*Electronic Journal of Differential Equations*, Vol. 2021 (2021), No. 76, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

# TRAVELING WAVES FOR UNBALANCED BISTABLE EQUATIONS WITH DENSITY DEPENDENT DIFFUSION

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ABSTRACT. We study the existence and qualitative properties of traveling wave solutions for the unbalanced bistable reaction-diffusion equation with a rather general density dependent diffusion coefficient. In particular, it allows for singularities and/or degenerations as well as discontinuities of the first kind at a finite number of points. The reaction term vanishes at equilibria and it is a continuous, possibly non-Lipschitz function. We prove the existence of a unique speed of propagation and a unique traveling wave profile (up to translation) which is a non-smooth function in general. In the case of the power-type behavior of the diffusion and reaction near equilibria we provide detailed asymptotic analysis of the profile.

## 1. INTRODUCTION

We are concerned with the traveling wave solutions of the bistable equation

$$\begin{gathered}\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\Big(d(u)\Big|\frac{\partial u}{\partial x}\Big|^{p-2}\frac{\partial u}{\partial x}\Big)+g(u),\quad (x,t)\in\mathbb{R}\times\mathbb{R}_+,\\ u(x,t)=U(x-ct)\end{gathered}\tag{1.1}$$

for a speed of propagation $c\in\mathbb{R}$. Here $\mathbb{R}_+:=[0,+\infty)$, $1<p<+\infty$ and the properties of the density dependent diffusion coefficient $d=d(s)$ as well as the reaction term $g=g(s)$ will be specified later.

If $p=2$, $d\equiv 1$ and $g:[0,1]\to\mathbb{R}$ is a smooth function such that $g(0)=g(s_*)=g(1)=0$, $g(s)<0$ for $s\in(0,s_*)$, $g(s)>0$ for $s\in(s_*,1)$, equation (1.1) is studied, e.g., in [1, 2]. The authors of these articles explain how the mathematical modeling of diploid individuals (homozygote and heterozygote) in population dynamics leads to the bistable equation. If in addition the reaction term $g$ satisfies the *unbalanced bistable* condition

$$\int_0^1 g(s)\,\mathrm{d}s>0,\tag{1.2}$$

the mathematical model describes the so called *heterozygote inferior* case. If $g(s)\le 0$ instead of $g(s)<0$ for $s\in(0,s_*)$, the bistable equation models the flame propagation in chemical reactor theory. In contrast with the population dynamics model, where $u$ denotes the relative density of the population of one allele, in the combustion model $u$ represents a normalized temperature and $s_*$ represents

2010 *Mathematics Subject Classification.* 35Q92, 35C07, 34A12, 35K92.
*Key words and phrases.* Density dependent diffusion; unbalanced bistable reaction term; degenerate and singular diffusion; traveling wave; degenerate non-Lipschitz reaction.

Submitted August 22, 2021. Published September 14, 2021.

a critical temperature at which an exothermic reaction starts (see, e.g., [10]). The bistable equation with reaction term like above was also suggested in [4] as a model for a nerve which has been treated with certain toxins. In [11] this equation serves as a model for a bistable active transmission line. Other possible interpretations may be found in [13].

In this paper we focus on more general, in particular non-smooth (even non-Lipschitz) reaction term $g$ as well as on the density dependent diffusion coefficient $d$ which can be singular and/or degenerate near the equilibria 0 and 1 and discontinuous in a finite number of points in the interval $(0,1)$. We show the existence of a traveling wave solution if the classical unbalanced bistable condition (1.2) is replaced by the following more general one:

$$\int_0^1 \big(d(s)\big)^{\frac{1}{p-1}} g(s)\,\mathrm{d}s > 0. \tag{1.3}$$

In our previous work [7] we proved that if equality holds in (1.3) then (1.1) possesses nonconstant stationary solutions (also called *standing wave solutions*) "connecting" the equilibria 0 and 1. We also studied qualitative properties of these stationary solutions, in particular, the lack of smoothness and the asymptotic properties near the equilibria 0 and 1. In contrast with the stationary case, inequality (1.3) renders the time dependent traveling wave solutions of the form

$$u(x,t) = U(x-ct),$$

where $U = U(z)$, $z \in \mathbb{R}$, is the profile of the traveling wave and $c \in \mathbb{R}$ is the speed of its propagation. Density dependent diffusion coefficient which is discontinuous is studied, e.g., in [12] to model the temperature field in a wire of superconducting material carrying an electrical current and immersed in a bath at constant temperature. Convergence of the solution of the initial value problem for equation in (1.1) with $d \equiv 1$ to a traveling wave is studied, e.g., in [9] for $C^1$ reaction term $g$ or in [6] for reaction term $g$ which is only one-sided Lipschitz continuous.

Our results concerning the existence and uniqueness of the profile $U$ and speed $c$ extend and generalize those from [1, Theorem 4.2], [2, Theorem 4.1], [5, Theorem 3.1]. The asymptotic analysis of $U$ near 0 and 1 extends that from [5, Section 6].

## 2. Preliminaries

Let $g : [0,1] \to \mathbb{R}$, $g \in C[0,1]$ be such that $g(0) = g(s_*) = g(1) = 0$ for $s_* \in (0,1)$ and

$$g(s) \le 0,\ s \in (0,s_*), \quad g(s) > 0,\ s \in (s_*,1).$$

The diffusion coefficient $d : [0,1] \to \mathbb{R}$ is supposed to be nonnegative lower semicontinuous and $d > 0$ in $(0,1)$. There exist $0 = s_0 < s_1 < s_2 < \dots < s_n < s_{n+1} = 1$ such that $d\big|_{(s_i,s_{i+1})} \in C(s_i,s_{i+1})$, $i = 0,\dots,n$, and $d$ has discontinuity of the first kind (finite jump) at $s_i$, $i = 1,\dots,n$.

We introduce the moving coordinate $z = x - ct$ and write $u(x,t) = U(x-ct) = U(z)$. For the sake of simplicity we write $(\cdot)'$ instead of $\frac{\mathrm{d}}{\mathrm{d}z}(\cdot)$. Then (1.1) transforms into

$$\Big(d(U(z))|U'(z)|^{p-2}U'(z)\Big)' + cU'(z) + g(U(z)) = 0\,. \tag{2.1}$$

Let $U : \mathbb{R} \to [0,1]$ be a monotone continuous function. We denote

$$M_U := \{z \in \mathbb{R} : U(z) = s_i,\ i = 1,2,\dots,n\},$$

$$N_U := \{z \in \mathbb{R} : U(z) = 0 \text{ or } U(z) = 1\}.$$

Then $M_U$ and $N_U$ are closed sets, $M_U$ is a union of a finite number of points or intervals,

$$N_U = (-\infty, z_0] \cup [z_1, +\infty),$$

where $-\infty \le z_0 < z_1 \le +\infty$ and we use the convention $(-\infty, z_0] = \emptyset$ if $z_0 = -\infty$ and $[z_1, +\infty) = \emptyset$ if $z_1 = +\infty$. Below we introduce the definition of a monotone solution of (2.1).

