently, we obtain that

$$u > x + c + 2t_0.$$

We consider a graph C corresponding to a component of $G - X$. By the maximality of X , we note that C is good, and therefore it is t_0 -tough. Furthermore, C belongs to \mathcal{G} (since \mathcal{G} is hereditary). Consequently, C is Hamiltonian (or a copy of K_1 or K_2); and thus we can consider a Hamilton path of C .

Since $u \geq x + c$, we can use the additional universal vertices and join the vertices of X and the Hamilton paths of the components of $G - X$ into a Hamilton cycle of G' . \square

We note that the constructions of [71, 10] (see Figure 2.2 and Figures 3.4, 3.5) produce graphs with arbitrarily many universal vertices and, in fact, these

3.3. Representing known non-Hamiltonian graphs of high toughness

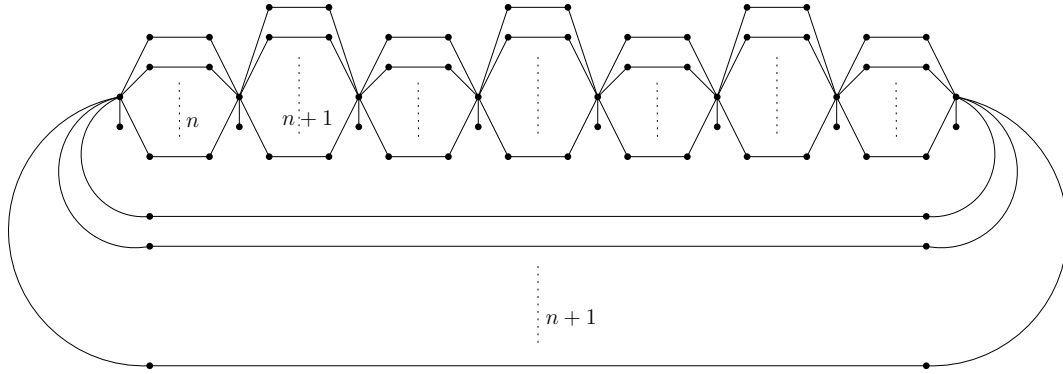


Figure 3.7: An outline of the preimage graphs related to the graphs depicted in Figure 3.1. We consider the line graphs of the outlined graphs, and we extend these line graphs by adding two universal vertices. We note that the resulting graphs are precisely the graphs depicted in Figure 3.1.

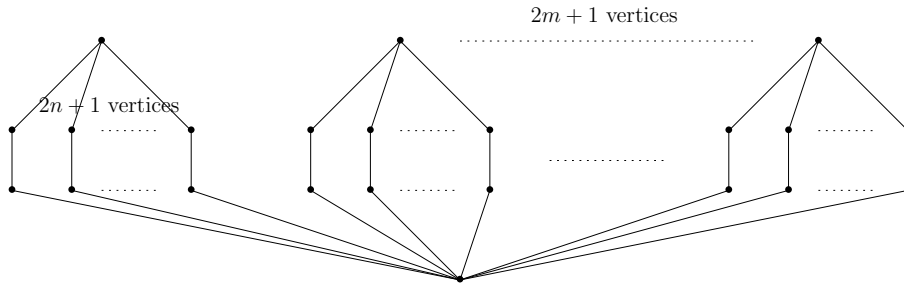


Figure 3.8: An outline of the preimage graphs related to the graphs depicted in Figure 3.2. We consider the line graphs of these graphs, and we extend them by adding m universal vertices, and we note that the resulting graphs are the graphs depicted in Figure 3.2.

vertices increase the bounds on toughness attained by the constructions. As noted in [25] (in relation to the constructions of [10]), one of the crucial properties of the building block is that (when disconnecting the whole graph) each block contributes by adding a relatively small number of components. In addition, some suggestions for trying to improve this construction were made in [25].

On the other hand, studying the construction of [10] and its modifications, it seems that the bound of ‘almost’ $\frac{9}{4}$ might be the limit of this method. Furthermore, even if the construction could be improved, it seems that the potential improvement by using this method is limited. (For the sake of discussion, let us suppose that we use the same construction and a more suitable building block with two distinguished vertices. For instance, we consider a graph obtained by using $2n + 1$ building blocks and n universal vertices. Clearly, after removing all distinguished vertices and all universal vertices the resulting graph has at least $2n + 1$ components. Thus, the toughness of the constructed graph is at most

$\frac{5n+2}{2n+1}$.) Therefore, the question is: are there other approaches to constructing such graphs? An additional motivation for this question arises from possible modifications of such constructions, see for instance Section 3.4.

In relation to Conjecture 1.7 and Question 1.9 (see Section 1.4), we remark that by adding a universal vertex to a graph which contains k independent vertices, we obtain $K_{1,k}$ as an induced subgraph. Thus, we should also think of different constructions when trying to approach these problems.

3.4 No walks and no trestles

We recall that every Hamiltonian graph is 1-tough. Furthermore, Jackson and Wormald [60] observed that every graph which contains a k -walk is $\frac{1}{k}$ -tough. Regarding Conjecture 1.13, the best available lower bound is due to Ellingham and Zha [40] who constructed graphs with no k -walk and toughness ‘almost’ $\frac{8k+1}{4k(2k-1)}$ for every $k \geq 1$. (In particular, there are graphs with no 2-walk and toughness ‘almost’ $\frac{17}{24}$.) Actually, the graphs are obtained by a simple modification of the construction of [10]. The building block (the graph L depicted in Figure 3.3) is extended by adding $k - 1$ pendant edges to each of the distinguished vertices and their neighbours (see Figure 3.9), and fewer additional universal vertices are used in the construction.

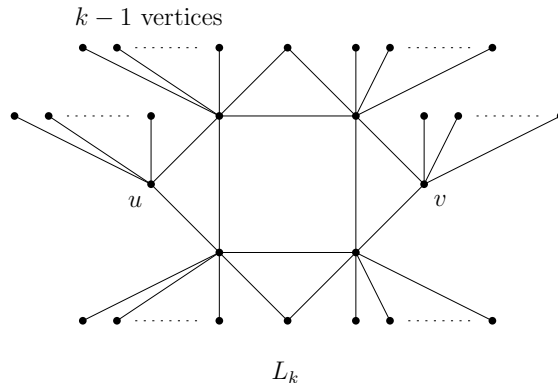


Figure 3.9: The building block L_k used in the construction of [40]. Clearly, the graph L_k has no walk from u to v which uses every vertex at least once and at most k times.

Similarly, Tkáč and Voss [97] observed that every graph with a k -trestle is $\frac{2}{k}$ -tough. In relation to Conjecture 1.14, we note that there are graphs with no k -trestle and toughness greater than 1 for every $k \geq 3$. For instance, in a discussion with Jakub Teska, we observed that such graphs of toughness ‘almost’ $\frac{k+1}{k}$ (see Figure 3.10) can be obtained by a basic modification of the construction of [71].

3.5. Shortness of longest cycles

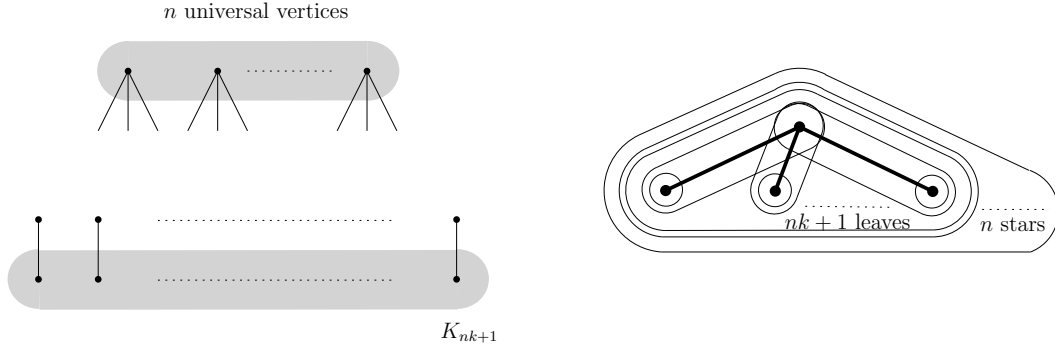


Figure 3.10: A construction of graphs with no k -trestle for every $k \geq 2$ (left) and their intersection representations (right). We note that with increasing n the toughness of these graphs goes to $\frac{k+1}{k}$.

3.5 Shortness of longest cycles

In this section, we discuss graphs whose longest cycles are relatively short (compared to the number of vertices of the graph).

Studying non-Hamiltonian 3-connected planar graphs, Grünbaum and Walther [54] introduced the so-called shortness exponent (and shortness coefficient and shortness index) as a way of measuring the shortness of longest cycles in graphs belonging to a given class. We recall that the *shortness exponent* of a class of graphs Γ is the \liminf , taken over all infinite sequences (G_n) of non-isomorphic graphs of Γ (for n going to infinity), of the logarithm of the length of a longest cycle in G_n to base equal to the number of vertices of G_n .

In addition, Grünbaum and Walther [54] presented upper bounds on the shortness exponent for numerous subclasses of the class of 3-connected planar graphs, and they remarked that the upper bound for the class of 3-connected planar graphs itself had been presented formerly by Moser and Moon [80] who used a slightly different notation. (We recall that 4-connected planar graphs are Hamilton-connected [94].) Furthermore, they suggested to study the introduced parameters for other classes of graphs, not necessarily planar or 3-connected. Later, Chen and Yu [33] showed that every 3-connected planar graph G contains a cycle of length at least $|V(G)|^{\log_3 2}$; in combination with the bound of [80], it follows that the shortness exponent of this class equals $\log_3 2$. Many of the results considering the shortness exponent and similar parameters are to be found surveyed in [84].

In the remainder of this section, we mention results considering planar graphs of certain toughness. In [55], Harant showed that the shortness exponent of the class of $\frac{3}{2}$ -tough planar graphs is less than 1. (We recall that every planar graph of toughness greater than $\frac{3}{2}$ is Hamilton-connected.) Similarly, Owens [83] presented non-Hamiltonian maximal planar graphs of toughness arbitrarily close to $\frac{3}{2}$. (In

fact, Owens showed that the shortness coefficient of this class of graphs is less than 1. Simply spoken, the class contains infinitely many graphs such that the number of vertices which are missed by a longest cycle of the graph is at least linear in the number of vertices of the graph.) In addition, Owens asked whether there exists a non-Hamiltonian maximal planar graph of toughness exactly $\frac{3}{2}$. This question is still open. Considering maximal planar graphs of smaller toughness, it was shown that their longest cycles can be even shorter. In particular, Harant and Owens [56] argued that the shortness exponent of the class of $\frac{5}{4}$ -tough maximal planar graphs is at most $\log_9 8$, and Tkáč [96] showed that it is at most $\log_6 5$ for the class of 1-tough maximal planar graphs. Similarly, considering the class of 1-tough chordal planar graphs Böhme et al. [23] argued that the shortness exponent is at most $\log_9 8$.

In [63] (see Appendix B), we formalize the ideas of the commonly used construction for bounding the shortness exponent, and we improve and generalize the results of [56, 96] as follows:

Theorem 3.2. *Let σ be the shortness exponent of the class of maximal planar graphs under certain toughness restriction.*

- (1) *If the graphs are $\frac{5}{4}$ -tough, then σ is at most $\log_{30} 22$.*
- (2) *If the graphs are $\frac{8}{7}$ -tough, then σ is at most $\log_6 5$.*
- (3) *If the toughness of the graphs is greater than 1, then σ equals $\log_3 2$.*

Regarding item (3) of Theorem 3.2, we remark that we show that the value $\log_3 2$ is the upper bound, and the sharpness of the bound follows from [33].

In the following section, we improve the results of [23]. In particular, we show that for the class of 1-tough planar 3-trees, the shortness exponent is at most $\log_{30} 22$ (a similar bound is shown in item (1) of Theorem 3.2). We view the value $\log_{30} 22$ as reflecting the shortness of the longest cycles in graphs of the considered classes, but also as reflecting how the parameter itself is defined (for more details, see Appendix B).

3.6 Short longest paths of 1-tough chordal planar graphs and k -trees

In this section, we revisit the topic studied in Section 2.4, and we present additional new results considering long paths and toughness of k -trees and chordal planar graphs. We note that the results included in this section are also available in the manuscript [64]. The author would like to thank Jakub Teska for helpful discussions resulting in a weaker version of Theorem 3.4, which partly motivated this study.

We recall that (in addition to the result of Theorem 2.10) Böhme et al. [23] presented 1-tough chordal planar graphs whose longest cycles are relatively short; and they argued the following:

Theorem 3.3. *The shortness exponent of the class of 1-tough chordal planar graphs is at most $\log_9 8$.*

We improve the bound of Theorem 3.3 in two different ways. We present 1-tough chordal planar graphs and 1-tough planar 3-trees whose longest paths and cycles are relatively short.

In particular, for every $\varepsilon > 0$, there exists a 1-tough chordal planar graph G (on at least 3 vertices) whose longest path has less than $|V(G)|^\varepsilon$ vertices. We work with the characterization of trees whose square has a Hamilton path by Gould [52] and a similar characterization considering a Hamilton cycle by Harary and Schwenk [58]. Furthermore, we show that the square of every subcubic tree is a 1-tough chordal planar graph (we use Theorem 1.11 and Propositions 3.8 and 3.9). Consequently, we adjust the result of Theorem 3.3 as follows:

Theorem 3.4. *The shortness exponent of the class of 1-tough chordal planar graphs is 0.*

The proof of Theorem 3.4 is included in this section. We remark that the graphs constructed in [23] are 3-connected, so the bound $\log_9 8$ of Theorem 3.3 also applies to the shortness exponent of the class of 1-tough planar 3-trees (see Proposition 2.19). We use the standard construction for bounding the shortness exponent (for more details regarding this construction, see for instance [84] or [63]), and we improve this bound by the following:

Theorem 3.5. *The shortness exponent of the class of 1-tough planar 3-trees is at most $\log_{30} 22$.*

The proof of Theorem 3.5 is also included in this section. We recall that, in addition to the result of Theorem 2.9, Broersma, Xiong and Yoshimoto constructed 1-tough k -trees which have no Hamilton cycle for every $k \geq 3$. We extend the construction used for proving Theorem 3.5, and we remark that there are k -trees of toughness greater than 1 whose longest paths are relatively short for every $k \geq 4$. (Meanwhile, 3-trees of toughness greater than 1 are Hamilton-connected by Theorem 2.11.) This remark slightly improves the lower bound on toughness of non-Hamiltonian k -trees presented in [30].

Aiming for Theorem 3.3, we start by recalling the Hamiltonian properties of squares of trees. (We remark that for $k \geq 2$, the square of every k -tree is Hamilton-connected by [35, 44].) We recall that Gould [52] characterized trees whose square has a Hamilton path. For convenience, we restate this characterization by considering forbidden subgraphs, and we provide a short proof along similar lines.

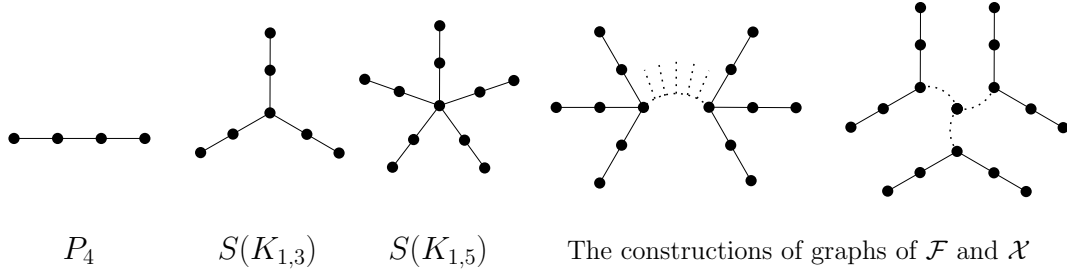


Figure 3.11: The graphs P_4 , $S(K_{1,3})$, $S(K_{1,5})$ and the families of graphs \mathcal{F} and \mathcal{X} . The graphs of \mathcal{F} are obtained from two copies of $S(K_{1,3})$ by joining their central vertices with a path (possibly an edge) and adding one pendant edge to each interior vertex of this path. The graphs of \mathcal{X} are obtained from three copies of P_5 and from a tree containing exactly three leaves by identifying each of these leaves with the central vertex of one P_5 .

In Theorem 3.6, we collect this restated characterization and a similar result of Harary and Schwenk [58] considering Hamilton cycles and a trivial result considering Hamilton-connectedness. The forbidden subgraphs are depicted in Figure 3.11.

Theorem 3.6. *Let T be a tree. Then the following statements are satisfied:*

- (1) T^2 is Hamilton-connected (a complete graph) if and only if T is P_4 -free,
- (2) T^2 is Hamiltonian if and only if T (on at least 3 vertices) is $S(K_{1,3})$ -free,
- (3) T^2 has a Hamilton path if and only if T is $S(K_{1,5})$ -free, \mathcal{F} -free and \mathcal{X} -free.

Proof. We show (1). We suppose that a path $v_1v_2v_3v_4$ is a subgraph of T . We note that $\{v_2, v_3\}$ is a cut in T^2 , so there is no Hamilton path between v_2 and v_3 . On the other hand, if T is P_4 -free, then T^2 is a complete graph, and thus Hamilton-connected.

We recall that (2) was shown in [58].

We show (3). We suppose that T contains a subtree S such that either S is $S(K_{1,5})$ or it belongs to $\mathcal{F} \cup \mathcal{X}$. We observe that S^2 has no Hamilton path. Furthermore, we consider a path in T^2 , and we suppress every vertex of this path not belonging to S , and we note that the resulting graph is a path in S^2 . Consequently, the resulting path does not contain all vertices of S , and so neither does the considered path in T^2 . Thus, T^2 has no Hamilton path.

Since T is \mathcal{X} -free, there is a path containing the central vertices of all subtrees $S(K_{1,3})$, and we let $P = v_1v_2 \dots v_k$ be a longest such path. We consider the components of the forest obtained from T by removing all edges of P ; and for every $i = 1, 2, \dots, k$, we let T_i denote the subtree of T corresponding to the component containing v_i .

We consider the vertices of P and their neighbours in T . Since T is $S(K_{1,5})$ -free, every vertex of P has at most four non-leaf neighbours. Furthermore, it has

at most two such neighbours outside P (since P is chosen as longest). So T_i is $S(K_{1,3})$ -free for every $i = 1, 2, \dots, k$. We say a vertex of P is *green* if it has no neighbour outside P , we say it is *yellow* if it has a neighbour outside P and it has at most one non-leaf neighbour outside P , and we say it is *red* if it has two such non-leaf neighbours.

For every $i = 1, 2, \dots, k$, we shall find a particular Hamilton path H_i in T_i^2 ; and we shall join the paths H_1, H_2, \dots, H_k and obtain a Hamilton path of T^2 . Considering a vertex v_i of P , we note the following.

If v_i is green, then v_i is the only vertex of T_i , and we view this vertex as the (trivial) path H_i .

If v_i is yellow, then v_i is incident with an edge e_i such that e_i is an end-edge of a longest path of T_i (that is, e_i is incident with a leaf of this path). By [58, Theorem 2], T_i^2 has a Hamilton cycle containing e_i ; and we view it as a Hamilton path H_i whose ends are v_i and its neighbour in T_i .

We show that if v_i is red, then T_i^2 has a Hamilton path whose both ends are neighbours of v_i in T_i . We let u_i be a non-leaf neighbour of v_i in T_i , and we consider the two subtrees corresponding to the components of $T_i - v_i u_i$ and we add an auxiliary copy of v_i to the subtree which contains u_i . By a similar argument as we used for the yellow vertices, the square of each of these two subtrees has a Hamilton path between v_i and its neighbour in T_i ; and we glue these two paths by identifying v_i with its copy and obtain a desired path H_i .

We consider a subpath of P between two red vertices. Since T is \mathcal{F} -free, every such subpath contains a green vertex. Thus, we observe that we can join the paths H_1, H_2, \dots, H_k and obtain a Hamilton path in T^2 . \square

Now, we focus on long paths in 1-tough chordal planar graphs. We shall use Theorem 3.6 and show the following:

Proposition 3.7. *For every n_0 , there exists a 1-tough chordal planar graph on $n > n_0$ vertices whose longest cycle has $4 \log_2 \frac{n+2}{3}$ vertices and whose longest path has $2(\log_2 \frac{n+2}{3})^2 + 2$ vertices.*

In particular, the first part of Proposition 3.7 immediately implies the result of Theorem 3.4.

Proof of Theorem 3.4. We consider an infinite sequence of non-isomorphic graphs given by Proposition 3.7. We recall that a graph on n vertices belonging to this sequence has a longest cycle on $4 \log_2 \frac{n+2}{3}$ vertices. Consequently, the considered shortness exponent is at most $\lim_{n \rightarrow \infty} \log_n(4 \log_2 \frac{n+2}{3})$. \square

In addition to Theorem 3.6, we shall need several simple statements. First, we recall that by Theorem 1.11, the square of a k -connected graph is k -tough. Also we recall that (as observed by Fulkerson and Gross [48]) a graph G is chordal if and only if it has a *perfect elimination ordering*, that is, an ordering (v_1, v_2, \dots, v_n)

of all vertices of G such that v_i is a simplicial vertex of G_i for every $i = 1, 2, \dots, n$, where G_i is the subgraph of G induced by $\{v_1, v_2, \dots, v_i\}$. We note the following:

Proposition 3.8. *The square of a tree is a chordal graph.*

Proof. Clearly, a perfect elimination ordering of the tree is a perfect elimination ordering of its square. \square

We note that Harary et al. [57] characterised graphs whose squares are planar. Working with squares of trees, we can use a simpler restricted version of this characterisation. We recall that a graph is *subcubic* if it has no vertex of degree greater than 3.

Proposition 3.9. *Let T be a tree. Then T^2 is planar if and only if T is subcubic.*

Proof. Being planar, T^2 is K_5 -free, and thus T has no vertex of degree greater than 3.

We prove the reverse implication by induction on the number of vertices. Clearly, every graph on at most four vertices is planar. We choose a leaf v of T ; and we let N denote the set of all its neighbours in T^2 , and we note that $3 \geq |N| \geq 2$ since T is subcubic and since we can assume that it has at least five vertices.

We show that there is a facial cycle containing all vertices of N in every planar embedding of $(T - v)^2$. Clearly, $(T - v)^2$ is 2-connected, so every edge is contained in some facial cycle which proves the claim for the case $|N| = 2$. In case $|N| = 3$, we use the fact that if a cycle contains the edge incident with two vertices of N which are non-adjacent in T , then it contains all vertices of N . Thus, there is a facial cycle containing all vertices of N . We embed v inside this facial cycle and obtain a planar embedding of T^2 . \square

We recall that a *cubic tree* is a tree which is subcubic and has no vertex of degree 2. Finally, we construct graphs having the properties stated in Proposition 3.7.

Proof of Proposition 3.7. We let T be a cubic tree (on at least 4 vertices) having a vertex such that the distances from this vertex to every leaf are the same; and we let r denote this distance. By Theorem 1.11 and Propositions 3.8 and 3.9, T^2 is a 1-tough chordal planar graph.

We let n denote the number of vertices of T . By simple counting arguments, we get that $n = 3 \cdot 2^r - 2$ (that is, $r = \log_2 \frac{n+2}{3}$) and that a largest $S(K_{1,3})$ -free subtree of T has $4r$ vertices. Furthermore, T is $S(K_{1,5})$ -free and \mathcal{F} -free (since it is a cubic tree), and a largest \mathcal{X} -free subtree of T has $2(r^2 + 1)$ vertices. By Theorem 3.6, a longest cycle of T^2 has $4 \log_2 \frac{n+2}{3}$ vertices and its longest path has $2(\log_2 \frac{n+2}{3})^2 + 2$ vertices. \square

Now, we focus on long paths in 1-tough planar 3-trees. In order to prove Theorem 3.5, we show the following:

Proposition 3.10. *Let n be a non-negative integer and let $c(n) = 1 + 62(1 + 22 + \dots + 22^n)$. Then there exists a 1-tough planar 3-tree H_n on $1 + 70(1 + 30 + \dots + 30^n)$ vertices whose longest cycle has $c(n)$ vertices and whose longest path has $c(n) + 2 + 2 \sum_{i=0}^{n-1} c(i)$ vertices.*

We note that the desired result follows as a corollary of Proposition 3.10.

Proof of Theorem 3.5. We consider the sequence of graphs H_1, H_2, \dots given by Proposition 3.10; and for every $n \geq 0$, we let $f(n)$ denote the number of vertices of H_n . Clearly,

$$f(n) = 1 + \frac{70}{29}(30^{n+1} - 1) \quad \text{and} \quad c(n) = 1 + \frac{62}{21}(22^{n+1} - 1).$$

Thus,

$$\lim_{n \rightarrow \infty} \log_{f(n)} c(n) = \log_{30} 22$$

and the considered shortness exponent is at most $\log_{30} 22$. \square

We construct the graphs H_n and prove Proposition 3.10. We remark that, as well as in [23], we shall use the standard construction for bounding the shortness exponent; the improvement of the bound comes with a choice of a more suitable starting graph H_0 . The reasoning behind this choice is similar to the one applied in [63] (see Appendix B).

We consider the graph H_0 constructed in Figure 3.12; and we let u_1, u_2, u_3 denote the vertices of its outer face in the present embedding. Clearly, there are 30 vertices of degree 3 in H_0 ; and we call these vertices *white*.

For every $n \geq 0$, we let H_{n+1} be a graph obtained from H_n by replacing every white vertex of H_n with a copy of H_0 and by adding edges which connect vertices u_i (for $i = 1, 2, 3$) of this copy to $4 - i$ neighbours of the replaced vertex. We note the following:

Proposition 3.11. *For every $n \geq 0$, the graph H_n is a planar 3-tree.*

Proof. Following Figure 3.12, we let u_1, u_2, \dots, u_{71} denote the vertices of H_0 . Clearly, $\{u_1, u_2, u_3\}$ induces K_3 , and we consider adding vertices u_4, u_5, \dots, u_{71} in sequence (in this order), and we observe that H_0 is a 3-tree by definition.

We view the replacement of a white vertex by a copy of H_0 as identifying this white vertex with the vertex u_1 of this copy and adding vertices u_2, u_3, \dots, u_{71} of this copy in sequence, and we note that H_n is a 3-tree for every $n \geq 0$.

We consider the planar embedding of H_0 given by Figure 3.12. When replacing a white vertex by a copy of H_0 , we proceed in two steps. First, we remove the white vertex, and we note that its neighbourhood induces a facial cycle. Next, we embed a copy of H_0 inside this facial cycle, and we observe that the additional edges can be embedded as non-crossing. Consequently, H_n is planar for every $n \geq 0$. Alternatively, the planarity can be observed using Proposition 2.20. \square

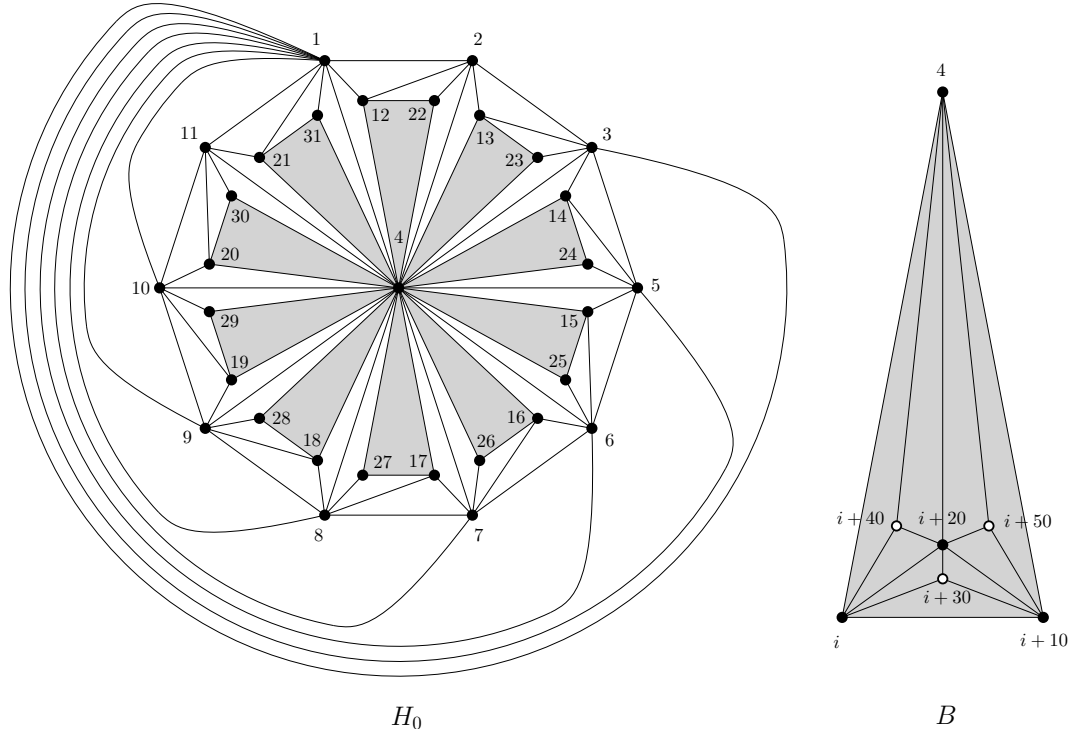


Figure 3.12: The graph B and the construction of the graph H_0 . The graph H_0 is obtained by replacing each of the highlighted triangles (of the graph depicted on the left) with a copy of B in the natural way (by identifying the vertices of the highlighted triangle with the vertices of degree 5 in B). The numbers represent the ordering of vertices of H_0 .

To verify the toughness of the graphs H_n , we shall use the following lemma shown in [63] (see Appendix B).

Lemma 3.12. *For $i = 1, 2$, let G_i^+ and G_i be t -tough graphs such that G_i is obtained by removing vertex v_i from G_i^+ . Let U be a graph obtained from the disjoint union of G_1 and G_2 by adding new edges such that the minimum degree of the bipartite graph $(N(v_1), N(v_2))$ is at least t . Then U is t -tough.*

In order to apply Lemma 3.12, we determine the toughness of H_0^+ , that is, the graph obtained from H_0 by adding one auxiliary vertex x adjacent to u_1, u_2 and u_3 .

Proposition 3.13. *The graphs H_0^+ and H_0 are 1-tough.*

Proof. We consider a separating set S of vertices of H_0^+ . If u_4 belongs to a component of $H_0^+ - S$, then every other component has exactly one vertex, and we note that $|S| > c(H_0^+ - S)$.

We assume that u_4 belongs to S . Except for u_4 , the vertices adjacent to a white vertex are called *black*. Except for u_4 and x , the non-white and non-black

vertices are called *blue*. We shall use a discharging argument. We assign charge 1 to every component of $H_0^+ - S$, and we distribute all assigned charge among the vertices of S according to the following rules.

- The component containing x (if there is such) gives its charge to u_4 .
- The total charge of all components which consist exclusively of white vertices is distributed equally among black vertices of S .
- The total charge of all remaining components is distributed equally among blue vertices of S .

We observe that every vertex of S receives charge at most 1, that is, $|S| \geq c(H_0^+ - S)$. Thus, H_0^+ is 1-tough. Consequently, H_0 is 1-tough by Proposition 2.13. \square

Proposition 3.14. *For every $n \geq 0$, the graph H_n is 1-tough.*

Proof. By Proposition 3.13, H_0^+ and H_0 are 1-tough. We consider the iterative construction of H_n replacing white vertices in sequence. We use Lemma 3.12 and show that the graph in each iteration is 1-tough. The graph at a current step plays the role of G_1^+ and the replaced vertex the role of v_1 , and H_0^+ and H_0 play the role of G_2^+ and G_2 . Using Lemma 3.12 repeatedly, we conclude that H_n is 1-tough. \square

We recall the standard construction for bounding the shortness exponent (this construction produces graphs whose longest cycles are relatively short). The idea of the construction is formalized in the following definition and in Lemma 3.15 which is proven in [63] (see Appendix B).

An *arranged block* is a 5-tuple (G_0, j, W, O, k) where G_0 is a graph, j is the number of vertices of G_0 , and W and O are disjoint sets of vertices of G_0 such that the vertices of W are simplicial and independent and O induces a complete graph and such that every cycle in G_0 contains at most k vertices of W .

Lemma 3.15. *Let (G_0, j, W, O, k) be an arranged block such that $k \geq 1$. For every $n \geq 1$, let G_n be a graph obtained from G_{n-1} by replacing every vertex of W with a copy of G_0 (which contains W and O), and by adding arbitrary edges which connect the neighbourhood of the replaced vertex with the set O of the copy of G_0 replacing this vertex. Then G_n has $1 + (j - 1)(1 + |W| + \dots + |W|^n)$ vertices and its longest cycle has at most $1 + (\ell - 1)(1 + k + \dots + k^n)$ vertices where $\ell = j - |W| + k$.*

Finally, we show that the constructed graphs H_n have all properties stated in Proposition 3.10.

Proof of Proposition 3.10. By Propositions 3.11 and 3.14, H_n is a 1-tough planar 3-tree. By simple counting arguments, we get that H_n has $1 + 70(1 + 30 + \dots + 30^n)$ vertices.

We observe that a path in H_0 contains at most $22 + z$ white vertices where z is the number of white ends of the path.

In particular, every cycle in H_0 contains at most 22 white vertices. By Lemma 3.15, a longest cycle in H_n has at most $c(n)$ vertices.

We let $p(n) = c(n) + 2 + 2 \sum_{i=0}^{n-1} c(i)$ and $w(n) = 22^{n+1} + 2 \sum_{i=0}^n 22^i$. For the sake of induction, we show that every path in H_n has at most $p(n)$ vertices, and furthermore it contains at most $w(n)$ white vertices (a similar idea is used in [63]). We note that the claim is satisfied for $n = 0$, and we proceed by induction on n .

We consider a path P in H_n , and for every newly added copy of H_0 , we suppress all but one vertex of the copy and we replace the remaining vertex (if there is such) by the corresponding replaced vertex of H_{n-1} ; and we let P' be the resulting graph. Since the neighbourhood of every replaced vertex induces a complete graph, P' is a path, and we view P' as a path in H_{n-1} . By the hypothesis, P' contains at most $w(n-1)$ white vertices. Thus, P visits at most $w(n-1)$ of the newly added copies of H_0 .

Similarly, we choose an arbitrary newly added copy of H_0 , and we suppress all vertices of P not belonging to this copy. Since $\{u_1, u_2, u_3\}$ induces a complete graph, the resulting graph is a path in H_0 (possibly empty or trivial). Considering such paths for all newly added copies of H_0 and the set of all their ends, we note that at most two white vertices belong to this set (since P is a path). Thus, in total these paths contain at most $63 \cdot w(n-1) + 2$ vertices. Furthermore, P contains at most $w(n)$ white vertices.

We note that

$$p(n) = p(n-1) - w(n-1) + 63 \cdot w(n-1) + 2.$$

Consequently, a longest path of H_n has at most $p(n)$ vertices.

We extend the earlier observation as follows. In fact, there are paths in H_0 containing $22 + z$ white and all non-white vertices such that all non-white ends belong to $\{u_1, u_2\}$. Using these paths, we observe that H_n has a cycle on $c(n)$ vertices and a path on $p(n)$ vertices. \square

We remark that for every $k \geq 4$, there are k -trees of toughness greater than 1 whose longest paths are relatively short. For brevity, we omit enumerating the exact length of these paths.

We consider the 1-tough 3-trees H_n given by Proposition 3.10. Clearly, adding a universal vertex to a k -tree gives a $(k+1)$ -tree. For every $k \geq 4$ and every $n \geq 0$, we let $H_{n,k}$ denote the graph obtained by adding $k-3$ universal vertices to H_n ; and we note that $H_{n,k}$ is a k -tree of toughness greater than 1.