**Definition 2.1.** A monotone continuous function $U : \mathbb{R} \to [0, 1]$ is called a solution of (2.1) if

(a) For any $z \notin M_U \cup N_U$ the derivative $U'(z)$ exists and it is finite. For $z \in \operatorname{int} M_U \cup \operatorname{int} N_U$ we have $U'(z) = 0$.

(b) For any $z \in \partial M_U$ there exist finite one sided derivatives $U'(z-)$, $U'(z+)$ and

$$L(z) := |U'(z-)|^{p-2}U'(z-) \lim_{\xi \to z-} d(U(\xi)) = |U'(z+)|^{p-2}U'(z+) \lim_{\xi \to z+} d(U(\xi)).$$

(c) Function $v : \mathbb{R} \to \mathbb{R}$ defined by

$$v(z) := \begin{cases} d(U(z))|U'(z)|^{p-2}U'(z), & z \notin M_U \cup N_U, \\ 0, & z \in N_U \cup \operatorname{int} M_U, \\ L(z), & z \in \partial M_U \end{cases}$$

is continuous and for any $z, \hat{z} \in \mathbb{R}$,

$$v(\hat{z}) - v(z) + c\,(U(\hat{z}) - U(z)) + \int_z^{\hat{z}} g(U(\xi))\,\mathrm{d}\xi = 0\,. \tag{2.2}$$

Moreover, $\lim_{z\to\pm\infty} v(z) = 0$ if either
$\lim_{z\to-\infty} U(z) = 1$ and $\lim_{z\to+\infty} U(z) = 0$, or
$\lim_{z\to-\infty} U(z) = 0$ and $\lim_{z\to+\infty} U(z) = 1$.

**Remark 2.2.** Constant functions $U \equiv k$, where $k$ is such that $g(k) = 0$, are solutions of (2.1). In particular, $U \equiv 0$, $U \equiv 1$ and $U \equiv s_*$ are solutions.

**Remark 2.3.** Let $z \notin M_U \cup N_U, \hat{z} = z + h, h \ne 0$. Divide (2.2) by $h$ and let $h \to 0$. Then, by Definition 2.1 (a), the derivative $U'(z)$ exists and

$$v'(z) + cU'(z) + g(U(z)) = 0. \tag{2.3}$$

In particular, $v$ is differentiable in $z$.

**Remark 2.4.** Let $U$ be a solution of (2.1) in the sense of Definition 2.1 such that either $U(z) \to 1$ as $z \to -\infty$ and $U(z) \to 0$ as $z \to +\infty$ or $U(z) \to 0$ as $z \to -\infty$ and $U(z) \to 1$ as $z \to +\infty$. If $d$ is not continuous in $(0, 1)$ then $M_U \ne \emptyset$ and either $M_U = \partial M_U$ (i.e., $\operatorname{int} M_U = \emptyset$) or else $\operatorname{int} M_U \ne \emptyset$. In the former case $M_U = \{\xi_1, \dots, \xi_n\}$, where $U(\xi_i) = s_i$, $i = 1, \dots, n$. In the latter case there exist $1 \le k \le n$ and $1 \le i_1 < i_2 < \dots < i_k \le n$ such that $U \equiv s_{i_j}$ on some interval $[a_{i_j}, b_{i_j}]$, $a_{i_j} < b_{i_j}$, $j = 1, \dots, k$, and $\operatorname{int} M_U = \cup_{j=1}^n (a_{i_j}, b_{i_j})$. The equation (2.1) is satisfied pointwise in $\operatorname{int} M_U$ and it follows from the continuity of $v$ that if $a_{i_j} > -\infty$ or $b_{i_j} < +\infty$ we have $U'(a_{i_j}) = 0$ or $U'(b_{i_j}) = 0$, $j = 1, \dots, k$, respectively, because $d(s) > 0$, $s \in (0, 1)$.

If $z_0 > -\infty$ then $U'(z_0-) = 0$ and $U'(z_0+)$ exists finite or infinite. Similarly, if $z_1 < +\infty$ then $U'(z_1+) = 0$ and $U'(z_1-)$ exists finite or infinite.

**Remark 2.5.** Let $p = 2$, $d \equiv 1$ and $g \in C^1[0,1]$. Let $U = U(z)$ be a solution in the sense of Definition 2.1. Then $M_U = \emptyset$, $N_U = \emptyset$, and (2.1) holds pointwise, i.e., $U \in C^2(\mathbb{R})$ and it is a classical solution. For more general $d$ we have to employ the first integral (2.2) because of the lack of differentiability of a solution $U$.

## 3. Equivalent first order ODE

We will look for monotone traveling waves $U = U(z)$ satisfying boundary conditions

$$\lim_{z\to-\infty} U(z) = 1 \quad \text{and} \quad \lim_{z\to+\infty} U(z) = 0. \tag{3.1}$$

Let $U : \mathbb{R} \to [0,1]$ be a monotone nonincreasing solution of the BVP (2.1), (3.1) such that $U$ is strictly decreasing at any point $z \in \mathbb{R}$ where $U(z) \in (0,1)$. Then there exist $-\infty \le z_0 < z_1 \le +\infty$ such that $U(z) = 1$, $z \in (-\infty, z_0]$ and $U(z) = 0, z \in [z_1, +\infty)$. Moreover, $M_U = \{\xi_1, \xi_2, \dots, \xi_n\}$ where $U(\xi_i) = s_i$, $i = 1, 2, \dots, n$. In particular, $\operatorname{int} M_U = \emptyset$ and $M_U = \partial M_U$, see Remark 2.4. For all $z \notin M_U \cup N_U$ we have $U'(z) < 0$ and for all $z \in M_U$ we have $U'(z-) < 0$ and $U'(z+) < 0$. The function $U$ is continuous and piecewise $C^1$ in the sense that $U\big|_{(\xi_i,\xi_{i+1})} \in C^1(\xi_i, \xi_{i+1})$. Therefore, there exists strictly decreasing inverse function $U^{-1} : (0,1) \to (z_0, z_1)$, $z = U^{-1}(U)$, such that $U^{-1}\big|_{(s_i,s_{i+1})} \in C^1(s_i, s_{i+1})$, $i = 0, 1, \dots, n$ and the limits

$$\lim_{U\to s_i-} \frac{\mathrm{d}}{\mathrm{d}U} U^{-1}(U), \quad \lim_{U\to s_i+} \frac{\mathrm{d}}{\mathrm{d}U} U^{-1}(U)$$

exist and are finite for $i = 1, 2, \dots, n$. Hence, we make the change of variables (cf. [7, 8])

$$w(U) = v(U^{-1}(U)), \quad U \in (0,1). \tag{3.2}$$

It follows from Remark 2.3 that $w = w(U)$ is a piecewise $C^1$-function in $(0,1)$,

$$w\big|_{(s_i,s_{i+1})} \in C^1(s_i, s_{i+1}), \quad i = 0, 1, \dots, n,$$

with finite limits $\lim_{U\to s_i-} w'(U)$, $\lim_{U\to s_i+} w'(U)$, $i = 1, 2, \dots, n$. Therefore, for any $z \in (\xi_i, \xi_{i+1})$ and $U \in (s_i, s_{i+1})$, $i = 0, 1, \dots, n$, we have

$$\frac{\mathrm{d}}{\mathrm{d}z} v(z) = \frac{\mathrm{d}}{\mathrm{d}z} w(U(z)) = \frac{\mathrm{d}w}{\mathrm{d}U}\,(U(z))\,U'(z). \tag{3.3}$$

From $v(z) = -d(U(z))|U'(z)|^{p-1}$ we deduce that

$$U'(z) = -\Big|\frac{v(z)}{d(U(z))}\Big|^{p'-1}, \quad p' = \frac{p}{p-1}\,. \tag{3.4}$$

From (3.2), (3.3) and (3.4),

$$\frac{\mathrm{d}v}{\mathrm{d}z} = -\frac{\mathrm{d}w}{\mathrm{d}U}\,(U(z))\,\Big|\frac{v(z)}{d(U(z))}\Big|^{p'-1} = -\frac{\mathrm{d}w}{\mathrm{d}U}\Big|\frac{w(U)}{d(U)}\Big|^{p'-1}.$$

Therefore, equation (2.3), namely

$$v'(z) + cU'(z) + g(U(z)) = 0, \quad z \in (\xi_i, \xi_{i+1}),$$

becomes

$$-\frac{\mathrm{d}w}{\mathrm{d}U}\Big|\frac{w(U)}{d(U)}\Big|^{p'-1} - c\Big|\frac{w(U)}{d(U)}\Big|^{p'-1} + g(U) = 0, \quad U \in (s_i, s_{i+1}),$$

$i = 0, 1, \dots, n$. This is equivalent to

$$|w|^{p'-1}\frac{\mathrm{d}w}{\mathrm{d}U} = -c|w|^{p'-1} + (d(U))^{p'-1}\, g(U), \tag{3.5}$$

or

$$\frac{1}{p'}\frac{\mathrm{d}}{\mathrm{d}U}|w|^{p'} = c|w|^{p'-1} - (d(U))^{p'-1}\, g(U). \tag{3.6}$$

Set $f(U) = (d(U))^{\frac{1}{p-1}}\, g(U)$ and write $t$ instead of $U$, and $y(t) = |w(t)|^{p'}$. Then (3.6) becomes

$$y'(t) = p'\big[c\,(y(t))^{1/p} - f(t)\big], \quad t \in (0,1) \setminus \cup_{i=1}^{n}\{s_i\}. \tag{3.7}$$

From (3.1) and Definition 2.1(c) we deduce that $v(z) \to 0$ as $z \to z_0+$ or $z \to z_1-$ which is equivalent to $\lim_{U\to 0+} w(U) = \lim_{U\to 1-} w(U) = 0$. Therefore, $y = y(t)$ satisfies the boundary conditions

$$y(0) = y(1) = 0. \tag{3.8}$$

On the other hand, let us suppose that $y = y(t)$ is a positive solution of (3.7), (3.8). Set $w(s) := -(y(s))^{1/p'}$. Then $w$ satisfies (3.5) and (3.6). For $U \in (0,1)$ set

$$z(U) = z(s_*) - \int_{s_*}^{U} \Big|\frac{d(s)}{w(s)}\Big|^{\frac{1}{p-1}}\, \mathrm{d}s. \tag{3.9}$$

Then the function $z = z(U)$ is continuous strictly decreasing and maps the interval $(0,1)$ onto $(z_0, z_1)$, where $-\infty \le z_0 < z_1 \le +\infty$. Let us denote by $U : (z_0, z_1) \to (0,1)$ the inverse function to $z = z(U)$. Then $U(z(s_*)) = s_*$, $U$ is continuous strictly decreasing,

$$\lim_{z\to z_0+} U(z) = 1 \quad \text{and} \quad \lim_{z\to z_1-} U(z) = 0.$$

Let $z \in (\xi_i, \xi_{i+1})$, $i = 0, 1, \dots, n$, where $U(\xi_i) = s_i$, $i = 0, 1, \dots, n, n+1$. Then from (3.9) we deduce

$$\frac{\mathrm{d}U}{\mathrm{d}z} = \frac{1}{\frac{\mathrm{d}z(U)}{\mathrm{d}U}} = -\Big|\frac{w(U)}{d(U)}\Big|^{\frac{1}{p-1}}, \quad U \in (s_i, s_{i+1}),$$

i.e., $U \in C^1(\xi_i, \xi_{i+1})$, $U'(z) < 0$ and

$$-\, d(U(z))\Big|\frac{\mathrm{d}U(z)}{\mathrm{d}z}\Big|^{p-1} = w(U(z)) =: v(z), \tag{3.10}$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}z}\Big[d(U(z))\Big|\frac{\mathrm{d}U}{\mathrm{d}z}\Big|^{p-2}\frac{\mathrm{d}U}{\mathrm{d}z}\Big] = \frac{\mathrm{d}}{\mathrm{d}z}w(U(z)) = \frac{\mathrm{d}w}{\mathrm{d}U}\,\frac{\mathrm{d}U(z)}{\mathrm{d}z}\,. \tag{3.11}$$

From (3.5), (3.10) we deduce that

$$\begin{aligned}
\frac{\mathrm{d}w}{\mathrm{d}U} &= -|w(U)|^{-(p'-1)}\Big(-c|w(U)|^{p'-1} + (d(U))^{p'-1}\, g(U)\Big)\\
&= -c + |w(U)|^{-(p'-1)}\,(d(U))^{p'-1}\, g(U)\\
&= -c + (d(U(z)))^{-(p'-1)}\,\Big|\frac{\mathrm{d}U(z)}{\mathrm{d}z}\Big|^{-(p-1)(p'-1)}\,(d(U(z)))^{p'-1}\, g(U(z))\\
&= -c + \Big|\frac{\mathrm{d}U(z)}{\mathrm{d}z}\Big|^{-1} g(U(z)).
\end{aligned}$$

Let us substitute this into (3.11):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\big[d(U(z))\big|\frac{\mathrm{d}U}{\mathrm{d}z}\big|^{p-2}\frac{\mathrm{d}U}{\mathrm{d}z}\big] &= \big[-c+\big|\frac{\mathrm{d}U(z)}{\mathrm{d}z}\big|^{-1}g(U(z))\big]\frac{\mathrm{d}U(z)}{\mathrm{d}z}\\ &= -c\frac{\mathrm{d}U(z)}{\mathrm{d}z}-g(U(z)),\end{aligned}$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}z}\big[d(U(z))\big|\frac{\mathrm{d}U}{\mathrm{d}z}\big|^{p-2}\frac{\mathrm{d}U}{\mathrm{d}z}\big]+c\frac{\mathrm{d}U(z)}{\mathrm{d}z}+g(U(z))=0,\quad z\in(\xi_i,\xi_{i+1}),$$

$i=0,1,\dots,n$. It follows from (3.10) and the continuity of $U$ that

$$\lim_{z\to z_0+}d(U(z))|U'(z)|^{p-2}U'(z)=\lim_{z\to z_1-}d(U(z))|U'(z)|^{p-2}U'(z)=0$$

and the following one sided limits are finite

$$\lim_{z\to\xi_i-}d(U(z))|U'(z)|^{p-2}U'(z)=\lim_{z\to\xi_i+}d(U(z))|U'(z)|^{p-2}U'(z),$$

$i=1,2,\dots,n$. Since $U$ is monotone decreasing in $(z_0,z_1)$, we have

$$\lim_{z\to\xi_i-}d(U(z))=\lim_{s\to s_i+}d(s)\quad\text{and}\quad\lim_{z\to\xi_i+}d(U(z))=\lim_{s\to s_i-}d(s).$$

Therefore, $U$ satisfies the transition condition

$$|U'(\xi_i-)|^{p-2}\,U'(\xi_i-)\lim_{s\to s_i+}d(s)=|U'(\xi_i+)|^{p-2}\,U'(\xi_i+)\lim_{s\to s_i-}d(s).$$

We summarize the above reasoning in the following equivalence.