We consider a path in $H_{n,k}$. We remove the universal vertices of $H_{n,k}$ from this path, and we view the resulting forest (whose components are paths) as a subgraph of H_n . By Proposition 3.10, every path of this forest is relatively short. Consequently, we observe that for every $k \geq 4$, there exists n_0 such that if $n \geq n_0$,

3.6. Short longest paths of 1-tough chordal planar graphs and k -trees

then a longest path in $H_{n,k}$ is relatively short. (We note that the same idea can be applied to the graphs constructed in [23].)

Chapter 4

Conclusion

As recalled in Chapter 1, there is a number of results and open problems closely related to Conjecture 1.1; and we view these as providing additional motivation for the study of toughness and Hamiltonicity in graphs.

In Chapters 2 and 3, we reviewed different approaches towards Conjecture 1.1. In Chapter 2, we focused on partial results on Conjecture 1.1 in restricted classes of graphs (for instance, in interval graphs, chordal graphs, circular arc graphs, in k -trees and chordal planar graphs, or in graphs defined by forbidden subgraphs). In particular, in Section 2.6 we discussed the ‘underlying structure’ of some of these graphs which seems to be an important ingredient for proving the results. In Section 2.9, we noted a similar ‘duality character’ of the main tools which are used for proving some of the results. In Chapter 3, we analysed constructions of non-Hamiltonian graphs of relatively high toughness. In particular, in Section 3.3 we discussed common features of some of these constructions, and we present alternative views on the constructions.

Conjecture 1.1 is still open, and over the past 45 years it has motivated the study of a number of related topics. In this study, different ideas were introduced and eventually some of the related questions were solved. The problem appears to be challenging also in the sense that it is non-trivial even when restricted to some classes of graphs which are generally considered as being well-studied (for instance, to chordal graphs or $K_{1,3}$ -free graphs). Considering Conjecture 1.1 (in its general setting), it seems that our understanding of the problem is not deep enough. In particular, regarding the known partial results, the restrictions on the considered classes of graphs seem to be relatively strong. Furthermore, we recall that Conjecture 1.14 is not resolved for any k (which is seemingly a much weaker version of the problem). On the other hand, considering the constructions which provide lower bounds to Conjecture 1.1 and to related problems, it seems that many of these constructions are based on similar ideas. It is not clear whether these ideas could be used for further improving the lower bound; however, it seems that the potential of such improvement is limited. Anyway, it should be interesting to see different approaches to constructing non-Hamiltonian graphs of

relatively high toughness.

4.1 New results

We present new results related to Conjecture 1.1. The results are included in the thesis, and they are also available separately in two papers [65, 63] and two manuscripts [64, 86]. The two papers are to be found appended to the thesis. The main results of the manuscripts are included in Sections 2.4, 3.6 and 2.9. The papers and manuscripts are authored or co-authored by the author of the present thesis. In particular, the first of the appended papers is a joint work of the author and the adviser of the thesis.

In the first paper [65] (see Appendix A), we introduce a technique for assigning vertices of a particular graph to regions of the ‘underlying structure’ of the graph and using the assignment to construct a Hamilton cycle. The main tool for finding the assignment is the hypergraph extension of Hall’s theorem (by Aharoni and Haxell [2]). In particular, we consider a chordal graph and we assign its vertices to subtrees of its underlying tree. In case no desired assignment is found, we can obtain a particular division of the underlying tree which gives a disconnecting of the chordal graph into relatively many components. As the main result, we show that every 10-tough chordal graph is Hamilton-connected which improves the before known bound proven by Chen et al. [32]. As an outline to the paper, the reader is invited to consult Propositions 2.6 and 2.21 which present simplified versions of the proof technique.

In the second paper [63] (see Appendix B), we formalize the ideas of the standard construction for bounding the shortness exponent of a class of graphs. As the main results, we present constructions of maximal planar graphs of relatively high toughness whose longest cycles are relatively short. With these constructions, we improve the upper bound of Harant and Owens [56] on the shortness exponent of the class of $\frac{5}{4}$ -tough maximal planar graphs, and we improve and generalize a similar result of Tkáč [96] who considered 1-tough maximal planar graphs.

In the first manuscript [64] (submitted), we unify the view on toughness and Hamiltonicity of chordal planar graphs and of k -trees (studied separately by Böhme et al. [23] and by Broersma, Xiong and Yoshimoto [30], respectively). We show that every k -tree of toughness greater than $\frac{k}{3}$ is Hamilton-connected for $k \geq 3$ (and consequently, chordal planar graphs of toughness greater than 1 are Hamilton-connected). This improves and generalizes the results of [23, 30]. In addition, we present constructions of graphs whose longest paths (and longest cycles) are relatively short. Namely, we construct 1-tough chordal planar graphs, 1-tough planar 3-trees, and k -trees of toughness greater than 1 for $k \geq 4$. These constructions improve the bounds presented in [23, 30]. The results are included in Sections 2.4 and 3.6. We view the present techniques as an alternative ap-

proach to the study of toughness and Hamiltonicity of k -trees conducted in [30]. In addition to the results included in the manuscript, we present yet another approach to this study, see Corollary 2.8.

In the second manuscript [86] (which we are preparing for submission), we apply the Max-flow min-cut theorem and Hall's marriage theorem and we present additional partial results on Conjecture 1.1. The results are included in Section 2.9. In particular, we consider graphs obtained from an initial graph by adding copies of vertices in sequence (generalizing so-called C_5^* -graphs studied in [27]), and we show that toughness at least 1 ensures the Hamiltonicity for certain 'cactus-like' graphs. We remark that this result can be viewed as 'deciding Hamiltonicity by Max-flow min-cut theorem' for the considered class of graphs. In fact, the proof translates to an algorithm which provides a certificate for this decision. Namely, if we find a 'nice' saturating flow, then we construct a Hamilton cycle; if we find either a 'not nice' saturating flow or a min-cut (if no saturating flow exists), then we provide a set which shows that the graph is not 1-tough. In addition, we consider so-called multisplit graphs (generalizing split graphs and bisplit graphs), and we show that 2-tough multisplit graphs are Hamilton-connected. (We note that the computational complexity of determining toughness of C_p^* -graphs and multisplit graphs is also studied in [86]; we do not include these results in the present thesis.)

4.2 Questions for further research

Throughout the thesis, we discuss several questions related to Conjecture 1.1. More questions and open problems can be found, for instance, in [8, 25]. In the context of the present thesis, the author's interest is in the following three topics.

The main topic is related to the concept of H -graphs (discussed in Section 2.6). We recall that every 10-tough chordal graph is Hamilton-connected [65] (see Appendix A). Furthermore, a simplified version of the proof technique can be applied to circular arc graphs (see Proposition 2.21). In addition, we recall that chordal graphs and circular arc graphs can be viewed as H -graphs. Namely, every chordal graph is an H -graph where H is a tree, and vice versa; and circular arc graphs are precisely H -graphs where H is a cycle. We consider a class of graphs \mathcal{H} such that some graph of \mathcal{H} contains a cycle, and for every tree, there is a graph of \mathcal{H} which contains the tree as a subgraph; and we note that the class of all H -graphs such that H belongs to \mathcal{H} is a superclass of the classes of chordal graphs and circular arc graphs. Considering various choices of such class \mathcal{H} (which provides the underlying graphs), the question is: can we apply a similar proof technique (or a different one) and obtain partial results on Conjecture 1.1? In particular, how do we adapt the technique for the class \mathcal{H} chosen as the class of all cacti which do not contain many cycles? What about if \mathcal{H} is the class of all graphs whose number of cycles is bounded by a constant? Can we solve the problem for

\mathcal{H} chosen as the class of all cacti, or as the class of all outerplanar graphs, or as the class of all series-parallel graphs? This topic partly motivates the following question.

As remarked in [65], the present bound on toughness (ensuring Hamiltonicity for chordal graphs) is still far from the lower bound of ‘almost’ $\frac{7}{4}$ proven in [10]. Studying the non-Hamiltonian chordal graphs constructed in [10] and their intersection representations (see Figure 3.5), it seems that the tight bound might be less than 2 (which is not in accordance with Conjecture 2.4). Hence the question is: can we improve the result of [65]? Moreover, are $\frac{7}{4}$ -tough chordal graphs Hamiltonian?

Lastly, we recall the best available lower bounds to Conjectures 1.1 and 1.13 (the corresponding constructions are outlined in Sections 3.2 and 3.4). The first bound is due to the graphs with no Hamilton path and toughness arbitrarily close to $\frac{9}{4}$ constructed in [10]. In Section 3.3, we present an alternative view on this construction (and we remark that the view also applies to the constructions of graphs with no 2-factor and toughness ‘almost’ 2 presented in [41, 13]). Studying the construction of [10] and its modifications, it is not clear whether the same method can be used for further improving the lower bound. On the other hand, it seems that the potential of this method is not unlimited. It seems natural to ask the following: are there other approaches to constructing such graphs? Regarding Conjecture 1.13, we recall that the construction of [10] was modified in [40] obtaining graphs with no k -walk and relatively high toughness for every $k \geq 2$ (which provide the best available lower bounds to Conjecture 1.13). In particular, considering the case $k = 2$ of Conjecture 1.13 (that is, every 1-tough graph has a 2-walk), the obtained lower bound is ‘almost’ $\frac{17}{24}$. On the other hand, considering graphs with no k -walk and relatively high toughness, it seems that there could be other methods (different from [10]) for constructing such graphs. The last question is: can we improve some of these two lower bounds?

4.3 List of authored and co-authored papers

We conclude this chapter with the list of papers and manuscripts on which the author has been working during his Ph.D. studies. The list also contains items which are not related to the topic of the present thesis; these items are highlighted in grey. The present thesis is based on four items from the list. The two papers are to be found appended to the thesis, and the main results of the two manuscripts are included in the thesis in Sections 2.4, 3.6 and 2.9.

Journal papers

- Z. Dvořák, A. Kabela, T. Kaiser: Planar graphs have two-coloring number at most 8, *Journal of Combinatorial Theory, Series B* 130 (2018), 144–157.

4.3. List of authored and co-authored papers

- J. Ekstein, S. Fujita, A. Kabela, J. Teska: Bounding the distance among longest paths in a connected graph, *Discrete Mathematics* 341 (2018), 1155–1159.
- A. Kabela: An update on non-Hamiltonian $\frac{5}{4}$ -tough maximal planar graphs, *Discrete Mathematics* 341 (2018), 579–587.
- A. Kabela, T. Kaiser: 10-tough chordal graphs are Hamiltonian, *Journal of Combinatorial Theory, Series B* 122 (2017), 417–427.

Submitted manuscripts

- C. Brause, P. Holub, A. Kabela, Z. Ryjáček, I. Schiermeyer, P. Vrána: On forbidden subgraphs for $K_{1,3}$ -free perfect graphs, submitted.
- A. Kabela: Long paths and toughness of k -trees and chordal planar graphs, submitted, arXiv:1707.08026v2.

Manuscripts in preparation

- C. Brause, T. D. Doan, P. Holub, A. Kabela, Z. Ryjáček, I. Schiermeyer, P. Vrána: Forbidden subgraphs for perfect graphs in claw-free graphs of independence at least 4, in preparation.
- J. Hančl, A. Kabela, M. Opler, J. Sosnovec, R. Šámal: Improved bounds for the binary paint shop problem, in preparation.
- A. Kabela, Z. Ryjáček, P. Vrána: Equivalent formulation of Thomassen’s conjecture using Tutte paths in claw-free graphs, in preparation.
- H. Qi, H. J. Broersma, A. Kabela, E. Vumar: On toughness and hamiltonicity of multisplit and C_p^* -graphs, in preparation.

Bibliography

- [1] M. El Kadi Abderrezzak, E. Flandrin, Z. Ryjáček: Induced $S(K_{1,3})$ and Hamiltonian cycles in the square of a graph, *Discrete Mathematics* 207 (1999), 263–269.
- [2] R. Aharoni, P. E. Haxell: Hall’s theorem for hypergraphs, *Journal of Graph Theory* 35 (2000), 83–88.
- [3] J. Akiyama, M. Kano: *Factors and Factorizations of Graphs*, *Lecture Notes in Mathematics* 2031, Springer (2011).
- [4] N. Alon: Tough ramsey graphs without short cycles, *Journal of Algebraic Combinatorics* 4 (1995), 189–195.
- [5] S. Arora, B. Barak: *Computational Complexity, A Modern Approach*, Cambridge University Press (2009).
- [6] R. Balakrishnan, P. Paulraja: Chordal graphs and some of their derived graphs, *Congressus Numerantium* 53 (1986), 71–74.
- [7] D. Bauer, H. J. Broersma, E. Schmeichel: More progress on tough graphs – The Y2K report, in: Y. Alavi, D. Jones, D. R. Lick, J. Liu, editors, *Proceedings of the Ninth Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms and Applications*, *Electronic Notes in Discrete Mathematics* 11 (2002), 1–18.
- [8] D. Bauer, H. J. Broersma, E. Schmeichel: Toughness in graphs — A survey, *Graphs and Combinatorics* 22 (2006), 1–35.
- [9] D. Bauer, H. J. Broersma, J. van den Heuvel, H. J. Veldman: On Hamiltonian properties of 2-tough graphs, *Journal of Graph Theory* 18 (1994), 539–543.
- [10] D. Bauer, H. J. Broersma, H. J. Veldman: Not every 2-tough graph is Hamiltonian, *Discrete Applied Mathematics* 99 (2000), 317–321.
- [11] D. Bauer, S. L. Hakimi, E. Schmeichel: Recognizing tough graphs is NP-hard, *Discrete Applied Mathematics* 28 (1990), 191–195.

- [12] D. Bauer, G. Y. Katona, D. Kratsch, H. J. Veldman: Chordality and 2-factors in tough graphs, *Discrete Applied Mathematics* 99 (2000), 323–329.
- [13] D. Bauer, E. Schmeichel: Toughness, minimum degree, and the existence of 2-factors, *Journal of Graph Theory* 18 (1994), 241–256.
- [14] D. Bauer, E. Schmeichel, H. J. Veldman: Some recent results on long cycles in tough graphs, in: Y. Alavi, G. Chartrand, O. R. Oellermann, A. J. Schwenk, editors, *Graph Theory, Combinatorics, and Applications - Proceedings of the Sixth Quadrennial International Conference on the Theory and Applications of Graphs*, John Wiley and Sons (1991), 113–121.
- [15] D. Bauer, E. Schmeichel, H. J. Veldman: Cycles in tough graphs - updating the last four years, in: Y. Alavi, A. J. Schwenk, editors, *Graph Theory, Combinatorics, and Applications - Proceedings of the Seventh Quadrennial International Conference on the Theory and Applications of Graphs*, John Wiley and Sons (1995), 19–34.
- [16] D. Bauer, E. Schmeichel, H. J. Veldman: Progress on tough graphs - another four years, in: Y. Alavi, D. R. Lick, A. J. Schwenk, editors, *Proceedings of the Eighth Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms and Applications*, John Wiley and Sons (1999), 69–88.
- [17] D. Bauer, J. van den Heuvel, E. Schmeichel: Toughness and triangle-free graphs, *Journal of Combinatorial Theory, Series B* 65 (1995), 208–221.
- [18] P. Bedrossian: Forbidden subgraph and minimum degree conditions for Hamiltonicity, Ph.D. Thesis, Memphis State University (1991).
- [19] J. C. Bermond, J. C. Meyer: Graphe representatif des arretes d'un multigraphe, *Journal de Mathématiques Pures et Appliquées* 52 (1973), 299–308.
- [20] M. Biró, M. Hujter, Z. Tuza: Precoloring extension. I. Interval graphs, *Discrete Mathematics* 100 (1992), 267–279.
- [21] H. L. Bodlaender: A partial k -arboretum of graphs with bounded treewidth, *Theoretical Computer Science* 209 (1998), 1–45.
- [22] J. A. Bondy, U. S. R. Murty: *Graph Theory*, Graduate Texts in Mathematics 244, Springer (2008).
- [23] T. Böhme, J. Harant, M. Tkáč: More than one tough chordal planar graphs are Hamiltonian, *Journal of Graph Theory* 32 (1999), 405–410.
- [24] A. Brandstädt, P. L. Hammer, V. B. Le, V. V. Lozin: Bisplit graphs, *Discrete Mathematics* 299 (2005), 11–32.

- [25] H. J. Broersma: How tough is toughness?, *Bulletin of the European Association for Theoretical Computer Science* 117 (2015), 28–52.
- [26] H. J. Broersma, J. Fiala, P. A. Golovach, T. Kaiser, D. Paulusma, A. Proskurowski: Linear-time algorithms for scattering number and Hamilton-connectivity of interval graphs, *Journal of Graph Theory* 79 (2015), 282–299.
- [27] H. J. Broersma, V. Patel, A. Pyatkin: On toughness and Hamiltonicity of $2K_2$ -free graphs, *Journal of Graph Theory* 75 (2014), 244–255.
- [28] H. J. Broersma, Z. Ryjáček, P. Vrána: How many conjectures can you stand? A survey, *Graphs and Combinatorics* 28 (2012), 57–75.
- [29] H. J. Broersma, H. J. Veldman: Restrictions on induced subgraphs ensuring Hamiltonicity or pancyclicity of claw-free graphs, R. Bodendiek, editor, *Contemporary Methods in Graph Theory*, BI Wissenschaftsverlag (1990), 181–194.
- [30] H. J. Broersma, L. Xiong, K. Yoshimoto: Toughness and Hamiltonicity in k -trees, *Discrete Mathematics* 307 (2007), 832–838.
- [31] G. Chen, Y. Egawa, R. J. Gould, A. Saito: Forbidden pairs for k -connected Hamiltonian graphs, *Discrete Mathematics* 312 (2012), 938–942.
- [32] G. Chen, H. S. Jacobson, A. E. Kézdy, J. Lehel: Tough enough chordal graphs are Hamiltonian, *Networks* 31 (1998), 29–38.
- [33] G. Chen, X. Yu: Long cycles in 3-connected graphs, *Journal of Combinatorial Theory, Series B* 86 (2002), 80–99.
- [34] M. Chudnovsky, P. Seymour: The structure of claw-free graphs, in: B. S. Webb, editor, *Surveys in Combinatorics 2005*, London Mathematical Society Lecture Notes 327, Cambridge University Press (2005), 153–171.
- [35] G. Chartrand, A. M. Hobbs, H. A. Jung, S. F. Kapoor, C. St. J. A. Nash-Williams: The square of a block is Hamiltonian connected, *Journal of Combinatorial Theory, Series B* 16 (1974), 290–292.
- [36] V. Chvátal: Tough graphs and Hamiltonian circuits, *Discrete Mathematics* 5 (1973), 215–228.
- [37] Chvátal and Erdős: A note on Hamiltonian circuits, *Discrete Mathematics* 2 (1972), 111–113.
- [38] J. S. Deogun, D. Kratsch, G. Steiner: 1-tough cocomparability graphs are Hamiltonian, *Discrete Mathematics* 170 (1997), 99–106.

- [39] R. A. Duke: On the genus and connectivity of Hamiltonian graphs, *Discrete Mathematics* 2 (1972), 199–206.
- [40] M. N. Ellingham, X. Zha: Toughness, trees, and walks, *Journal of Graph Theory* 33 (2000), 125–137.
- [41] H. Enomoto, B. Jackson, P. Katerinis, A. Saito: Toughness and the existence of k -factors, *Journal of Graph Theory* 9 (1985), 87–95.
- [42] R. Faudree, E. Flandrin, Z. Ryjáček: Claw-free graphs — A survey, *Discrete Mathematics*, 164 (1997) 87–147.
- [43] R. J. Faudree, R. J. Gould: Characterizing forbidden pairs for Hamiltonian properties, *Discrete Mathematics* 173 (1997), 45–60.
- [44] R. J. Faudree, R. H. Schelp: The square of a block is strongly path connected, *Journal of Combinatorial Theory, Series B* 20 (1976), 47–61.
- [45] H. Fleischner: In the square of graphs, Hamiltonicity and pancyclicity, Hamiltonian Connectedness and panconnectedness are equivalent concepts, *Monatshefte für Mathematik* 82 (1976), 125–149.
- [46] H. Fleischner: The square of every two-connected graph is Hamiltonian, *Journal of Combinatorial Theory, Series B* 16 (1974), 29–34.
- [47] L. R. Ford, Jr., D. R. Fulkerson: Maximal flow through a network, *Canadian Journal of Mathematics* 8 (1956), 399–404.
- [48] D. R. Fulkerson, O. A. Gross: Incidence matrices and interval graphs, *Pacific Journal of Mathematics* 15 (1965), 835–855.
- [49] F. Gavril: The intersection graphs of subtrees in a tree are exactly the chordal graphs, *Journal of Combinatorial Theory, Series B* 16 (1974), 47–56.
- [50] T. Gerlach: Toughness and Hamiltonicity of a class of planar graphs, *Discrete Mathematics* 286 (2004), 61–65.
- [51] P. C. Gilmore, A. J. Hoffman: A characterization of comparability graphs and of interval graphs, *Canadian Journal of Mathematics* 16 (1964), 539–548.
- [52] R. J. Gould: Traceability in the square of a tree, *Journal of Combinatorics, Informatics and System Sciences* 8 (1983), 253–260.
- [53] R. J. Gould, M. S. Jacobson: Forbidden subgraphs and Hamiltonian properties of graphs, *Discrete Mathematics* 42 (1982), 189–196.

- [54] B. Grünbaum, H. Walther: Shortness exponents of families of graphs, *Journal of Combinatorial Theory* 14 (1973), 364–385.
- [55] J. Harant: Toughness and nonhamiltonicity of polyhedral graphs, *Discrete Mathematics* 113 (1993), 249–253.
- [56] J. Harant, P. J. Owens: Non-hamiltonian $\frac{5}{4}$ -tough maximal planar graphs, *Discrete Mathematics* 147 (1995), 301–305.
- [57] F. Harary, R. M. Karp, W. T. Tutte: A criterion for planarity of the square of a graph, *Journal of Combinatorial Theory* 2 (1967), 395–405.
- [58] F. Harary, A. Schwenk: Trees with Hamiltonian square, *Mathematika* 18 (1971), 138–140.
- [59] P. J. Heawood: Map-colour theorem, *Quarterly Journal of Mathematics* 24 (1890), 332–338.
- [60] B. Jackson, N. C. Wormald: k -walks of graphs, *The Australasian Journal of Combinatorics* 2 (1990), 135–146.
- [61] B. Jackson, N. C. Wormald: Long cycles and 3-connected spanning subgraphs of bounded degree in 3-connected $K_{1,d}$ -free graphs, *Journal of Combinatorial Theory, Series B* 63 (1995), 163–169.
- [62] H. A. Jung: On a class of posets and the corresponding comparability graphs, *Journal of Combinatorial Theory, Series B* 24 (1978), 125–133.
- [63] A. Kabela: An update on non-Hamiltonian $\frac{5}{4}$ -tough maximal planar graphs, *Discrete Mathematics* 341 (2018), 579–587.
- [64] A. Kabela: Long paths and toughness of k -trees and chordal planar graphs, [arXiv:1707.08026v2](https://arxiv.org/abs/1707.08026v2).
- [65] A. Kabela, T. Kaiser: 10-tough chordal graphs are Hamiltonian, *Journal of Combinatorial Theory, Series B* 122 (2017), 417–427.
- [66] T. Kaiser, D. Král', L. Stacho: Tough spiders, *Journal of Graph Theory* 56 (2007), 23–40.
- [67] T. Kaiser, P. Vrána: Hamilton cycles in 5-connected line graphs, *European Journal of Combinatorics* 33 (2012), 924–947.
- [68] K. Kawarabayashi, K. Ozeki: 5-connected toroidal graphs are Hamiltonian-connected, *SIAM Journal on Discrete Mathematics* 30 (2016), 112–140.
- [69] J. M. Keil: Finding Hamiltonian circuits in interval graphs, *Information Processing Letters* 20 (1985), 201–206.

- [70] A. V. Kostochka: A lower bound for the Hadwiger number of graphs by the average degree, *Combinatorica* 4 (1984), 307–316.
- [71] D. Kratsch, J. Lehel, H. Müller: Toughness, Hamiltonicity and split graphs, *Discrete Mathematics* 150 (1996), 231–245.
- [72] M. S. Krishnamoorthy: An NP-hard problem in bipartite graphs, *SIGACT News* 7 (1975), 26–26.
- [73] C. Lekkerkerker, J. Boland: Representation of a finite graph by a set of intervals on the real line, *Fundamenta Mathematicae* 51 (1962), 45–64.
- [74] L. Lesniak: Chvátal’s t_0 -tough conjecture, in: R. Gera, S. Hedetniemi, C. Larson, editors, *Graph Theory, Favorite Conjectures and Open Problems - 1*, Springer (2016), 135–147.
- [75] B. Li, H. J. Broersma, S. Zhang: Forbidden subgraphs for Hamiltonicity of 1-tough graphs, *Discussiones Mathematicae Graph Theory* 36 (2016), 915–929.
- [76] B. Li, P. Vrána: Forbidden pairs of disconnected graphs implying Hamiltonicity, *Journal of Graph Theory* 84 (2017), 249–261.
- [77] L. Markenзон, C. M. Justel, N. Paciorek: Subclasses of k -trees: Characterization and recognition, *Discrete Applied Mathematics* 154 (2006), 818–825.
- [78] M. M. Matthews, D. P. Sumner: Hamiltonian results in $K_{1,3}$ -free graphs, *Journal of Graph Theory* 8 (1984), 139–146.
- [79] B. Mohar, C. Thomassen: *Graphs on surfaces*, Johns Hopkins University Press (2001).
- [80] J. W. Moon, L. Moser: Simple paths on polyhedra, *Pacific Journal of Mathematics* 13 (1963), 629–631.
- [81] F. Neuman: On a certain ordering of the set of vertices of a tree, *Časopis pro pěstování matematiky* 89 (1964), 323–339.
- [82] Z. G. Nikoghosyan: Disconnected forbidden subgraphs, toughness and Hamilton cycles, *International Scholarly Research Notices Combinatorics* 2013 (2013), ID 673971.
- [83] P. J. Owens: Non-hamiltonian maximal planar graphs with high toughness, *Tatra Mountains Mathematical Publications* 18 (1999), 89–103.
- [84] P. J. Owens: Shortness parameters for polyhedral graphs, *Discrete Mathematics* 206 (1999), 159–169.

- [85] H. P. Patil: On the structure of k -trees, *Journal of Combinatorics, Information and System Sciences* 11 (1986), 57–64.
- [86] H. Qi, H. J. Broersma, A. Kabela, E. Vumar: On toughness and Hamiltonicity of multisplit and C_p^* -graphs, in preparation.
- [87] Z. Ryjáček: On a closure concept in claw-free graphs, *Journal of Combinatorial Theory, Series B* 70 (1997), 217–224.
- [88] P. Scheffler: Die Baumweite von Graphen als ein Maß für die Kompliziertheit algorithmischer Probleme, Ph.D. Thesis, Akademie der Wissenschaften der DDR (1989).
- [89] S. Shan: Hamiltonian cycles in 3-tough $2K_2$ -free graphs, arXiv:1706.09029v1.
- [90] W. K. Shih, T. C. Chern, W. L. Hsu: An $O(n^2 \log n)$ Algorithm for the Hamiltonian cycle problem on circular-arc graphs, *SIAM Journal on Computing* 21 (1992), 1026–1046.
- [91] J. Teska: Factors and cycles in graphs, Ph.D. Thesis, University of West Bohemia (2008).
- [92] R. Thomas, X. Yu: 4-connected projective-planar graphs are Hamiltonian, *Journal of Combinatorial Theory, Series B* 62 (1994), 114–132.
- [93] R. Thomas, X. Yu, W. Zang: Hamilton paths in toroidal graphs, *Journal of Combinatorial Theory, Series B* 94 (2005), 214–236.
- [94] C. Thomassen: A theorem on paths in planar graphs, *Journal of Graph Theory* 7 (1983), 169–176.
- [95] C. Thomassen: Reflections on graph theory, *Journal of Graph Theory* 10 (1986), 309–324.
- [96] M. Tkáč: On the shortness exponent of 1-tough, maximal planar graphs, *Discrete Mathematics* 154 (1996), 321–328.
- [97] M. Tkáč, H. J. Voss: On k -trestles in polyhedral graphs, *Discussiones Mathematicae Graph Theory* 22 (2002), 193–198.
- [98] W. T. Tutte: A short proof of the factor theorem for finite graphs, *Canadian Journal of Mathematics* 6 (1954), 347–352.
- [99] W. T. Tutte: A theorem on planar graphs, *Transactions of the American Mathematical Society* 82 (1956), 99–116.

- [100] H. J. Veldman: Existence of dominating cycles and paths, *Discrete Mathematics* 43 (1983), 281–296.
- [101] S. Win: On a connection between the existence of k -trees and the toughness of a graph, *Graphs and Combinatorics* 5 (1989), 201–205.
- [102] D. R. Woodall: The binding number of a graph and its Anderson number, *Journal of Combinatorial Theory, Series B* 15 (1973), 225–255.

Appendix A

10-tough chordal graphs are Hamiltonian

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10-tough chordal graphs are Hamiltonian [☆]



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ABSTRACT

Chen et al. (1998) proved that every 18-tough chordal graph has a Hamilton cycle. Improving upon their bound, we show that every 10-tough chordal graph is Hamiltonian (in fact, Hamilton-connected). We use Aharoni and Haxell's hypergraph extension of Hall's Theorem as our main tool.

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1. Introduction

We study Hamilton cycles and toughness in chordal graphs. Recall that following Chvátal [6], the *toughness* of a graph G is the minimum, taken over all separating sets S of vertices of G , of the ratio of $|S|$ to the number of components of $G - S$. If G is complete, the toughness is defined to be ∞ . We say that a graph is *t-tough* if its toughness

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is at least t . It is easy to observe that Hamiltonian graphs are 1-tough. In the reverse direction, Chvátal [6] conjectured the following:

Conjecture 1. *There exists t_0 such that every t_0 -tough graph (on at least 3 vertices) is Hamiltonian.*

Conjecture 1 is still open. The best available lower bound is due to Bauer, Broersma and Veldman [3] who constructed non-Hamiltonian graphs with toughness arbitrarily close to $\frac{9}{4}$.

Partial results related to Chvátal’s conjecture have been obtained in various restricted classes of graphs (see the survey [2] for details). A number of these results concern chordal graphs. For instance, it is known that (with the exception of K_1 and K_2) every chordal planar graph of toughness more than 1 is Hamiltonian [4], and so is every 1-tough interval graph [8] or every $\frac{3}{2}$ -tough split graph [9]. All of these results are tight.

Non-Hamiltonian chordal graphs with toughness arbitrarily close to $\frac{7}{4}$ were constructed in [3]. On the other hand, Chen et al. [5] showed that every 18-tough chordal graph is Hamiltonian. In this paper, we improve the bound as follows:

Theorem 2. *Every 10-tough chordal graph on at least 3 vertices is Hamiltonian.*

The construction of the Hamilton cycle is based on auxiliary graphs that are defined in Section 2 to encode the local structure of a given chordal graph. Our main tool is a hypergraph extension of Hall’s Theorem, due to Aharoni and Haxell [1]; its application is described in Section 3. The proof of Theorem 2 is given in Section 4. We conclude the paper in Section 5 discussing a strengthening of Theorem 2 to Hamilton-connectedness.

2. Tree representations and overspan graphs

For a graph H , let $V(H)$ denote the set of vertices, $E(H)$ the set of edges, $c(H)$ the number of components of H . By a well-known theorem of Gavril [7], for every chordal graph G there exists a *tree representation* of G — that is, a tree T_0 and a family \mathcal{F} of subtrees of T_0 such that G is isomorphic to the intersection graph of \mathcal{F} . For each vertex v of G , let F_v denote the corresponding subtree in \mathcal{F} .

For a given chordal graph G , we choose a tree representation (T_0, \mathcal{F}) such that the tree T_0 has minimal number of vertices. Thus, for each leaf of T_0 , there is a subtree in \mathcal{F} consisting of the leaf as its only vertex. We fix this tree representation and choose an independent set I in G that is maximal with the property that for each $v \in I$, F_v is a path all of whose vertices have degree at most 2 in T_0 . Moreover, we choose I such that for every $v \in I$, F_v contains no subtree of \mathcal{F} as a proper subgraph. For $v \in I$, a path F_v is called an *I -path*; it is *trivial* if it consists of a single vertex. To emphasize the distinction between the edges contained in I -paths and the other edges, we colour each edge of T_0 either *red* (if it belongs to some I -path) or *black* (otherwise).

Next, we fix the choice of the independent set I and we modify T_0 into a tree T which we call the base tree for G . (See Fig. 1 for an illustration.) One by one, we suppress each degree 2 vertex of T_0 that is not an endvertex of any I -path (a trivial I -path has one endvertex). The resulting tree T (the *base tree* for G) inherits a red–black colouring of edges. We observe that nontrivial I -paths in T_0 correspond one-to-one to red edges in T , furthermore the red edges form a matching and their endvertices are all of degree 2. Vertices of T_0 that exist also in T are called *substantial* (that is, substantial vertices are the endvertices of I -paths and vertices of degree at least 3). For further reference, let us state the following observation:

Proposition 3. *For every vertex v of G , the tree F_v contains a substantial vertex.*

Proof. To the contrary, suppose there is a vertex v such that the tree F_v contains no substantial vertex. That is, F_v contains neither a vertex whose degree in T_0 is at least 3 nor an endvertex of any I -path. In particular, $v \notin I$ and by the choice of I , F_v is not a proper subgraph of any I -path. Hence F_v does not intersect any I -path, so v is not adjacent to any vertex of I . We obtain a contradiction with the maximality of I . \square

We use T to construct a family of so-called *overspan graphs*, assigning one such graph A_e to each edge e of T (see Fig. 1). The vertex set of A_e is $V(G) \setminus I$. The graph A_e may contain loops; to avoid ambiguity, we point out that we view a loop as an edge of a special type. To describe the edges of A_e , we let r and s denote the endvertices of e . (Note that these are substantial vertices of T_0 .) The edge set of A_e is defined as follows:

- there is a loop on a vertex v if F_v contains the vertices r and s in T_0 ,
- vertices u and v are connected by an edge if $r \in V(F_u)$ and $s \in V(F_v)$ (or vice versa), and uv is an edge of G .

Furthermore, for each black edge e of T we assign to e an additional overspan graph which is a copy of A_e .

The family of overspan graphs for G is constructed for a particular tree representation (T_0, \mathcal{F}) and an independent set I . As the tree representation and the independent set are fixed, let us use the notation $\mathcal{A}(G)$ for the family of overspan graphs.

For $\mathcal{B} \subseteq \mathcal{A}(G)$, we define a graph $G_{\mathcal{B}}$ on vertex set $V(G) \setminus I$. The edge set of $G_{\mathcal{B}}$ is the union of the edge sets of all the graphs that belong to \mathcal{B} ; each edge is included at most once in this union. In case $\mathcal{B} = \mathcal{A}(G)$, we let the graph be denoted $G_{\mathcal{A}}$.

The reason for the name ‘overspan graph’ is that we view each edge of T as representing a gap that needs to be crossed by the desired Hamilton cycle, and the edges of the corresponding overspan graph encode the possible ways of doing so. We conclude this section by pointing out a connection between the family $\mathcal{A}(G)$ and the Hamiltonicity of G . In graphs with loops (such as the overspan graphs and their unions), we allow loops in matchings, as long as they are vertex-disjoint from the other edges of the matching.