**Proposition 3.1.** *A function $U:\mathbb{R}\to[0,1]$ is a monotone non-increasing solution of* (2.1), (3.1) *which is strictly decreasing at any point $z\in\mathbb{R}$ where $U(z)\in(0,1)$ if and only if $y:[0,1]\to\mathbb{R}$ is a positive solution of* (3.7), (3.8).

Thanks to this proposition we can study the first order problem (3.7), (3.8) and derive the existence result for (2.1), (3.1). Let us recall that there are "two unknowns" in the first order problem. Indeed, besides the positive solution $y=y(t)$ we also look for unknown speed of propagation $c\in\mathbb{R}$.

**Lemma 3.2.** *Let us assume that* (1.3) *holds and BVP* (3.7), (3.8) *has a positive solution. Then $c>0$.*

*Proof.* Let $y(t)>0, t\in(0,1)$ be a positive solution of (3.7), (3.8). Integrating (3.7) and using (3.8) we obtain

$$0=y(1)-y(0)=\int_0^1 y'(t)\,\mathrm{d}t=p'\Big[c\int_0^1 y(t)\,\mathrm{d}t-\int_0^1 f(t)\,\mathrm{d}t\Big].$$

Hence

$$c=\frac{\int_0^1(d(t))^{\frac{1}{p-1}}\,g(t)\,\mathrm{d}t}{\int_0^1 y(t)\,\mathrm{d}t}>0. \qquad \square$$

**Remark 3.3.** Let $d$ and $g$ be as in Section 2 and the following balanced condition holds

$$\int_0^1(d(s))^{\frac{1}{p-1}}\,g(s)\,\mathrm{d}s=0.$$

Then

$$y(t)=-p'\int_0^t(d(s))^{\frac{1}{p-1}}\,g(s)\,\mathrm{d}s,\quad t\in(0,1) \tag{3.12}$$

is a unique positive solution of (3.7), (3.8) with $c = 0$. The solution given by (3.12) leads to the standing wave. Its profile $U = U(x)$ satisfies

$$\left(d(U(x))|U'(x)|^{p-2}U'(x)\right)' + g(U(x)) = 0, \quad x \in \mathbb{R}.$$

Detailed analysis of these solutions for $g$ satisfying $g < 0$ in $(0, s_*)$ is given in [7].

## 4. Existence result

In this section we first concentrate on the existence result for the first order BVP (3.7), (3.8). More precisely, we prove that under appropriate assumptions on $f = f(t)$ there exists a unique real number $c > 0$ and an absolutely continuous function $y = y(t)$ such that $y(t) > 0, t \in (0,1)$, (3.8) holds and the equation (3.7) is satisfied in the sense of Carathéodory (see [3, Chapter 2]).

The following result is interesting on its own. In combination with Proposition 3.1 it is a tool to prove the existence and uniqueness of the traveling speed $c$ and monotone decreasing traveling wave profile $U$.

**Theorem 4.1.** *Let $f \in L^1(0,1)$, $f(t) \le 0$, $t \in (0, s_*)$, $f(t) > 0$, $t \in (s_*, 1)$, and*

$$\int_0^1 f(s)\,\mathrm{d}s > 0. \tag{4.1}$$

*Then there is a unique number $c > 0$ and an absolutely continuous function $y = y(t)$, $t \in [0,1]$, such that $y(0) = y(1) = 0$, $y(t) > 0, t \in (0,1)$, and*

$$y'(t) = p'\left[c\left(y^+(t)\right)^{1/p} - f(t)\right]$$

*for a.a. $t \in (0,1)$. Here $y^+ = \max\{y, 0\}$ denotes the positive part of $y$.*

We prove Theorem 4.1 using the concept of solution of the first order ODE in the sense of Carathéodory. For $(t,y,c) \in [0,1] \times \mathbb{R}^2$ and $f = f(t)$ we set

$$h(t,y,c) := p'\left[c\left(y^+\right)^{1/p} - f(t)\right]$$

and consider the following two initial value problems which depend on a parameter $c \in \mathbb{R}$:

$$y'(t) = h(t, y(t), c), \quad y(0) = 0 \tag{4.2}$$

and

$$y'(t) = h(t, y(t), c), \quad y(1) = 0. \tag{4.3}$$

In both cases we look for a solution $y = y(t)$, $t \in [0,1]$. Therefore, (4.2) is referred to as a *forward initial value problem*, while (4.3) is referred to as a *backward initial value problem*. Note that $f \in L^1(0,1)$ implies that $h = h(t,y,c)$ satisfies Carathéodory's conditions, i.e., for a.e. $t \in [0,1]$ fixed, $h(t,\cdot,\cdot)$ is continuous with respect to $y$ and $c$ and for every $y \in \mathbb{R}$ and $c \in \mathbb{R}$ fixed, $h(\cdot,y,c)$ is measurable with respect to $t$. In what follows, for any fixed $c \in \mathbb{R}$, $y_c = y_c(t)$ denotes the solution in the sense of Carathéodory of the forward and backward initial value problem (4.2) and (4.3), respectively. In particular, $y_c$ is absolutely continuous in $[0,1]$ and the equation holds a.e. in $[0,1]$. Let us mention the following global existence result.

**Lemma 4.2.** *Let $f \in L^1(0,1)$, $c \in \mathbb{R}$. Then there exists at least one global solution $y_c = y_c(t)$ of both (4.2) and (4.3) defined on the entire interval $[0,1]$.*

*Proof.* For any fixed $K > 0$ and $c \in \mathbb{R}$ there exists $m_{c,K} \in L^1(0,1)$ such that for any $y \in [-K, K]$ we have $|h(t, y, c)| \le m_{c,K}(t)$ for a.a. $t \in [0,1]$. This fact follows from the definition of $h$. But then according to [14, Theorem 10.XX] there exist solutions $y_c = y_c(t)$ of both (4.2) and (4.3) which are defined for all $t \in [0,1]$. $\square$

**Remark 4.3.** The uniqueness of the solution in the above lemma does not hold in general due to the fact that the function $y \mapsto c\,(y^+)^{1/p}, y \in \mathbb{R}$, does not satisfy the Lipschitz condition at 0. However, it is nondecreasing for $c > 0$ and non-increasing for $c < 0$. Therefore, it satisfies one-sided Lipschitz condition in either case and we have the following uniqueness results separately for the forward and backward initial value problems, depending on the sign of $c$.

**Lemma 4.4.** *Let $f \in L^1(0,1)$. If $c \le 0$ then* (4.2) *has exactly one solution $y_c = y_c(t), t \in [0,1]$. If $c \ge 0$ then* (4.3) *has exactly one solution $y_c = y_c(t), t \in [0,1]$.*

The proof of the above lemma follows from combination of Theorems 9.X and 10.XX in [14]. Thanks to the uniqueness result we also have continuous dependence of solutions on the parameter $c$.

**Lemma 4.5.** *Let $f \in L^1(0,1)$, $c_0 \ge 0$. Then $c \to c_0 \ne 0$ ($c \to 0_+$ if $c_0 = 0$) implies that solutions $y_c = y_c(t)$ of the backward initial value problem* (4.3) *converge uniformly in* $[0,1]$ *(i.e., in the topology of* $C[0,1]$*) to $y_{c_0}$. Similar result holds for $c_0 \le 0$ and the forward initial value problem* (4.2)*.*

The proof of the above lemma follows from the uniqueness result in Lemma 4.4 and [3, Theorems 4.1 and 4.2].

As we already observed in the proof of Lemma 3.2, the assumption (4.1) yields $c > 0$. For this reason, we further focus on parameters $c \in [0, +\infty)$ and the backward initial value problem (4.3). We know that for any $c \in [0, +\infty)$ there is a unique solution of (4.3), $y_c = y_c(t), t \in [0,1]$. Our goal is to show that there is unique $c_* > 0$ such that $y_{c_*}(t) > 0$, $t \in (0,1)$, $y_{c_*}(0) = 0$. To this end we have to investigate in more detail the dependence of the solution $y_c = y_c(t)$ of the backward initial value problem (4.3) on the parameter $c$.

Let us introduce the notion of the *defect* $P_c\varphi$ of a function $\varphi = \varphi(t)$ with respect to the differential equation $y' = h(t, y, c)$, see e.g. [14, §9.II]:

$$P_c\varphi := \varphi' - h(t, \varphi, c).$$

The following comparison argument is one of our basic tools.

**Lemma 4.6.** *Let $f \in L^1(\varrho, 1)$,$0 \le \varrho < 1$, $c \ge 0$, $\varphi(1) \le \psi(1)$, $P_c\varphi \ge P_c\psi$ a.e. in $[\varrho, 1]$. Then either $\varphi < \psi$ in $[\varrho, 1]$ or there exists $\xi \in [\varrho, 1]$ such that $\varphi = \psi$ in $[\xi, 1]$ and $\varphi < \psi$ in $(\varrho, \xi]$. In particular, $\varphi \le \psi$ in $[\varrho, 1]$.*

The proof of the above lemma follows from combination of Theorems 9.X and 10.XXI in [14].

**Corollary 4.7.** *Let $f$ be as in Theorem 4.1, and $0 \le c_1 < c_2$. Then*

$$y_{c_1}(t) > y_{c_2}(t), \quad t \in (0,1).$$

*In particular, we have the weak comparison at the terminal value 0: $y_{c_1}(0) \ge y_{c_2}(0)$.*

*Proof.* We have

$$P_{c_2}y_{c_1} = y'_{c_1} - h(t, y_{c_1}, c_2) = \underbrace{y'_{c_1} - h(t, y_{c_1}, c_1)}_{=0} + h(t, y_{c_1}, c_1) - h(t, y_{c_1}, c_2)$$

$$= p'(c_1 - c_2)\left(y_{c_1}^+\right)^{1/p} \le 0 = y_{c_2}' - h(t, y_{c_2}, c_2) = P_{c_2} y_{c_2}\,.$$

Then Lemma 4.6 with $\varrho = 0$ yields that either $y_{c_1} > y_{c_2}$ in $(0,1)$ or there exists $\xi \in [0,1]$ such that $y_{c_1} = y_{c_2}$ in $[\xi, 1]$ and $y_{c_1} > y_{c_2}$ in $(0, \xi)$. Since both $y_{c_1}$ and $y_{c_2}$ solve the backward initial value problem, we subtract the equation in (4.3) for $c = c_2$ from that for $c = c_1$ and obtain

$$p'(c_1 - c_2)\left(y_{c_1}^+\right)^{1/p} = 0 \quad \text{in } [\xi, 1],$$

i.e., $y_{c_1} = y_{c_2} \le 0$ in $[\xi, 1]$. But from (4.1) and (4.3) we then obtain

$$y_{c_1}(\xi) = y_{c_2}(\xi) = p' \int_\xi^1 f(\sigma)\,\mathrm{d}\sigma > 0$$

if $\xi < 1$. Therefore, $\xi = 1$ and $y_{c_1} > y_{c_2}$ in $(0,1)$. Using Lemma 4.2 and extending $y_{c_1}$, $y_{c_2}$ continuously to 0, we end up with $y_{c_1}(0) \ge y_{c_2}(0)$. □

**Remark 4.8.** Unfortunately, the comparison argument above does not allow us to conclude the strict inequality $y_{c_1}(0) > y_{c_2}(0)$ in general. For this reason the proof of uniqueness of $c_*$ is more involved and requires more detailed analysis of the solution of the backward initial value problem (4.3) at the terminal value 0.

**Corollary 4.9.** *Let $f \in L^1(0,1)$, $f(t) \le 0$, $t \in (0, s_*)$, $f(t) > 0$, $t \in (s_*, 1)$ and $\tilde f(t) = 0$, $t \in (0, s_*)$, $\tilde f(t) = f(t)$, $t \in (s_*, 1)$. Let $c \ge 0$ and $\tilde y_c = \tilde y(t)$, $t \in [0,1]$ be a solution of* (4.3) *with $f$ replaced by $\tilde f$. Then $y_c \le \tilde y_c$ in $(0,1]$.*

*Proof.* Set $\tilde h(t, y, c) := p'\left[c(y^+(t)^{1/p}) - \tilde f(t)\right]$. Then $\tilde h \le h$ and so

$$\begin{aligned} P_c \tilde y_c = \tilde y_c' - h(t, \tilde y_c, c) &= \underbrace{\tilde y_c' - \tilde h(t, \tilde y_c, c)}_{=0} + \tilde h(t, \tilde y_c, c) - h(t, \tilde y_c, c) \\ &\le 0 = y_c' - h(t, y_c, c) = P_c y_c \quad \text{a.e. in} \quad (0,1). \end{aligned}$$

It then follows from Lemma 4.6 with $\varrho = 0$ that $y_c \le \tilde y_c$ in $[0,1]$. □

**Corollary 4.10.** *Let $f \in L^1(0,1)$, $f(t) \le 0$, $t \in (0, s_*)$, $f(t) > 0$, $t \in (s_*, 1)$. Then there exists $c_\# > 0$ such that $y_{c_\#}(0) < 0$.*

*Proof.* Let $\tilde f$ be as in Corollary 4.9. Since $\tilde f = 0$ in $(0, s_*)$, we have

$$\tilde y_c' = p' c \left(\tilde y_c^+(t)\right)^{1/p} \quad \text{a.e. in } (0, s_*). \tag{4.4}$$