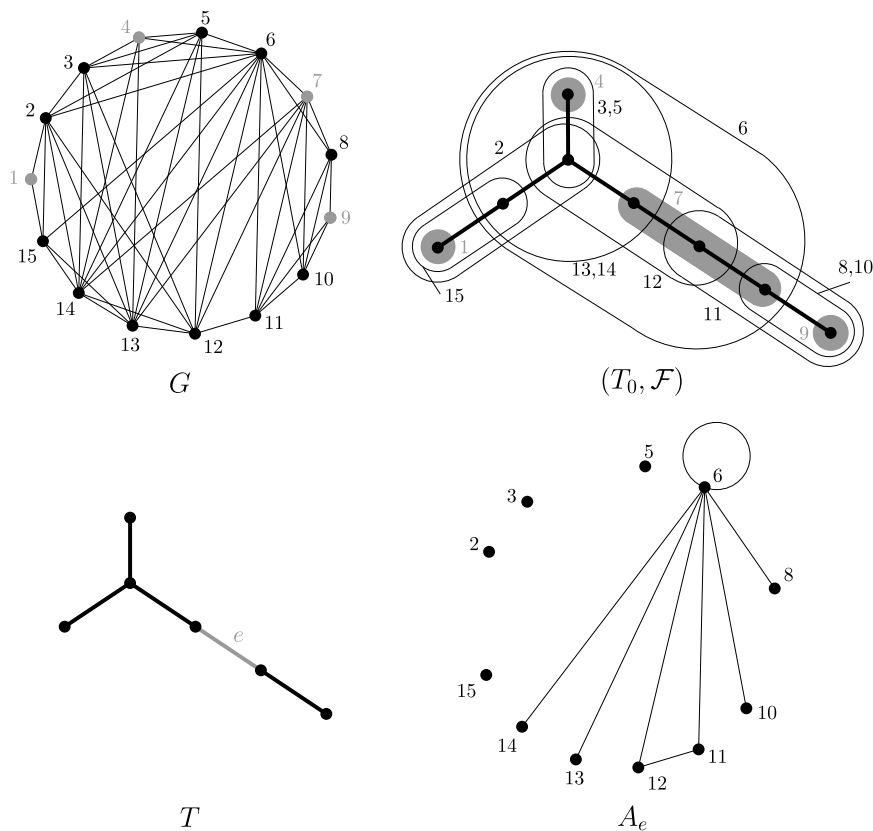


Fig. 1. A chordal graph G , a tree representation (T_0, \mathcal{F}) , a base tree T and an overspan graph A_e assigned to a red edge e . In the tree representation (T_0, \mathcal{F}) (top-right), the ovals depict subtrees of the tree T_0 that belong to \mathcal{F} . The subtrees of T_0 and vertices of A_e are indexed by the same integer as the corresponding vertices of G . In grey, we highlight the vertices of the set I in G (top-left), the I -paths in T_0 (top-right) and the red edge e in T (bottom-left).

Lemma 4. *Let G be a chordal graph on at least 3 vertices and let $\mathcal{A}(G)$ be the family of overspan graphs for G (with respect to a tree representation of G and an independent set I). Assume that we can choose one edge from each graph in $\mathcal{A}(G)$ in such a way that the chosen edges form a matching in $G_{\mathcal{A}}$. Then G is Hamiltonian.*

Proof. Let M be the set of chosen edges that form a matching in $G_{\mathcal{A}}$. We assume T has m edges ($m \geq 1$), and we fix an Euler tour $e_0, e_1, \dots, e_{2m-1}$ in the symmetric orientation of T . With every directed edge $e_i = t_i t'_i$ of the tour, we associate a pair of subtrees (F_i, F'_i) of \mathcal{F} as follows.

For every edge e of T there are two corresponding directed edges, say e_i and e_j , in the symmetric orientation. We discuss two cases: either e is black or it is red. If e is black, then there are two assigned graphs in $\mathcal{A}(G)$, say A_{e_i} and A_{e_j} . By the assumption of the lemma, for A_{e_i} there is a chosen edge of M , namely a simple edge uv or a loop v , and we consider a pair of subtrees (F_u, F_v) or (F_v, F_v) of \mathcal{F} (recall the definition of edges of overspan graphs). We associate e_i with this pair of subtrees of \mathcal{F} and associate e_j with the pair of subtrees of \mathcal{F} obtained analogously for A_{e_j} .

Similarly, if e is red, then there is one assigned graph in $\mathcal{A}(G)$ and a chosen edge of M , which gives a pair of subtrees of \mathcal{F} and we associate e_i with this pair. To find the associated pair for e_j , we recall that a red edge of T is obtained from a non-trivial I -path in T_0 . We let F_v denote this non-trivial I -path related to e and we associate e_j with the pair (F_v, F_v) .

We observe that no subtree of \mathcal{F} is used in more than one associated pair, considering that the edges of M form a matching in $G_{\mathcal{A}}$ and vertices of I are not included in $G_{\mathcal{A}}$.

We traverse the Euler tour $e_0, e_1, \dots, e_{2m-1}$ edge by edge, and as we go we build a sequence of subtrees of \mathcal{F} as follows. When traversing the edge $e_i = t_i t'_i$ of the tour, we extend the sequence by adding subtrees of the associated pair (F_i, F'_i) . In particular, we add the subtrees in the order F_i, F'_i such that $t_i \in V(F_i)$ and $t'_i \in V(F'_i)$. We obtain a sequence $S = F_0, F'_0, F_1, F'_1, \dots, F_{2m-1}, F'_{2m-1}$. By the definition of S , every two consecutive subtrees have a vertex in common (the first and last subtrees are also considered consecutive), and we shall preserve this property even as we further modify the sequence.

Now, we extend the sequence so as to include all subtrees of \mathcal{F} . For every subtree of \mathcal{F} that is not in S , we choose one of its substantial vertices arbitrarily; and we call it the distinguished vertex of this subtree. (This is possible due to [Proposition 3](#).) For every vertex t of T in sequence, we consider an edge of the tour incident with t and directed towards t , say e_i , and we note that $t \in F'_i$ and $t \in F_{i+1 \pmod{2m}}$. We extend the sequence by adding all subtrees with a distinguished vertex t as successors of F'_i in an arbitrary order.

Finally, we remove duplicities from the extended sequence. For every associated pair of subtrees (F_v, F_v) of \mathcal{F} that was obtained either using a loop in M or using a non-trivial I -path, we remove one copy of F_v from the extended sequence. In the resulting sequence, every subtree of \mathcal{F} occurs exactly once and every two consecutive subtrees have a vertex in common. The assumption $m \geq 1$ implies $|\mathcal{F}| \geq 3$, so the sequence of the corresponding vertices of G defines a Hamilton cycle.

To complete the proof, we observe that if T has no edge, then G is Hamiltonian since it is a complete graph. \square

3. Hall’s theorem for hypergraphs

In this section, we recall an extension of Hall’s Theorem to hypergraphs due to Aharoni and Haxell [\[1\]](#). We use this result as a tool to verify the condition in [Lemma 4](#).

In accordance with [\[1\]](#), we define a hypergraph as a set of subsets of a ground set. (In particular, multiple hyperedges are not allowed.) Let $\mathcal{A} = \{H_1, H_2, \dots, H_m\}$ be a family of hypergraphs. A *system of disjoint representatives* for \mathcal{A} is a function $f : \mathcal{A} \rightarrow \bigcup_{i=1}^m H_i$ such that for all distinct $i, j \in \{1, \dots, m\}$, $f(H_i)$ is a hyperedge of H_i and $f(H_i) \cap f(H_j) = \emptyset$. For $\mathcal{B} \subseteq \mathcal{A}$, let $\bigcup \mathcal{B}$ denote a hypergraph obtained as a union of hypergraphs in \mathcal{B} ; each hyperedge is included at most once in this union. Recall that a

matching in a hypergraph is a collection of pairwise disjoint hyperedges. A corollary of the main result of [1] is stated here as the following theorem.

Theorem 5. *Let \mathcal{A} be a family of n -uniform hypergraphs. A sufficient condition for the existence of a system of disjoint representatives for \mathcal{A} is that for every $\mathcal{B} \subseteq \mathcal{A}$, there exists a matching in $\bigcup \mathcal{B}$ of size greater than $n(|\mathcal{B}| - 1)$.*

The nontrivial direction of Hall's Theorem for graphs follows directly from the $n = 1$ case of Theorem 5. In the argument, we shall use the next case, $n = 2$, where the members of \mathcal{A} are graphs. Indeed, we intend to apply Theorem 5 to the family of overspan graphs $\mathcal{A}(G)$, which we regard as hypergraphs with hyperedges of size 1 (loops) and 2 (non-loops). Recall that the *rank* of a hypergraph is the maximum size of its hyperedge. Theorem 5 easily extends to non-uniform hypergraphs as follows:

Corollary 6. *Let \mathcal{A} be a family of hypergraphs of rank at most n . If for every $\mathcal{B} \subseteq \mathcal{A}$, there exists a matching in $\bigcup \mathcal{B}$ of size greater than $n(|\mathcal{B}| - 1)$, then there exists a system of disjoint representatives for \mathcal{A} .*

Proof. For every hypergraph $H \in \mathcal{A}$, we define an n -uniform hypergraph H^+ by adding $n - k$ new vertices for every hyperedge of size k and extending it to size n . We let \mathcal{A}^+ denote the resulting family of hypergraphs, and for a subfamily $\mathcal{B} \subseteq \mathcal{A}$ we let \mathcal{B}^+ denote the corresponding subfamily of \mathcal{A}^+ .

By the natural correspondence of hyperedges, $\bigcup \mathcal{B}^+$ contains a matching of size greater than $n(|\mathcal{B}| - 1)$, for every $\mathcal{B}^+ \subseteq \mathcal{A}^+$. Since $\bigcup \mathcal{B}^+$ is an n -uniform hypergraph, by Theorem 5 there is a system of disjoint representatives for \mathcal{A}^+ , and hence also for \mathcal{A} . \square

Recall that the *matching number* $\nu(H)$ is the size of a maximum matching in graph H . The following is a reformulation of Lemma 4:

Lemma 7. *Let G be a chordal graph on at least 3 vertices. If for every $\mathcal{B} \subseteq \mathcal{A}(G)$, the matching number of $G_{\mathcal{B}}$ is greater than $2|\mathcal{B}| - 2$, then G is Hamiltonian.*

Proof. We view $G_{\mathcal{B}}$ as a hypergraph of rank at most 2. For any $\mathcal{B} \subseteq \mathcal{A}(G)$, in fact $G_{\mathcal{B}}$ is the same hypergraph as $\bigcup \mathcal{B}$. By Corollary 6 there exists a system of disjoint representatives for $\mathcal{A}(G)$. The edges in the system form a matching in $G_{\mathcal{A}}$. By Lemma 4, the graph G is Hamiltonian. \square

4. Vertex covers of the overspan graphs and toughness

Throughout this section, G is a chordal graph, (T_0, \mathcal{F}) is a tree representation and I is an independent set used for the construction of a base tree T , \mathcal{A} is an associated family of overspan graphs and $G_{\mathcal{B}}$ is the union of graphs in $\mathcal{B} \subseteq \mathcal{A}$, all defined as in

Section 2. In addition, we say an edge e of T is a \mathcal{B} -edge if the overspan graph assigned to e belongs to \mathcal{B} .

We concluded Section 3 with Lemma 7 that provides a sufficient condition for the Hamiltonicity of G in terms of the matching numbers $\nu(G_{\mathcal{B}})$. As a next step, we relate the matching number of $G_{\mathcal{B}}$ to its vertex cover number. Recall that the *vertex cover* of a graph H is a set of its vertices such that every edge of H is incident with a vertex in this set. The *vertex cover number* $\tau(H)$ is the size of a minimum vertex cover of H . By the classical theorem of König, $\nu(H) = \tau(H)$ for every bipartite graph H . We show that the same equality holds for $G_{\mathcal{B}}$.

Lemma 8. *The graph $G_{\mathcal{B}}$ satisfies $\nu(G_{\mathcal{B}}) = \tau(G_{\mathcal{B}})$.*

Proof. We remove from $G_{\mathcal{B}}$ all vertices incident with a loop, and let $G_{\mathcal{B}}^*$ denote the resulting graph.

First, we show that $G_{\mathcal{B}}^*$ is bipartite. We let B denote the set of all \mathcal{B} -edges of T . By definition, a vertex u of $G_{\mathcal{B}}^*$ is also a vertex of G and there is a related subtree F_u in the tree representation. By Proposition 3, F_u contains a substantial vertex. Furthermore, u is not incident with a loop in $G_{\mathcal{B}}$, so F_u does not contain both endvertices of any edge of B . Hence F_u contains substantial vertices from just one component of $T - B$.

In T , we contract every edge that is not in B and we let T' denote the resulting tree. Vertices of T' correspond one-to-one to components of $T - B$. For every vertex u of $G_{\mathcal{B}}^*$, we associate u with a vertex of T' such that the corresponding component of $T - B$ contains all substantial vertices of F_u .

Let u and v be vertices adjacent in $G_{\mathcal{B}}^*$. By the definition of $G_{\mathcal{B}}$, there is an edge xy in B such that $x \in V(F_u)$ and $y \in V(F_v)$. The vertex of T' associated with u (with v) is obtained by contracting all edges of the component of $T - B$ containing x (containing y , respectively). As x and y are adjacent in T , the associated vertices are adjacent in T' . The association of vertices is a graph homomorphism from $G_{\mathcal{B}}^*$ to a tree, thus $G_{\mathcal{B}}^*$ is bipartite.

Since $\nu(H) \leq \tau(H)$ holds for every graph H , it suffices to prove $\nu(G_{\mathcal{B}}) \geq \tau(G_{\mathcal{B}})$ for the graph $G_{\mathcal{B}}$. By König's theorem, $\nu(G_{\mathcal{B}}^*) = \tau(G_{\mathcal{B}}^*)$ since $G_{\mathcal{B}}^*$ is bipartite. A matching in $G_{\mathcal{B}}^*$ extended with all the loops forms a matching in $G_{\mathcal{B}}$. A vertex cover in $G_{\mathcal{B}}^*$ extended with all the vertices incident with a loop in $G_{\mathcal{B}}$ forms a vertex cover in $G_{\mathcal{B}}$. Hence $\nu(G_{\mathcal{B}}) \geq \tau(G_{\mathcal{B}})$. \square

In the analysis of the toughness of the chordal graph G , we shall use the following technical lemma on trees:

Lemma 9. *Let T be a tree. For $i \in \{0, 1, 2\}$, let $E_i \subseteq E(T)$ be such that every edge of E_i is incident with exactly i vertices of degree at most 2. For every $\frac{1}{3} \leq k \leq \frac{1}{2}$, the graph $T - (E_0 \cup E_1 \cup E_2)$ has at least $1 + k|E_0| + (1 - k)|E_1| + |E_2|$ components that contain a vertex whose degree in T is at most 2.*

Proof. Let $E_* = E_0 \cup E_1 \cup E_2$. For a tree T and a subset E of its edge set, let $c_2(T - E)$ denote the number of components of the forest $T - E$ that contain a vertex whose degree in the tree T is at most 2.

We proceed by induction on the number of vertices of degree 2. Suppose T contains no such vertex. (Thus, the only vertices of degree at most 2 are the leaves of T .) If $|E_2| \geq 1$, then T is a tree on 2 vertices and the statement holds, so in addition we can suppose $|E_2| = 0$. We consider all components of $T - E_*$ that contain a leaf of T . For each such component, we contract all edges in the subtree of T that corresponds to this component, and if the resulting vertex is not a leaf, then we add a new leaf adjacent to this vertex; we let T' denote the resulting tree. We let ℓ denote the number of leaves of T' . Since T contains no vertex of degree 2, we have $\ell = c_2(T - E_*)$. Furthermore, T' contains no vertex of degree 2. By an easy inductive argument, such a tree has at most $2\ell - 2$ vertices, and therefore at most $2\ell - 3$ edges. In conjunction with $|E_2| = 0$, this implies the following bound:

$$2\ell - 3 \geq |E_0| + |E_1|. \tag{1}$$

The absence of degree 2 vertices in T implies that every edge of E_1 is incident with a leaf in T , which yields

$$\ell \geq |E_1|. \tag{2}$$

To show that $c_2(T - E_*) \geq 1 + k|E_0| + (1 - k)|E_1|$, we consider the right hand side of this inequality in the form $1 + k(|E_0| + |E_1|) + (1 - 2k)|E_1|$. By (1) and (2), we have for $\frac{1}{3} \leq k \leq \frac{1}{2}$,

$$\begin{aligned} 1 + k(|E_0| + |E_1|) + (1 - 2k)|E_1| &\leq 1 + k(2\ell - 3) + (1 - 2k)\ell \\ &= 1 - 3k + \ell \leq \ell = c_2(T - E_*). \end{aligned}$$

Thus, the lemma holds for a tree that contains no vertex of degree 2.

Suppose that T contains a vertex u of degree 2. We let T_1 and T_2 be the two subtrees of T such that u is the only common vertex of T_1 and T_2 and every vertex of T is in T_1 or T_2 . We observe that for $i \in \{0, 1, 2\}$ and $j \in \{1, 2\}$, every edge in $E_i^j = E_i \cap E(T_j)$ is incident with exactly i vertices of degree at most 2 in T_j , so by induction the statement holds for T_j with the sets of edges E_i^j playing the role of E_i . The trees T_1 and T_2 have no common edge, so $|E_i| = |E_i^1| + |E_i^2|$ for $i \in \{0, 1, 2\}$, and we have:

$$\begin{aligned} c_2(T - E_*) &= c_2(T_1 - E_*) + c_2(T_2 - E_*) - 1 \\ &\geq 1 + k|E_0^1| + (1 - k)|E_1^1| + |E_2^1| + 1 + k|E_0^2| + (1 - k)|E_1^2| + |E_2^2| - 1 \\ &= 1 + k|E_0| + (1 - k)|E_1| + |E_2|. \quad \square \end{aligned}$$

In relation to an edge e of T , we say that two vertices u, v of G form an e -enclosing pair if there is a pair of substantial vertices $s \in V(F_u)$ and $t \in V(F_v)$ such that s and t are in different components of $T - e$.

Lemma 10 is a key part of the argument relating vertex covers of $G_{\mathcal{B}}$ to disconnecting sets of G .

Lemma 10. *Let e be an edge of T and let A_e be the overspan graph assigned to e . Let C be a vertex cover of A_e . We define a set $S \subseteq V(G)$ as follows: In case e is a black edge, let $S = C$, or in case e is a red edge, let x be the corresponding vertex of I and let $S = C \cup \{x\}$. If vertices u, v of $G - S$ form an e -enclosing pair, then u and v are in different components of $G - S$.*

Proof. We first claim that $G - S$ consists of vertices whose corresponding subtree in \mathcal{F} contains substantial vertices from exactly one component of $T - e$. Let r, s be the (substantial) vertices incident with the edge e in T . Let w be a vertex of G such that F_w contains a substantial vertex from each component of $T - e$. Hence F_w contains r and s . We show that $w \in S$. If $w \in I$, then e is a red edge and $w = x$. If $w \in V(G) \setminus I$, then by the construction of A_e , w is incident with a loop in A_e , hence $w \in C$. For every vertex of $G - S$, the corresponding subtree in \mathcal{F} does not contain a vertex from each component of $T - e$. The claim follows from **Proposition 3**. Moreover, observe that if two vertices are adjacent in $G - S$, then the two corresponding subtrees in \mathcal{F} contain vertices from the same component of $T - e$.

Let u, v be vertices of $G - S$ that form an e -enclosing pair. So F_u contains vertices from one component of $T - e$ and F_v contains vertices from the other component. In particular, $u \neq v$. Let U be the set of all vertices of $G - S$ such that the corresponding subtrees in \mathcal{F} contain substantial vertices from the same component of $T - e$ as the subtree F_u , and let V be the set of vertices of $G - S$ that are not in U . We conclude that there is no edge from U to V in $G - S$, hence there is no path from u to v . The vertices u and v are in different components of $G - S$. \square

We are now ready to prove **Theorem 2**, showing that every 10-tough chordal graph on at least 3 vertices is Hamiltonian.

Proof of Theorem 2. Let G be a 10-tough chordal graph on at least 3 vertices, and for the sake of the contradiction suppose that G is not Hamiltonian. By **Lemma 7**, there is a subfamily $\mathcal{B}_0 \subseteq \mathcal{A}$ such that $\nu(G_{\mathcal{B}_0}) \leq 2|\mathcal{B}_0| - 2$ and by **Lemma 8**, we also have $\tau(G_{\mathcal{B}_0}) \leq 2|\mathcal{B}_0| - 2$. Let C be a minimum vertex cover of $G_{\mathcal{B}_0}$; we fix C and extend \mathcal{B}_0 to a maximal subfamily \mathcal{B} such that C is a vertex cover of $G_{\mathcal{B}}$. Clearly, $|C| \leq 2|\mathcal{B}| - 2$. We produce a separating set $S \subseteq V(G)$ demonstrating that G is not 10-tough; to find it, we augment C as follows.

Let B be the set of all \mathcal{B} -edges of T . Let E' be the set of all red edges of B such that none of the adjacent (black) edges of T belongs to B . Every red edge e corresponds to

an I -path, say F_{v_e} ; let X' be the set of all vertices v_e of G such that $e \in E'$. We set $S = C \cup X'$ and show that it has the required properties.

Let E_* be the set of all black edges that belong to B . For $i \in \{0, 1, 2\}$, let $E_i \subseteq E_*$ consist of edges incident with exactly i vertices whose degree in T is at most 2. Clearly, $|E_*| = |E_0| + |E_1| + |E_2|$. A black edge of E_i is adjacent to at most i red edges, and every red edge in $B \setminus E'$ is adjacent to a black edge of B , hence $|B \setminus (E_* \cup E')| \leq |E_1| + 2|E_2|$. By the definition of \mathcal{A} , there are two overspan graphs assigned to every black edge, hence $|\mathcal{B}| \leq 2|E_0| + 3|E_1| + 4|E_2| + |E'|$. We bound the size of the separating set S :

$$|S| = |C| + |X'| \leq 2|\mathcal{B}| - 2 + |E'| < 4|E_0| + 6|E_1| + 8|E_2| + 3|E'|. \tag{3}$$

In order to bound the number of components $c(G - S)$, let us start with $c(G - C)$. Observe that for every substantial vertex of degree at most 2, there is an I -path that contains this vertex. Furthermore, every trivial I -path contains exactly one substantial vertex and every non-trivial I -path contains exactly two substantial vertices that are connected by a red edge in T . Note that T with the sets of edges E_0, E_1, E_2 fit the criteria of Lemma 9, which we apply with $k = \frac{2}{5}$. Consequently, the graph $T - E_*$ has more than $\frac{2}{5}|E_0| + \frac{3}{5}|E_1| + |E_2|$ components that contain a vertex whose degree in T is at most 2. Associate one vertex v of I with each of these components such that the component contains substantial vertices of F_v . For any pair of these associated vertices, there is an edge e of E_* such that the vertices form an e -enclosing pair. By Lemma 10 these vertices are in different components of $G - C$. We obtain $c(G - C) > \frac{2}{5}|E_0| + \frac{3}{5}|E_1| + |E_2|$.

We continue by bounding $c(G - S)$. For every vertex $v_e \in X'$, there is a corresponding edge $e \in E'$ and the overspan graph A_e . Let d, d' denote the edges adjacent to e in T . Let us consider the graph A_d . (The argument for $A_{d'}$ is symmetric.) By the definition of E' , we have $A_d \notin \mathcal{B}$. Due to the maximality of \mathcal{B} the set C is not a vertex cover of the graph $G_{\mathcal{B} \cup \{A_d\}}$. Thus, the graph A_d contains an edge e_0 (a simple edge or a loop) such that no vertex incident with this edge is in C . In T , the edges d and e have a common substantial vertex, say t . Choose a vertex u of G such that $t \in V(F_u)$ and u is incident with e_0 in A_d . Since $t \in V(F_{v_e})$, the vertices u and v_e are adjacent in G . Observe that $u \notin C \cup I$. Similarly, there is a substantial vertex t' and a vertex $u' \in V(G) \setminus (C \cup I)$ such that $t' \in V(F_{u'})$ and $t' \in V(F_{v_e})$. The vertices u and u' form an e -enclosing pair. The three vertices u, v_e, u' are in the same component of the graph $G - C$. By Lemma 10, removing the vertex v_e disconnects this component into two components such that u is in one of them and u' is in the other. Removing the vertices of X' from $G - C$ increases the number of components by $|X'|$. Therefore we obtain:

$$c(G - S) > \frac{2}{5}|E_0| + \frac{3}{5}|E_1| + |E_2| + |E'|. \tag{4}$$

Comparing (3) and (4), we find that G is not 10-tough. We obtain a contradiction proving Theorem 2. \square

We remark that the bound of [Theorem 2](#) is still far from the lower bound of ‘almost’ $\frac{7}{4}$ proven in [\[3\]](#), and there seems to be ample room for further improvements.

5. Toughness and Hamilton-connectedness

With a little extra work, one can use the method of this paper to obtain a slightly stronger result than [Theorem 2](#), namely that any 10-tough chordal graph G is Hamilton-connected. (Recall that this means that for any two vertices u, v of G , there is a Hamilton path from u to v .)

Assume that the vertices u and v are given. Let us sketch the main modifications required to show that G admits a Hamilton path from u to v :

- in [Lemma 4](#), we additionally assume that the matching chosen from the graphs in $\mathcal{A}(G)$ is incident with neither u nor v ,
- in the proof of [Lemma 4](#), the Euler tour is replaced by a trail from a vertex of F_u to a vertex of F_v spanning all the vertices of T ,
- to find a matching as above, it is sufficient to increase the bound on the matching number of the graph $G_{\mathcal{B}}$ in [Lemma 7](#) by two, to $2|\mathcal{B}|$.

By inspecting inequality [\(3\)](#), one can see that the proof of [Theorem 2](#) works just the same even with the strengthened assumption in [Lemma 7](#).

We hope that the interested reader will be able to reconstruct the argument from this account.

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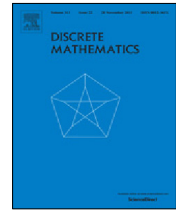
References

- [1] R. Aharoni, P.E. Haxell, Hall’s theorem for hypergraphs, *J. Graph Theory* 35 (2000) 83–88.
- [2] D. Bauer, H.J. Broersma, E. Schmeichel, Toughness in graphs — a survey, *Graphs Combin.* 22 (2006) 1–35.
- [3] D. Bauer, H.J. Broersma, H.J. Veldman, Not every 2-tough graph is Hamiltonian, *Discrete Appl. Math.* 99 (2000) 317–321.
- [4] T. Böhme, J. Harant, M. Tkáč, More than one tough chordal planar graphs are Hamiltonian, *J. Graph Theory* 32 (1999) 405–410.
- [5] G. Chen, H.S. Jacobson, A.E. Kézdy, J. Lehel, Tough enough chordal graphs are Hamiltonian, *Networks* 31 (1998) 29–38.
- [6] V. Chvátal, Tough graphs and hamiltonian circuits, *Discrete Math.* 5 (1973) 215–228.
- [7] F. Gavril, The intersection graphs of subtrees in a tree are exactly the chordal graphs, *J. Combin. Theory Ser. B* 16 (1974) 47–56.
- [8] J.M. Keil, Finding Hamiltonian circuits in interval graphs, *Inform. Process. Lett.* 20 (1985) 201–206.
- [9] D. Kratsch, J. Lehel, H. Müller, Toughness, hamiltonicity and split graphs, *Discrete Math.* 150 (1996) 231–245.

Appendix B

An update on non-Hamiltonian $\frac{5}{4}$ -tough maximal planar graphs

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An update on non-Hamiltonian $\frac{5}{4}$ -tough maximal planar graphs

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ABSTRACT

Studying the shortness of longest cycles in maximal planar graphs, we improve the upper bound on the shortness exponent of the class of $\frac{5}{4}$ -tough maximal planar graphs presented by Harant and Owens (1995). In addition, we present two generalizations of a similar result of Tkáč who considered 1-tough maximal planar graphs (Tkáč, 1996); we remark that one of these generalizations gives a tight upper bound. We fix a problematic argument used in both mentioned papers.

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1. Introduction

We continue the study of non-Hamiltonian graphs with the property that removing an arbitrary set of vertices disconnects the graph into a relatively small number of components (compared to the size of the removed set). In the present paper, we construct families of maximal planar such graphs whose longest cycles are short (compared to the order of the graph).

More formally, the properties which we study are the toughness of graphs and the shortness exponent of classes of graphs (both introduced in 1973). We recall that following Chvátal [5], the *toughness* of a graph G is the minimum, taken over all separating sets X of vertices in G , of the ratio of $|X|$ to $c(G - X)$ where $c(G - X)$ denotes the number of components of the graph $G - X$. The toughness of a complete graph is defined as being infinite. We say that a graph is t -tough if its toughness is at least t .

Along with the definition of toughness, Chvátal [5] conjectured that there is a constant t_0 such that every t_0 -tough graph (on at least three vertices) is Hamiltonian. As a lower bound on t_0 , Bauer et al. [2] presented graphs with toughness arbitrarily close to $\frac{3}{4}$ which contain no Hamilton path (and thus, they are non-Hamiltonian). While remaining open for general graphs, Chvátal's conjecture was confirmed in several restricted classes of graphs; and also various relations among the toughness of a graph and properties of its cycles are known. We refer the reader to the extensive survey on this topic [1].

Clearly, every graph (on at least five vertices) of toughness greater than $\frac{3}{2}$ is 4-connected, so every such planar graph is Hamiltonian by the classical result of Tutte [14]. On the other hand, Harant [8] showed that not every $\frac{3}{2}$ -tough planar graph is Hamiltonian; and furthermore, the shortness exponent of the class of $\frac{3}{2}$ -tough planar graphs is less than 1.

We recall that following Grünbaum and Walther [7], the *shortness exponent* of a class of graphs Γ is the \liminf , taken over all infinite sequences G_n of non-isomorphic graphs of Γ (for n going to infinity), of the logarithm of the length of a longest cycle in G_n to base equal to the order of G_n .

Introducing this notation, Grünbaum and Walther [7] also presented upper bounds on the shortness exponent for numerous subclasses of the class of 3-connected planar graphs. Furthermore, they remarked that the upper bound for

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the class of 3-connected planar graphs itself was presented earlier by Moser and Moon [10] who used a slightly different notation. Later, Chen and Yu [4] showed that every 3-connected planar graph G contains a cycle of length at least $|V(G)|^{\log_3 2}$; in combination with the bound of [10], it follows that the shortness exponent of this class equals $\log_3 2$. A number of results considering the shortness exponent and similar parameters are surveyed in [12].

Considering the class of maximal planar graphs under a certain toughness restriction, Owens [11] presented non-Hamiltonian maximal planar graphs of toughness arbitrarily close to $\frac{3}{2}$. Harant and Owens [9] argued that the shortness exponent of the class of $\frac{5}{4}$ -tough maximal planar graphs is at most $\log_9 8$. Improving the bound $\log_7 6$ presented by Dillencourt [6], Tkáč [13] showed that it is at most $\log_6 5$ for the class of 1-tough maximal planar graphs.

In the present paper, we show the following.

Theorem 1. *Let σ be the shortness exponent of the class of maximal planar graphs under a certain toughness restriction.*

- (i) *If the graphs are $\frac{5}{4}$ -tough, then σ is at most $\log_{30} 22$.*
- (ii) *If the graphs are $\frac{8}{7}$ -tough, then σ is at most $\log_6 5$.*
- (iii) *If the toughness of the graphs is greater than 1, then σ equals $\log_3 2$.*

We note that $\log_9 8 > \log_{30} 22$, that is, the statement in item (i) of [Theorem 1](#) improves the result of [9]. Furthermore, items (ii) and (iii) provide two different generalizations of the result of [13] since $\frac{8}{7} > 1$ and $\log_6 5 > \log_3 2$.

We remark that we fix a problem in a technical lemma presented in [9, Lemma 1]. The fixed version of this lemma (see [Lemma 12](#)) is applied to prove the present results.

2. Structure of the proof

In order to prove [Theorem 1](#), we shall construct three families of graphs whose properties are summarized in the following proposition.

Proposition 2. *For every $i = 1, 2, 3$ and every non-negative integer n , there exists a maximal planar graph $F_{i,n}$ on $f_i(n)$ vertices whose longest cycle has $c_i(n)$ vertices where*

- (i) $f_1(n) = 1 + 101(1 + 30 + \dots + 30^n)$ and $c_1(n) = 1 + 93(1 + 22 + \dots + 22^n)$ and $F_{1,n}$ is $\frac{5}{4}$ -tough,
- (ii) $f_2(n) = 1 + 14(1 + 6 + \dots + 6^n)$ and $c_2(n) = 1 + 13(1 + 5 + \dots + 5^n)$ and $F_{2,n}$ is $\frac{8}{7}$ -tough,
- (iii) $f_3(n) = 4 + 5(1 + 3 + \dots + 3^n)$ and $c_3(n) = 3 \cdot 2^{n+3} - 9n - 15$ and the toughness of $F_{3,n}$ is greater than 1.

Before constructing the graphs $F_{i,n}$, we point out that the use of [Proposition 2](#) leads directly to the main results of the present paper.

Proof of Theorem 1. We consider an infinite sequence of non-isomorphic graphs $F_{1,n}$ given by item (i) of [Proposition 2](#); and we recall that they are $\frac{5}{4}$ -tough maximal planar graphs. Furthermore, we have

$$f_1(n) = 1 + \frac{101}{29}(30^{n+1} - 1) \quad \text{and} \quad c_1(n) = 1 + \frac{93}{21}(22^{n+1} - 1).$$

It follows that

$$\lim_{n \rightarrow \infty} \log_{f_1(n)} c_1(n) = \log_{30} 22.$$

Thus, the considered shortness exponent is at most $\log_{30} 22$.

Using similar arguments and considering items (ii) and (iii) of [Proposition 2](#), we obtain the desired upper bounds.

Clearly, if G is a maximal planar graph (on at least four vertices), then it is 3-connected. By a result of [4], G contains a cycle of length at least $|V(G)|^{\log_3 2}$. In combination with the upper bound obtained due to item (iii) of [Proposition 2](#), we obtain that for the class of maximal planar graphs of toughness greater than 1, the shortness exponent equals $\log_3 2$. \square

In the remainder of the present paper, we construct the families of graphs having the properties described in [Proposition 2](#). Basically, we proceed in four steps. First, we introduce relatively small graphs $F_{i,0}$ called ‘building blocks’ in [Section 3](#), and we observe key properties of their longest cycles. We use these building blocks to construct larger graphs $F_{i,n}$ in [Section 4](#), and we show that their longest cycles are short. In [Section 5](#), we study the toughness of the building blocks. The toughness of the graphs $F_{i,n}$ is shown in [Sections 6 and 7](#).

We remark that the graphs $F_{1,n}$ and $F_{2,n}$ are obtained using a standard construction for bounding the shortness exponent (see for instance [7,6,9,13] or [3]); the improvement of the known bounds comes with the choice of suitable building blocks. In addition, we formalize the key ideas of this construction to make them more accessible for further usage.

The construction of graphs $F_{3,n}$ can be viewed as a simple modification of the construction used in [10] (yet the toughness and longest cycles of the constructed graphs are different).