Assume that there exist $c_n \to +\infty$ such that $\tilde y_{c_n} \ge 0$ in $[0,1]$. Then $\tilde y_c^+ = \tilde y_c$ and separating variables in (4.4) yields

$$(\tilde y_{c_n}(t))^{1/p'} = (\tilde y_{c_n}(s_*))^{1/p'} + c_n(t - s_*), \quad t \in [0, s_*), \tag{4.5}$$

and

$$\tilde y_{c_n}'(t) = p'\big[c_n\,(\tilde y_{c_n}(t))^{1/p} - f(t)\big], \quad t \in [s_*, 1),$$

with $\tilde y_{c_n}(1) = 0$. Therefore

$$\tilde y_{c_n}(s_*) \le p' \int_{s_*}^1 f(\sigma)\,\mathrm{d}\sigma < +\infty. \tag{4.6}$$

Then we conclude from (4.5), (4.6) that there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we must have $\tilde y_{c_n}(0) < 0$. But Corollary 4.9 yields that $y_{c_n}(0) \le \tilde y_{c_n}(0) < 0$. Therefore, we may set $c_\# = c_n$, $n \ge n_0$. □

**Corollary 4.11.** *Let $f \in L^1(0,1)$, $f(t) \le 0$, $t \in (0,s_*)$, $f(t) > 0$, $t \in (s_*,1)$ and $y_c = y_c(t)$ be a solution of (4.3) with $c > 0$. If $y_c(0) \ge 0$ then $y_c(t) > 0$ for $t \in (0,1)$.*

*Proof.* We have

$$P_c 0 = 0 - h(t,0,c) = p' f(t) \ge 0 = y_c' - h(t,y_c,c) = P_c y_c \quad \text{a.e. in } [s_*,1].$$

Then by Lemma 4.6 with $\varrho = s_*$ either $y_c > 0$ in $[s_*,1)$ or there exists $\xi \in [s_*,1]$ such that $y_c = 0$ in $[\xi,1]$ and $y_c > 0$ in $[s_*,\xi)$. In the latter case

$$0 = y_c(\xi) = p'\Big[c\int_1^{\xi} \big(y_c^+(t)\big)^{1/p}\,\mathrm{d}t - \int_1^{\xi} f(t)\,\mathrm{d}t\Big] = p'\int_{\xi}^1 f(t)\,\mathrm{d}t$$

forces $\xi = 1$ (note that $f > 0$ in $(s_*,1)$). Therefore, $y_c > 0$ in $[s_*,1)$. Assume that $y_c$ vanishes at $(0,1)$ and $\eta \in (0,s_*)$ be its largest zero. Then

$$y_c'(t) = p'\Big[c(y_c^+(t))^{1/p} - f(t)\Big] \ge p' c (y_c^+(t))^{1/p}, \quad \text{for a.e. } t \in [0,\eta].$$

Separating variables and integrating over $[t,\eta]$,

$$(y_c^+(t))^{1/p'} \le -c(\eta - t), \quad t \in [0,\eta].$$

In particular, we have

$$\big(y_c^+(0)\big)^{1/p'} \le -c\eta < 0,$$

a contradiction with $y_c(0) \ge 0$. □

*Proof of Theorem 4.1.* It follows from the assumptions on $f$ that

$$y_0(t) = p'\int_t^1 f(\sigma)\,\mathrm{d}\sigma > 0$$

for all $t \in [0,1)$. In particular, $y_0(0) > 0$. On the other hand, from Corollary 4.10 there exists $c_\# > 0$ such that $y_{c_\#}(0) < 0$. The continuous dependence on parameter $c$ in Lemma 4.5, intermediate value theorem and the monotonicity of function $\mathcal{S} : c \mapsto y_c(0)$ in Corollary 4.7 imply that there exist $0 < c_1 \le c_2 < c_\#$ such that $\mathcal{S}(c) = 0$ for all $c \in [c_1,c_2]$, $\mathcal{S}(c) > 0$, $c < c_1$ and $\mathcal{S}(c) < 0$, $c > c_2$. Below we derive the strong comparison argument which shows that, actually, $c_1 = c_2$. Indeed, let $c_1 < c_2$. By Corollaries 4.7 and 4.11 we have

$$y_{c_1}(t) > y_{c_2}(t) > 0 \quad \text{for } t \in (0,1). \tag{4.7}$$

Notice that for $c \in (c_1,c_2)$ we also have

$$y_{c_1}'(t) = p'\big[c_1\,(y_{c_1}(t))^{1/p} - f(t)\big] \le p'\big[c\,(y_{c_1}(t))^{1/p} - f(t)\big], \tag{4.8}$$

$$y_{c_2}'(t) = p'\big[c_2\,(y_{c_2}(t))^{1/p} - f(t)\big] \ge p'\big[c\,(y_{c_2}(t))^{1/p} - f(t)\big], \tag{4.9}$$

for a.e. $t \in (0,1)$. Set $z_1 = (y_{c_1})^{1/p'} > 0$, $z_2 = (y_{c_2})^{1/p'} > 0$. Then $z_1 > z_2$ in $(0,1)$ and it follows from (4.8) and (4.9) that

$$z_1'(t) \le c - \frac{f(t)}{(z_1(t))^{\frac{1}{p-1}}}\,, \tag{4.10}$$

$$z_2'(t) \ge c - \frac{f(t)}{(z_2(t))^{\frac{1}{p-1}}}\,, \tag{4.11}$$

for a.e. $t \in (0,1)$. Let us subtract (4.11) from (4.10) and restrict on the interval $(0,s_*)$. Then

$$(z_1(t)-z_2(t))' \le -f(t)\Big(\frac{1}{(z_1(t))^{\frac{1}{p-1}}} - \frac{1}{(z_2(t))^{\frac{1}{p-1}}}\Big)$$

and

$$\begin{aligned}&(z_1(t)-z_2(t))\,(z_1(t)-z_2(t))'\\ &\le -f(t)\Big(\frac{1}{(z_1(t))^{\frac{1}{p-1}}} - \frac{1}{(z_2(t))^{\frac{1}{p-1}}}\Big)\,(z_1(t)-z_2(t)) \le 0\end{aligned}$$

for a.e. $t \in (0,s_*)$ (notice that $f(t) \le 0$ in $(0,s_*)$). Hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,(z_1(t)-z_2(t))^2 \le 0 \quad \text{for a.e. } t \in (0,s_*). \tag{4.12}$$

Since $z_1(0)=z_2(0)=0$, it follows from (4.12) that $z_1(t)=z_2(t)$, $t \in (0,s_*)$, i.e., $y_{c_1}(t)=y_{c_2}(t)$, $t\in(0,s_*)$. However, this contradicts (4.7). Therefore, $c_1=c_2$ and $c_*=c_1=c_2$ is the unique value of $c$ for which $y_c(0)=0$. As mentioned above, $y_{c_*}(t)>0$, $t\in(0,1)$, and $y_{c_*}$ is the unique solution of the backward initial value problem (4.3). The proof is complete. □

**Theorem 4.12.** *Let $d$ and $g$ be as in Section 2 and (1.3) holds. Then there is a unique value of $c=c_*$ and unique non-increasing traveling wave profile $U=U(z)$, $z\in\mathbb{R}$, such that $U$ solves the BVP (2.1), (3.1). Furthermore, $c_*>0$ and*