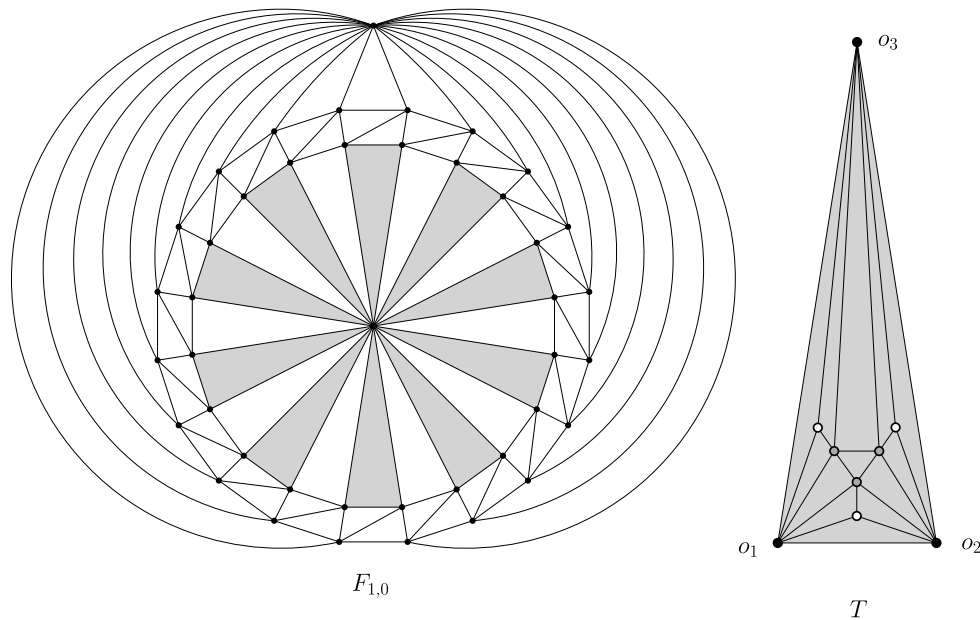


Fig. 1. The graph T and the construction of the graph $F_{1,0}$. The graph $F_{1,0}$ is obtained by replacing each of the highlighted triangles (of the graph on the left) with a copy of T in the natural way (by identifying the vertices of the highlighted triangle with vertices o_1, o_2, o_3 of T).

3. Building blocks

We start by considering the graph T depicted in Fig. 1 which plays an important role in the latter constructions. We let o_1, o_2 , and o_3 be its vertices of degree 6.

A T -region of a graph G is an induced subgraph isomorphic to T with a distinction of vertices referring to o_1, o_2, o_3 as to *outer* vertices and to the remaining vertices as to *inner* vertices, and with the property that no inner vertex is adjacent to a vertex of $G - T$. Similarly, we define an H -region for a given graph H and a given distinction of its vertices.

We view the T -regions as replacing triangles of a graph with copies of T in the natural way. The basic idea of the present constructions is that if these triangles share many vertices, then every cycle in the resulting graph misses many vertices. We formalize this idea in Proposition 3.

We recall that a vertex is called *simplicial* if its neighbourhood induces a complete graph.

Proposition 3. *Let R be a T -region of a graph G and let C be a cycle containing all three simplicial vertices of R and a vertex of $G - R$. Then the subgraph of C induced by the vertices of R is a path containing all outer vertices of R ; two of them as its ends.*

Proof. Clearly, R has three simplicial vertices none of which is an outer vertex. The statement follows from the fact that every simplicial vertex has only one neighbour which is not an outer vertex. \square

Aiming for the graph $F_{1,0}$ (the building block for constructing the graphs $F_{1,n}$), we consider a graph with a number, say r , of T -regions which all share one outer vertex and which are otherwise disjoint. By Proposition 3, every cycle in this graph contains at most $2r + 2$ of the $3r$ simplicial vertices belonging to these T -regions. (We view the building block used in [9] simply as choosing $r = 3$.) With hindsight, we remark that the used construction leads to the upper bound $\log_{3r}(2r + 2)$ on the shortness exponent; so we minimize this function over all integers $r \geq 3$, that is, we choose $r = 10$.

We let $F_{1,0}$ be the graph depicted in Fig. 1, and we note that $F_{1,0}$ is a maximal planar graph. Clearly, $F_{1,0}$ has 30 simplicial vertices, and we colour these vertices white. We recall that every cycle in $F_{1,0}$ contains at most 22 white vertices.

Furthermore, we let $F_{2,0}$ be the graph depicted in Fig. 2. Clearly, $F_{2,0}$ is a maximal planar graph having 6 simplicial vertices; and we colour these vertices white. We use Proposition 3, and we observe that a cycle in $F_{2,0}$ contains at most 5 white vertices. Lastly, we let $F_{3,0}$ be the graph T .

For every $i = 1, 2, 3$, we define the outer face of $F_{i,0}$ as given by the present embedding (see Fig. 1 and Fig. 2). In the following section, we shall use the blocks $F_{i,0}$ and construct the graphs $F_{i,n}$.

4. Families of tree-like structured graphs

We recall the standard construction used for bounding the shortness exponent, and we formalize it with the following definition and Lemma 4.

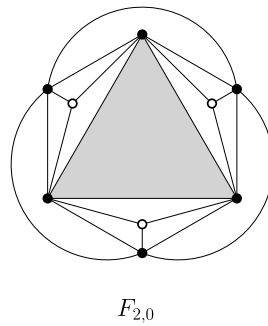


Fig. 2. The construction of the graph $F_{2,0}$. The highlighted triangle represents a subgraph T .

An *arranged block* is a 5-tuple (G_0, j, W, O, k) where G_0 is a graph, j is the number of vertices of G_0 , and W and O are disjoint sets of vertices of G_0 such that the vertices of W are simplicial and independent and O induces a complete graph and such that every cycle in G_0 contains at most k vertices of W .

Lemma 4. Let (G_0, j, W, O, k) be an arranged block such that $k \geq 1$. For every $n \geq 1$, let G_n be a graph obtained from G_{n-1} by replacing every vertex of W with a copy of G_0 (which contains W and O), and by adding arbitrary edges which connect the neighbourhood of the replaced vertex with the set O of the copy of G_0 replacing this vertex. Then G_n has $1 + (j - 1)(1 + |W| + \dots + |W|^n)$ vertices and its longest cycle has at most $1 + (\ell - 1)(1 + k + \dots + k^n)$ vertices where $\ell = j - |W| + k$.

Proof. We note that G_n contains $|W|^{n+1}$ vertices of W . For the sake of induction, we prove the statement with an additional claim that every cycle in G_n contains at most k^{n+1} vertices of W . Clearly, the statement and the claim are satisfied for $n = 0$, and we proceed by induction on n .

We note that the difference in the order of G_n and G_{n-1} equals $(j - 1)|W|^n$. Thus, G_n has $1 + (j - 1)(1 + |W| + \dots + |W|^n)$ vertices.

We let C be a cycle in G_n , and we view this cycle simply as a sequence of vertices. For every newly added copy of G_0 , we remove from C all but one vertex of the copy and we replace the remaining vertex (if there is such) by the corresponding replaced vertex of G_{n-1} ; and we let C' denote the resulting sequence. Clearly, if C' has at most two vertices, then C visits at most one of the newly added copies of G_0 . If C' has at least three vertices, then C' defines a cycle in G_{n-1} (since the neighbourhood of every vertex of W in G_{n-1} induces a complete graph); and C' contains at most k^n vertices of W (by the induction hypothesis). Thus, C visits at most k^n of the newly added copies of G_0 .

Similarly, we choose an arbitrary newly added copy of G_0 , and we remove from C all vertices not belonging to this copy. We note that the resulting sequence either contains at most two vertices (belonging to O) or it is a cycle in G_0 (since O induces a complete graph). Thus, a cycle in G_n contains at most k vertices belonging to W of one copy of G_0 . Furthermore, a cycle contains at most $j - |W| + k$ vertices of one such copy.

Consequently, C contains at most k^{n+1} vertices of W . Furthermore, the length of C minus the length of a longest cycle in G_{n-1} is at most $(j - |W| + k - 1)k^n$ which concludes the proof. \square

For $i = 1, 2$, we consider this construction for the graph $F_{i,0}$ playing the role of G_0 and the set of its white vertices playing the role of W and the set of vertices of its outer face playing the role of O . For every added copy of $F_{i,0}$, we join the vertices of its outer face to the neighbourhood of the corresponding replaced vertex by adding six edges in such a way that the new edges form a 2-regular bipartite graph (that is, a cycle of length 6). We let $F_{i,n}$ be the resulting graphs, and we observe that they are maximal planar graphs. For instance, see the graph $F_{2,1}$ depicted in Fig. 3.

We start verifying that the constructed graphs (for $i = 1, 2$) have the desired properties.

Corollary 5. For every $i = 1, 2$ and every non-negative integer n , the graph $F_{i,n}$ has $f_i(n)$ vertices and its longest cycle has $c_i(n)$ vertices where

- (i) $f_1(n) = 1 + 101(1 + 30 + \dots + 30^n)$ and $c_1(n) = 1 + 93(1 + 22 + \dots + 22^n)$,
- (ii) $f_2(n) = 1 + 14(1 + 6 + \dots + 6^n)$ and $c_2(n) = 1 + 13(1 + 5 + \dots + 5^n)$.

Proof. The order of the graphs and the upper bound on the length of their longest cycles follow from Lemma 4.

We note that a longest cycle in $F_{1,0}, F_{2,0}$ has 94, 14 vertices, respectively. Furthermore, there is a longest cycle which contains an edge of the outer face. Clearly, by removing this edge from the cycle we obtain a path whose ends are vertices of the outer face. We consider a longest cycle in $F_{i,n-1}$ and we extend it to a cycle in $F_{i,n}$ using these paths. The following observation shows that such an extension is possible. For an arbitrary pair, say A , of neighbours of one replaced vertex and an arbitrary pair, say B , of vertices of the outer face of the corresponding $F_{i,0}$ (used for replacing this vertex), the bipartite graph (A, B) has a perfect matching.

A simple counting argument gives that $F_{i,n}$ contains a cycle of length $c_i(n)$, for every $i = 1, 2$ and every n . \square

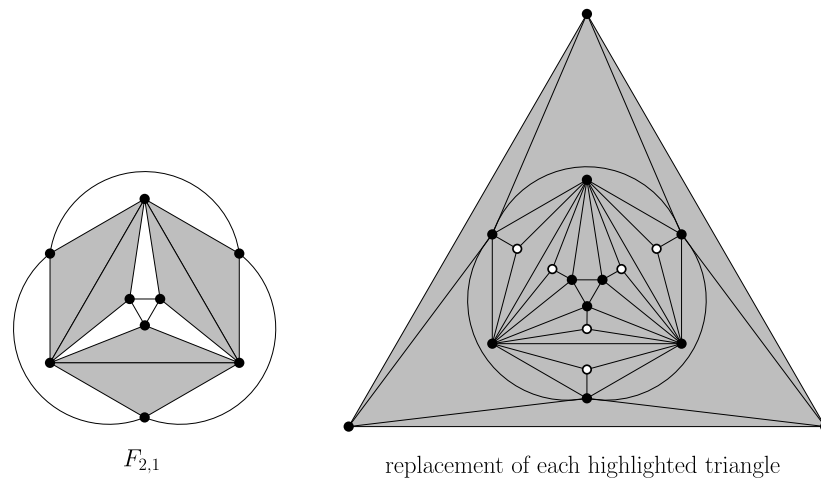


Fig. 3. The construction of the graph $F_{2,1}$. The graph $F_{2,1}$ is obtained from the smaller graph (left) by replacing each of its highlighted triangles with the larger graph (right) in the natural way. (This corresponds to replacing white vertices of $F_{2,0}$ with copies of $F_{2,0}$ and adding edges as in the present construction.)

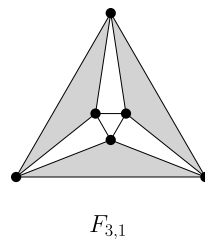


Fig. 4. The construction of the graph $F_{3,1}$. Each highlighted triangle represents a subgraph T .

We use a slightly different construction to get the graphs $F_{3,n}$. The building block $F_{3,0}$ is the graph T whose simplicial vertices are coloured white. We view a subgraph induced by a white vertex and its neighbourhood as a K_4 -region, that is, a subgraph isomorphic to K_4 whose white vertex has degree 3 in the whole graph, and the neighbours of the white vertex are called *outer vertices* of the K_4 -region.

For $n \geq 1$, we let $F_{3,n}$ be a graph obtained from $F_{3,n-1}$ by replacing every K_4 -region of $F_{3,n-1}$ with a T -region in the natural way (the outer vertices of the K_4 -region are the outer vertices of the T -region); and we note that they are maximal planar graphs. For instance, the graph $F_{3,1}$ is depicted in Fig. 4. We proceed by the following proposition.

Proposition 6. For every non-negative integer n , the graph $F_{3,n}$ has $4 + 5(1 + 3 + \dots + 3^n)$ vertices and its longest cycle has $3 \cdot 2^{n+3} - 9n - 15$ vertices.

Proof. We verify the order of $F_{3,n}$ using induction on n . Clearly, $F_{3,0}$ has 9 vertices, and we note that the difference in the order of $F_{3,n}$ and $F_{3,n-1}$ equals $5 \cdot 3^n$. Thus, $F_{3,n}$ has $4 + 5(1 + 3 + \dots + 3^n)$ vertices.

We show the length of a longest cycle using a slightly technical argument. We let $s_i(n)$ denote the length of a longest cycle in $F_{3,n}$ which contains i edges of the outer face (in the embedding which follows naturally from the construction). For the sake of induction, we prove the following equalities:

$$\begin{aligned}
 s_0(n) &= 3s_1(n-1) - 3 \\
 s_1(n) &= 2s_2(n-1) + s_1(n-1) - 3 \\
 s_2(n) &= 2s_2(n-1)
 \end{aligned}
 \tag{1}$$

and

$$\begin{aligned}
 s_0(n) &= 3 \cdot 2^{n+3} - 9n - 15 \\
 s_1(n) &= 2^{n+4} - 3n - 7 \\
 s_2(n) &= 2^{n+3}.
 \end{aligned}
 \tag{2}$$

Clearly, $s_0(0) = s_1(0) = 9$ and $s_2(0) = 8$, and (using Proposition 3) we note that $s_0(1) = 24$, $s_1(1) = 22$ and $s_2(1) = 16$; so the equalities are satisfied for $n = 1$. We assume that they are satisfied for $n - 1$ and we prove them for n .

For $n \geq 1$, we view $F_{3,n}$ as the graph obtained from $F_{3,0}$ by replacing each of its K_4 -regions with an $F_{3,n-1}$ -region; and we view a cycle, say C , of $F_{3,n}$ as a sequence of vertices. We consider one of the $F_{3,n-1}$ -regions, say R , and we remove from C all vertices not belonging to R ; and we let C' be the resulting sequence. We observe that C' either is a cycle or it has at most two vertices. Furthermore, if C visits a vertex not belonging to R , then C' (if it has at least three vertices) is a cycle in R containing at least one edge of its outer face.

Clearly, a longest cycle (in $F_{3,n}$) whose all vertices belong to the same $F_{3,n-1}$ -region has $s_0(n-1)$ vertices, and we observe that a longest cycle visiting more than one of the regions has $3s_1(n-1) - 3$ vertices. We use (2) for $s_0(n-1)$ and $s_1(n-1)$, and we note that

$$3 \cdot 2^{n+2} - 9(n-1) - 15 < 3(2^{n+3} - 3(n-1) - 7) - 3.$$

So we get $s_0(n) = 3s_1(n-1) - 3$. By similar arguments, we get $s_1(n) = 2s_2(n-1) + s_1(n-1) - 3$ and $s_2(n) = 2s_2(n-1)$. Consequently, we can use (1) for $s_i(n)$, and we obtain

$$s_0(n) = 3s_1(n-1) - 3 = 3(2^{n+3} - 3(n-1) - 7) - 3 = 3 \cdot 2^{n+3} - 9n - 15$$

and

$$s_1(n) = 2s_2(n-1) + s_1(n-1) - 3 = 2 \cdot 2^{n+2} + 2^{n+3} - 3(n-1) - 7 - 3 = 2^{n+4} - 3n - 7$$

and

$$s_2(n) = 2s_2(n-1) = 2 \cdot 2^{n+2} = 2^{n+3}.$$

Thus, the equalities are satisfied for n . In particular, a longest cycle in $F_{3,n}$ has $3 \cdot 2^{n+3} - 9n - 15$ vertices. \square

With Corollary 5 and Proposition 6 on hand, we shall focus on the toughness of the constructed graphs.

5. Toughness of the extended blocks

In this section, we study the toughness of $F_{i,0}$ (for $i = 1, 2, 3$) and of its extension $F_{i,0}^+$ (for $i = 1, 2$) which is a graph obtained by adding a vertex adjacent to all vertices of the outer face of $F_{i,0}$. We shall use the following two propositions.

Proposition 7. *Adding a simplicial vertex to a graph does not increase its toughness.*

Proof. We let x be a simplicial vertex of a graph G^+ and we let $G = G^+ - x$ and we let S be a set of vertices of G . We note that $c(G^+ - S) \geq c(G - S)$, and the statement follows. \square

Proposition 8. *Let R be a T -region of a graph G and let I, O be the set of all inner, outer vertices of R , respectively. Let $t \geq 1$ and let S be a set of vertices of G such that $c(G - S) > \frac{1}{t}|S|$. If $|S \cap O| \geq 2$, then there is a separating set $S' = (S \setminus I) \cup A$ such that $c(G - S') > \frac{1}{t}|S'|$ where A is chosen as follows.*

- If $|S \cap O| = 2$, then A consists of the non-simplicial vertex of I which is the common neighbour of the vertices of $S \cap O$.
- If $|S \cap O| = 3$, then A consists of two non-simplicial vertices of I .

Proof. In both cases, we modify S as suggested; and we let S' be the resulting set. Clearly, S' is a separating set and $c(G - S') - c(G - S) \geq 0$, and we observe that

$$c(G - S') - c(G - S) \geq |S'| - |S|.$$

Since $t \geq 1$, we have either $0 > \frac{1}{t}(|S'| - |S|)$ or $|S'| - |S| \geq \frac{1}{t}(|S'| - |S|)$. Consequently, we obtain

$$c(G - S') - c(G - S) \geq \frac{1}{t}|S'| - \frac{1}{t}|S|.$$

We use $c(G - S) > \frac{1}{t}|S|$ and we conclude that $c(G - S') > \frac{1}{t}|S'|$. \square

We recall that a set D of vertices is *dominating* a graph G if every vertex of $G - D$ is adjacent to a vertex of D . We note that every pair of vertices of degree 6 is dominating T , so every separating set in T has at least two of these vertices. As a consequence of Proposition 8, we note the following.

Corollary 9. *The graph T is $\frac{3}{2}$ -tough.*

Furthermore, if a graph contains a T -region and a vertex not belonging to the T -region, then the toughness of the graph is at most $\frac{5}{4}$; and we show that this is the correct value for $F_{1,0}^+$ and $F_{1,0}$.

Proposition 10. *The graphs $F_{1,0}^+$ and $F_{1,0}$ are $\frac{5}{4}$ -tough.*

Proof. By Proposition 7, it suffices to show that $F_{1,0}^+$ is $\frac{5}{4}$ -tough. To the contrary, we suppose that there is a separating set S of vertices such that $c(F_{1,0}^+ - S) > \frac{4}{5}|S|$. We consider such S adjusted by using Proposition 8, in sequence, for all T -regions of $F_{1,0}^+$.

We let \mathcal{I} denote the set of all components of $F_{1,0}^+ - S$ consisting exclusively of inner vertices of some T -region. We note that the existence of such component implies that at least two outer vertices of the corresponding T -region belong to S (since every pair of outer vertices of T is dominating T). We let r_2, r_3 denote the number of T -regions whose exactly 2, 3 outer vertices belong to S , respectively. We let c denote the common vertex of all T -regions. Considering the inner vertices of the T -regions, we call simplicial such vertices *white* and the remaining such vertices *grey*. Except for c , the outer vertices of the T -regions are called *black*. The vertices adjacent to a black vertex but not belonging to a T -region are called *blue*. We let c' denote the vertex adjacent to all blue vertices and x denote the vertex of $F_{1,0}^+$ not belonging to $F_{1,0}$. We let \mathcal{B} denote the set of all components of $F_{1,0}^+ - S$ containing a black vertex or a blue vertex.

We shall use a discharging argument to avoid complicated inequalities. We assign charge 5 to every component of $F_{1,0}^+ - S$, and we aim to distribute all assigned charge among the vertices of S , and to show that every vertex of S receives charge at most 4, contradicting the assumption that $c(F_{1,0}^+ - S) > \frac{4}{5}|S|$. We pre-distribute the charge according to the following rules.

- Every grey vertex of S receives 4 of the total charge of the components of \mathcal{I} belonging to the same T -region as the grey vertex.
- For every T -region, the remaining charge of all components of \mathcal{I} belonging to this T -region is distributed equally among the black vertices of S belonging to this T -region.
- Every blue vertex of S receives as much of the total charge of components outside \mathcal{I} as possible (at most 4).

We note that after the pre-distribution, the charge of every component of \mathcal{I} is 0; and we focus on the remaining charge of the rest of the components.

If $c(F_{1,0}^+ - S) - |\mathcal{I}| \leq 1$, then we have $|\mathcal{I}| \geq 1$ and the remaining charge is at most 5. Consequently, $r_2 \geq 1$ or $r_3 \geq 1$, and in both cases, the vertices of S can still receive charge at least 5, a contradiction.

We assume that $c(F_{1,0}^+ - S) - |\mathcal{I}| \geq 2$. We show that $F_{1,0}^+ - S$ has no component containing c and no black vertex. The existence of such component implies that all black vertices belong to S , and these vertices can still receive $\frac{7}{2} \cdot 20$. A contradiction follows by counting the maximum possible number of components not belonging to \mathcal{I} .

Consequently, $F_{1,0}^+ - S$ has at most one component belonging to neither \mathcal{I} nor \mathcal{B} . If $|\mathcal{B}| \leq 1$, then there is such component (since $c(F_{1,0}^+ - S) - |\mathcal{I}| \geq 2$). Clearly, this component contains c' or x , that is, all blue vertices or c' and at least two blue vertices belong to S , and a contradiction follows.

We assume that $|\mathcal{B}| \geq 2$. On the other hand, considering the graph induced by black and blue vertices, we observe that the size of a maximum independent set of this graph is 13. Thus, $|\mathcal{B}| \leq 13$.

Since $|\mathcal{B}| \geq 2$, we note that at least $|\mathcal{B}|$ black and at least $|\mathcal{B}|$ blue vertices belong to S . We let d denote the number of blue vertices of S minus $|\mathcal{B}|$. We recall that $F_{1,0}^+ - S$ has at most $|\mathcal{B}| + 1$ components not belonging to \mathcal{I} . Thus, the remaining charge, which is yet to be distributed, is at most $|\mathcal{B}| + 5 - 4d$; and since $d \geq 0$, it is at most $|\mathcal{B}| + 5$.

If c does not belong to S , then the black vertices of S can still receive at least $\frac{7}{2}|\mathcal{B}|$. Clearly, $\frac{7}{2}|\mathcal{B}| \geq |\mathcal{B}| + 5$ since $|\mathcal{B}| \geq 2$, a contradiction.

We assume that c belongs to S . Clearly, c can receive 4. We note that the black vertices of S can still receive $3r_2 + r_3$; and since $r_2 + 2r_3 \geq |\mathcal{B}|$, they can receive at least $\frac{1}{2}|\mathcal{B}| + \frac{5}{2}r_2$ (thus, at least $\frac{1}{2}|\mathcal{B}|$).

If c' does not belong to S , then we consider the graph induced by c' and by the black and blue vertices, and we observe that $r_2 + d \geq |\mathcal{B}| - 1$ (since there are at least $|\mathcal{B}| - 1$ components of \mathcal{B} not containing c'). Consequently, we have $\frac{1}{2}|\mathcal{B}| + \frac{5}{2}r_2 + 4 > |\mathcal{B}| + 5 - 4d$ since $|\mathcal{B}| \geq 2$, a contradiction.

We assume that c' belongs to S , so c' can receive 4. If $d \geq 1$, then the remaining charge is at most $|\mathcal{B}| + 1$; and we have $\frac{1}{2}|\mathcal{B}| + 4 + 4 > |\mathcal{B}| + 1$ since $|\mathcal{B}| \leq 13$, a contradiction.

We assume that $d = 0$. If there is not a component containing x as its only vertex, then the remaining charge is $|\mathcal{B}|$. Similarly as above, we have $\frac{1}{2}|\mathcal{B}| + 4 + 4 > |\mathcal{B}|$, a contradiction.

We assume that $F_{1,0}^+ - S$ has a component consisting of x , so all neighbours of x belong to S . Since $d = 0$ and since $|\mathcal{B}| \geq 2$, we have $r_2 \geq 1$. If $|\mathcal{B}| \leq 11$, then we have $\frac{1}{2}|\mathcal{B}| + \frac{5}{2}r_2 + 4 + 4 \geq |\mathcal{B}| + 5$, a contradiction. If $|\mathcal{B}| = 12$, then the parity implies that $r_2 \geq 2$ or $r_3 \geq 6$, and a contradiction follows.

We assume that $|\mathcal{B}| = 13$. We consider the structure of \mathcal{B} and we observe that $r_3 \leq 4$, that is, $r_2 \geq 5$. Consequently, we get $\frac{1}{2}|\mathcal{B}| + \frac{5}{2}r_2 + 4 + 4 > |\mathcal{B}| + 5$, and we obtain the desired distribution of the assigned charge, a contradiction. Thus, $F_{1,0}^+$ is $\frac{5}{4}$ -tough. \square

We continue with the following.

Proposition 11. *The graphs $F_{2,0}^+$ and $F_{2,0}$ are $\frac{8}{7}$ -tough.*

Proof. By Proposition 7, it suffices to show the toughness of $F_{2,0}^+$ which we do via contradiction. We suppose that there is a separating set S of vertices in $F_{2,0}^+$ such that $c(F_{2,0}^+ - S) > \frac{7}{8}|S|$.

Since S is separating, we have $|S| \geq 3$. Consequently, we can assume that $c(F_{2,0}^+ - S) \geq 3$.

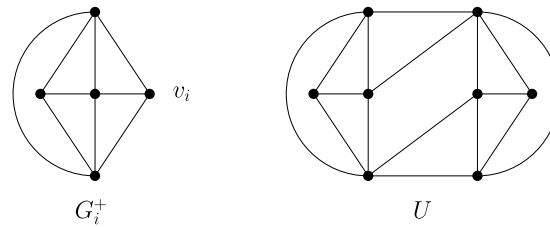


Fig. 5. A counterexample to a statement presented in [9, Lemma 1]. We consider the graphs G_i^+ and $G_i = G_i^+ - v_i$ for $i = 1, 2$, and the graph U . We note that U is obtained from the disjoint union of G_1 and G_2 by adding edges and every vertex of $N(v_1) \cup N(v_2)$ is incident with at least one new edge. Clearly, G_i^+ and G_i are $\frac{3}{2}$ -tough, but U is not.

To specify the structure of S , we consider the graph $F_{2,0}$. We note that $F_{2,0}$ contains two T -regions and the T -regions share their outer vertices; and we call these outer vertices *black* (as well as the corresponding vertices of $F_{2,0}^+$). We observe that every pair of black vertices is dominating $F_{2,0}$.

Since $c(F_{2,0}^+ - S) \geq 3$, we have that at least two black vertices belong to S . We note that $F_{2,0}^+$ contains one T -region (and its outer vertices are black), and we modify S using Proposition 8; and we let S' be the resulting set. Considering the possibilities, we observe that $c(F_{2,0}^+ - S') \leq \frac{7}{8}|S'|$, a contradiction. \square

We shall use the toughness of the building blocks $F_{i,0}$ (given by Propositions 10 and 11 and by Corollary 9) to show the toughness of the constructed graphs $F_{i,n}$.

6. Gluing tough graphs

We shall use the following lemma as the main tool for showing the toughness of graphs which are obtained by the standard construction for bounding the shortness exponent.

Lemma 12. For $i = 1, 2$, let G_i^+ and G_i be t -tough graphs such that G_i is obtained by removing vertex v_i from G_i^+ . Let U be a graph obtained from the disjoint union of G_1 and G_2 by adding new edges such that the minimum degree of the bipartite graph $(N(v_1), N(v_2))$ is at least t . Then U is t -tough.

Proof. We assume that $t > 0$ and that there exists a separating set of vertices in U . We let X be such a set and we let $X_i = X \cap V(G_i)$ for $i = 1, 2$. Clearly, $2 \leq c(U - X) \leq c(G_1 - X_1) + c(G_2 - X_2)$, and we use it to show that $c(U - X) \leq \frac{1}{t}|X|$.

If X_i is a separating set in G_i , then the toughness of G_i implies that $c(G_i - X_i) \leq \frac{1}{t}|X_i|$.

We suppose that X_1 is not separating in G_1 , and we observe that $c(U - X) \leq c(G_2 - X_2) + 1$. If $c(U - X) \leq c(G_2 - X_2)$, then X_2 is separating in G_2 , and thus, $c(G_2 - X_2) \leq \frac{1}{t}|X_2|$ and the desired inequality follows since $|X_2| \leq |X|$.

We assume that $c(U - X) = c(G_2 - X_2) + 1$. Clearly, if $N(v_2) \subseteq X_2$, then $c(G_2 - X_2) + 1 = c(G_2^+ - X_2)$; so X_2 is separating in G_2^+ and we have $c(G_2^+ - X_2) \leq \frac{1}{t}|X_2|$ and the inequality follows.

In addition, we assume that there is a vertex of $N(v_2)$ not belonging to X_2 . We recall that this vertex has at least $\lceil t \rceil$ neighbours in $N(v_1)$. Since $c(U - X) = c(G_2 - X_2) + 1$, we note that all these neighbours belong to X_1 . Thus, $|X_1| \geq t$ and we have $c(G_1 - X_1) \leq \frac{1}{t}|X_1|$.

Similarly, we get that if X_2 is not separating in G_2 , then $c(G_2 - X_2) \leq \frac{1}{t}|X_2|$. We conclude that (no matter whether X_i is separating or not) we have $c(G_i - X_i) \leq \frac{1}{t}|X_i|$ for both $i = 1, 2$; and the inequality follows. \square

We remark that a similar statement appears in [9, Lemma 1]; and it is used in [13] and [3]. (Compared to Lemma 12, the main difference is that the minimum degree of the considered bipartite graph is required to be at least 1.) The graphs depicted in Fig. 5 show that this statement is false. We view Lemma 12 as a fixed version of this statement, and we remark that Lemma 12 can be applied in the arguments of [9, 13] and [3]. We note that Lemma 12 can be viewed as a generalization of a similar statement (for 1-tough graphs) presented in [6, Lemma 4].

7. Toughness of the constructed graphs

In this section, we clarify that the graphs $F_{i,n}$ have the properties stated in Proposition 2.

Proof of Proposition 2. We recall that the order of the constructed graphs and the length of their longest cycles are given by Corollary 5 and by Proposition 6.

For every $i = 1, 2$, we show the toughness of the graphs $F_{i,n}$ using induction on n . The case $n = 0$ is verified by Propositions 10 and 11. By induction hypothesis, $F_{i,n-1}$ has the required toughness, and by Proposition 7, so does a graph obtained from $F_{i,n-1}$ by removing a simplicial vertex; we shall apply Lemma 12, and we view these two graphs as playing the role of G_i^+ and G_i and we view graphs $F_{i,0}^+$ and $F_{i,0}$ as playing the role of G_2^+ and G_2 . We consider the graph obtained from

$F_{i,n-1}$ by replacing one of its simplicial vertices by a copy of $F_{i,0}$ and by adding edges as in the present construction (we recall the construction of graphs $F_{i,n}$ for $i = 1, 2$; see Section 4). By Lemma 12, the resulting graph has the required toughness. Thus, we can replace a simplicial vertex of the resulting graph and apply Lemma 12 again; and repeating this argument, we obtain that $F_{i,n}$ has the required toughness.

Similarly, we show the toughness of $F_{3,n}$ by induction on n . By Corollary 9, the toughness of $F_{3,0}$ is greater than 1. We consider Lemma 13 (see below) applied on the graph $F_{3,n-1}$ playing the role of G (and then applied repeatedly on the resulting graph), and we obtain that the toughness of $F_{3,n}$ is greater than 1. \square

Similarly to Lemma 12, the following lemma can be used to construct large graphs from smaller ones while preserving certain toughness.

Lemma 13. *Let G be a graph of toughness greater than 1 which contains a K_4 -region and let G' be a graph obtained from G by replacing this K_4 -region by a T -region (in the natural way). Then the toughness of G' is greater than 1.*

Proof. We let X' be a separating set of vertices in G' , and we shall show that $c(G' - X') < |X'|$. We consider the set X obtained from X' by removing all inner vertices of the new T -region. For the sake of simplicity, we let $c = c(G' - X') - c(G - X)$ and $x = |X' \setminus X|$, and we note that it suffices to show that $c(G - X) + c < |X| + x$. Considering the choice of X , we observe that $c \leq x$. We conclude the proof by showing that $c(G - X) < |X|$. If X is separating in G , then the inequality is given by the toughness of G . We can assume that $c(G - X) = 1$. Consequently, $c(G' - X') > c(G - X)$, so at least two outer vertices of the new T -region belong to X' . Thus, they belong to X , that is, $|X| \geq 2$ and the inequality follows. \square

8. Note on longest paths

We remark that (using similar arguments as in Section 4 we obtain that) a longest path of $F_{i,n}$ has $p_i(n)$ vertices where

- (i) $p_1(n) = 2 + c_1(n) + 2\sum_{k=0}^{n-1} c_1(k)$,
- (ii) $p_2(n) = 1 + \operatorname{sgn}(n) + c_2(n) + c_2(n-1) + 2\sum_{k=0}^{n-2} c_2(k)$,
- (iii) $p_3(n) = 7 \cdot 2^{n+2} + 2\operatorname{sgn}(n) - 15n - 19$.

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References

- [1] D. Bauer, H.J. Broersma, E. Schmeichel, Toughness in graphs – A survey, *Graphs Combin.* 22 (2006) 1–35.
- [2] D. Bauer, H.J. Broersma, H.J. Veldman, Not every 2-tough graph is Hamiltonian, *Discrete Appl. Math.* 99 (2000) 317–321.
- [3] T. Böhme, J. Harant, M. Tkáč, More than one tough chordal planar graphs are Hamiltonian, *J. Graph Theory* 32 (1999) 405–410.
- [4] G. Chen, X. Yu, Long cycles in 3-connected graphs, *J. Combin. Theory Ser. B* 86 (2002) 80–99.
- [5] V. Chvátal, Tough graphs and hamiltonian circuits, *Discrete Math.* 5 (1973) 215–228.
- [6] M.B. Dillencourt, An upper bound on the shortness exponent of 1-tough, maximal planar graphs, *Discrete Math.* 90 (1991) 93–97.
- [7] B. Grünbaum, H. Walther, Shortness exponents of families of graphs, *J. Combin. Theory* 14 (1973) 364–385.
- [8] J. Harant, Toughness and nonhamiltonicity of polyhedral graphs, *Discrete Math.* 113 (1993) 249–253.
- [9] J. Harant, P.J. Owens, Non-hamiltonian $\frac{5}{3}$ -tough maximal planar graphs, *Discrete Math.* 147 (1995) 301–305.
- [10] J.W. Moon, L. Moser, Simple paths on polyhedra, *Pacific J. Math.* 13 (1963) 629–631.
- [11] P.J. Owens, Non-hamiltonian maximal planar graphs with high toughness, *Tatra Mt. Math. Publ.* 18 (1999) 89–103.
- [12] P.J. Owens, Shortness parameters for polyhedral graphs, *Discrete Math.* 206 (1999) 159–169.
- [13] M. Tkáč, On the shortness exponent of 1-tough, maximal planar graphs, *Discrete Math.* 154 (1996) 321–328.
- [14] W.T. Tutte, A theorem on planar graphs, *Trans. Amer. Math. Soc.* 82 (1956) 99–116.