(i) *there exist $-\infty \le z_0 < 0 < z_1 \le +\infty$ such that $U(z)=1$ for $z\in(-\infty,z_0]$, $U(z)=0$ for $z\in[z_1,+\infty)$;*
(ii) *$U$ is strictly decreasing in $(z_0,z_1)$, $U(0)=s_*$;*
(iii) *for $i=0,1,2,\dots,n,n+1$ let $\xi_i\in[z_0,z_1]$ be such that $U(\xi_i)=s_i$, then $U$ is a piecewise $C^1$- function in the sense that $U$ is continuous,*
$$U\big|_{(\xi_i,\xi_{i+1})} \in C^1(\xi_i,\xi_{i+1}), \quad i=0,1,\dots,n,$$
*and the limits $U'(\xi_i-) := \lim_{z\to\xi_i-} U'(z)$, $U'(\xi_i+) := \lim_{z\to\xi_i+} U'(z)$ exist are finite for all $i=1,2,\dots,n$;*
(iv) *for any $i=1,2,\dots,n$ the following transition condition holds:*
$$|U'(\xi_i-)|^{p-2}\,U'(\xi_i-)\lim_{s\to s_i+} d(s) = |U'(\xi_i+)|^{p-2}\,U'(\xi_i+)\lim_{s\to s_i-} d(s).$$

*Proof.* The existence and uniqueness of $c_*$ and $U$ follow directly from Theorem 4.1 and Proposition 3.1. The properties of $U$ are derived in the reasoning preceding the statement of Proposition 3.1. □

**Remark 4.13.** The inequality "$>$" in (1.3) was motivated by modeling heterzygote inferior case. The assumption

$$\int_0^1 (d(s))^{\frac{1}{p-1}}\,g(s)\,\mathrm{d}s < 0 \tag{4.13}$$

leads to negative traveling speed of propagation $c_*<0$ and it can be treated in a similar way. However, in this case the main tool is a shooting argument applied to the forward initial value problem and the strong comparison argument must be derived at the terminal value 1.

We can also prove similar results for increasing traveling wave profile $U$ satisfying

$$\lim_{z\to-\infty} U(z) = 0 \quad \text{and} \quad \lim_{z\to+\infty} U(z) = 1.$$

In this case the assumption (1.3) leads to $c_* < 0$ while (4.13) leads to $c_* > 0$, respectively.

**Remark 4.14.** Notice that condition $U(0) = s_*$ has just a normalizing character. Indeed, since the equation (2.1) is autonomous then given any $\xi \in \mathbb{R}$ the translation $V(z) = U(z-\xi)$, $z \in \mathbb{R}$, is also a solution of (2.1) which satisfies $V(\xi) = s_*$.

## 5. Asymptotic analysis of the traveling wave profile

In this section we focus on the asymptotic behavior of the traveling wave profile $U = U(z)$ as $z \to \pm\infty$. Similar asymptotic analysis for standing waves ($c = 0$) was done in [7]. Even though the main idea is the same also in the case $c \neq 0$, the analysis is much more involved and not so precise because the solution of equation (3.7) for $c \neq 0$ cannot be obtained in a closed form by simple integration of $f = f(t)$ as in the stationary case ($c = 0$), cf. [7].

For the sake of brevity, for $t_0 \in \mathbb{R}$ we write

$$h_1(t) \sim h_2(t) \text{ as } t \to t_0 \quad \text{if and only if} \quad \lim_{t\to t_0} \frac{h_1(t)}{h_2(t)} \in (0,+\infty).$$

### 5.1. Asymptotics near 1.

Let us assume that $g(t) \sim (1-t)^\gamma$ and $d(t) \sim (1-t)^\delta$ as $t \to 1-$ for some $\gamma > 0$ and $\delta \in \mathbb{R}$. Then, formally,

$$f(t) = (d(t))^{\frac{1}{p-1}}\, g(t) \sim (1-t)^{\gamma+\frac{\delta}{p-1}} \quad \text{as } t \to 1- \ .$$

The fact that $f \in L^1(0,1)$ then implies the following necessary condition for parameters $\gamma$ and $\delta$:

$$\gamma + \frac{\delta}{p-1} > -1. \tag{5.1}$$

It follows from (3.9) that the inverse function to a profile $U = U(z)$ corresponding to the speed $c > 0$ and normalized by $U(0) = s_*$ is given by

$$z(U) = -\int_{s_*}^{U} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\, dt, \quad U \in (0,1), \tag{5.2}$$

where $y_{c_*} = y_{c_*}(t)$ is the unique positive solution of (3.7), (3.8). In order to find the asymptotic behavior of $z = z(U)$ as $U \to 1-$ from (5.2) we need to establish the asymptotics of $y_{c_*} = y_{c_*}(t)$ as $t \to 1-$. The asymptotics of $U = U(z)$ as $z \to -\infty$ then would follow applying the inverse point of view to $z = z(U)$ for $U \to 1-$.

From our assumptions it follows that there exists $\theta > 0$ (small enough) such that both $d$ and $g$ are continuous in $(1-\theta, 1)$. Therefore, $f = f(t)$ is also continuous in $(1-\theta, 1)$ and hence $f(t) \sim (1-t)^{\gamma+\frac{\delta}{p-1}}$ is equivalent to

$$f(t) = \eta(t)(1-t)^{\gamma+\frac{\delta}{p-1}}, \quad t \in (1-\theta, 1),$$

where $\eta = \eta(t)$ is a continuous function in $(1-\theta, 1)$, $\lim_{t\to 1-} \eta(t) \in (0,+\infty)$.

In what follows we discuss different cases with respect to parameters $\gamma$, $\delta$ and $p$.

A. Let $\gamma+\frac{\delta}{p-1}\le\frac{1}{p-1}$. Then for $\kappa>0$ we set $y_\kappa(t)=\kappa(1-t)^{\gamma+\frac{\delta}{p-1}+1}$, $t\in(1-\theta,1)$ and calculate the defect

$$
\begin{aligned}
P_{c_*}y_\kappa &= y_\kappa' - p'[c_*\,(y_\kappa)^{\frac1p}-f(t)]\\
&= -\kappa\Big(\gamma+\frac{\delta}{p-1}+1\Big)(1-t)^{\gamma+\frac{\delta}{p-1}}\\
&\quad - p'\Big[c_*\kappa^{\frac1p}(1-t)^{\frac{\gamma+\frac{\delta}{p-1}+1}{p}}-\eta(t)(1-t)^{\gamma+\frac{\delta}{p-1}}\Big]\\
&= (1-t)^{\gamma+\frac{\delta}{p-1}}\Big[-\kappa\Big(\gamma+\frac{\delta}{p-1}+1\Big)+p'\eta(t)\Big]\\
&\quad -(1-t)^{\frac{\gamma+\frac{\delta}{p-1}+1}{p}}p'c_*\kappa^{\frac1p},
\end{aligned}
\tag{5.3}
$$

$t\in(1-\theta,1)$. Our assumption $\gamma+\frac{\delta}{p-1}\le\frac{1}{p-1}$ implies

$$
\gamma+\frac{\delta}{p-1}\le\frac{\gamma+\frac{\delta}{p-1}+1}{p}\,,
$$

and therefore the power $(1-t)^{\gamma+\frac{\delta}{p-1}}$ dominates the power $(1-t)^{\frac{\gamma+\frac{\delta}{p-1}+1}{p}}$ near 1. It then follows from (5.3) that we may distinguish between two cases:

A1. There exists $\underline{\kappa}\ll1$ so small that $P_{c_*}y_{\underline{\kappa}}>0=P_{c_*}y_{c_*}$ a.e. in $(1-\theta,1)$.

A2. There exists $\bar\kappa\gg1$ so large that $P_{c_*}y_{\bar\kappa}<0=P_{c_*}y_{c_*}$ a.e. in $(1-\theta,1)$.

**Case A1.** Let $\frac{\gamma-\delta+1}{p}<1$. It follows from Lemma 4.6 with $\varrho=1-\theta$ that

$$
y_{c_*}(t)\ge y_{\underline{\kappa}}(t)\quad\text{in } (1-\theta,1). \tag{5.4}
$$

From (5.2) and (5.4) we conclude that there exists $c_1>0$ such that

$$
\begin{aligned}
z_0=\lim_{U\to1-}z(U) &= -\int_{s_*}^1\frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac1p}}\,\mathrm{d}t\\
&\ge -\int_{s_*}^1\frac{(d(t))^{\frac{1}{p-1}}}{(y_{\underline{\kappa}}(t))^{\frac1p}}\,\mathrm{d}t\\
&\ge -c_1\int_{s_*}^1\frac{(1-t)^{\frac{\delta}{p-1}}}{(1-t)^{\frac{\gamma+\frac{\delta}{p-1}+1}{p}}}\,\mathrm{d}t\\
&= -c_1\int_{s_*}^1\frac{\mathrm{d}t}{(1-t)^{\frac{\gamma+\frac{\delta}{p-1}+1}{p}-\frac{\delta}{p-1}}}\\
&= -c_1\int_{s_*}^1\frac{\mathrm{d}t}{(1-t)^{\frac{\gamma-\delta+1}{p}}}>-\infty.
\end{aligned}
$$

**Case A2.** Let $\frac{\gamma-\delta+1}{p}\ge1$. It follows from Lemma 4.6 with $\varrho=1-\theta$ that

$$
y_{c_*}(t)\le y_{\bar\kappa}(t)\quad\text{in } (1-\theta,1). \tag{5.5}
$$

From (5.2) and (5.5) we conclude that there exists $c_2>0$ such that

$$
z_0=-\int_{s_*}^1\frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac1p}}\,\mathrm{d}t\le-\int_{s_*}^1\frac{(d(t))^{\frac{1}{p-1}}}{(y_{\bar\kappa}(t))^{\frac1p}}\,\mathrm{d}t
$$

$$\begin{aligned}
&\le -c_2 \int_{s_*}^1 \frac{(1-t)^{\frac{\delta}{p-1}}}{(1-t)^{\frac{\gamma+\frac{\delta}{p-1}+1}{p}}}\,\mathrm{d}t \\
&= -c_2 \int_{s_*}^1 \frac{\mathrm{d}t}{(1-t)^{\frac{\gamma-\delta+1}{p}}} = -\infty.
\end{aligned}$$

Taking into account (5.1), we may summarize these two cases as follows.

**Theorem 5.1.** *Let us assume $\gamma > 0$,*

$$-1 < \gamma + \frac{\delta}{p-1} \le \frac{1}{p-1}, \tag{5.6}$$

$$\frac{\gamma-\delta+1}{p} < 1, \tag{5.7}$$

*Then $z_0 > -\infty$. Let us assume $\gamma > 0$, (5.6), and*

$$\frac{\gamma-\delta+1}{p} \ge 1. \tag{5.8}$$

*Then $z_0 = -\infty$.*

B. Let $\gamma + \frac{\delta}{p-1} > \frac{1}{p-1}$. Then for $\kappa > 0$ we set $y_\kappa(t) = \kappa(1-t)^{p\left(\gamma+\frac{\delta}{p-1}\right)}$, $t \in (1-\theta, 1)$ and calculate

$$\begin{aligned}
P_{c_*} y_\kappa &= y'_\kappa - p'[c_* (y_\kappa)^{\frac{1}{p}} - f(t)] \\
&= -\kappa p\Big(\gamma + \frac{\delta}{p-1}\Big)(1-t)^{p\left(\gamma+\frac{\delta}{p-1}\right)-1} \\
&\quad - p'\big[c_*\kappa^{\frac{1}{p}}(1-t)^{\gamma+\frac{\delta}{p-1}} - \eta(t)(1-t)^{\gamma+\frac{\delta}{p-1}}\big] \\
&= -\kappa p\Big(\gamma + \frac{\delta}{p-1}\Big)(1-t)^{p\left(\gamma+\frac{\delta}{p-1}\right)-1} \\
&\quad - p'\big[c_*\kappa^{\frac{1}{p}} - \eta(t)\big](1-t)^{\gamma+\frac{\delta}{p-1}},
\end{aligned} \tag{5.9}$$

for $t \in (1-\theta, 1)$. Our assumption $\gamma(p-1) + \delta > 1$ implies

$$\gamma + \frac{\delta}{p-1} < p\left(\gamma + \frac{\delta}{p-1}\right) - 1,$$

and the power $(1-t)t^{\gamma+\frac{\delta}{p-1}}$ dominates the power $(1-t)^{p\left(\gamma+\frac{\delta}{p-1}\right)-1}$ near 1. It follows from (5.9) that we may distinguish between two cases:

B1. There exists $\underline{\kappa} \ll 1$ so small that $P_{c_*} y_{\underline{\kappa}} > 0 = P_{c_*} y_{c_*}$ a.e. in $(1-\theta, 1)$.

B2. There exists $\bar{\kappa} \gg 1$ so large that $P_{c_*} y_{\bar{\kappa}} < 0 = P_{c_*} y_{c_*}$ a.e. in $(1-\theta, 1)$.

**Case B1.** Let $\gamma < 1$. From Lemma 4.6 with $\varrho = 1-\theta$ we obtain

$$y_{c_*}(t) \ge y_{\underline{\kappa}}(t) \quad \text{in } (1-\theta, 1)$$

and therefore, similarly as in Case A1 we conclude that there exists $c_3 > 0$ such that

$$z_0 \ge -c_3 \int_{s_*}^1 \frac{(1-t)^{\frac{\delta}{p-1}}}{(1-t)^{\gamma+\frac{\delta}{p-1}}}\,\mathrm{d}t = -c_3 \int_{s_*}^1 \frac{\mathrm{d}t}{(1-t)^{\gamma}} > -\infty.$$

**Case B2.** Let $\gamma \ge 1$. From Lemma 4.6 with $\varrho = 1-\theta$ we obtain

$$y_{c_*}(t) \le y_{\bar{\kappa}}(t) \quad \text{in } (1-\theta, 1)$$

and as in Case A2 we conclude that there exists $c_4 > 0$ such that

$$z_0 \le -c_4 \int_{s_*}^{1} \frac{(1-t)^{\frac{\delta}{p-1}}}{(1-t)^{\gamma+\frac{\delta}{p-1}}}\,\mathrm{d}t = -c_4 \int_{s_*}^{1} \frac{\mathrm{d}t}{(1-t)^{\gamma}} = -\infty.$$

We summarize these two cases as follows.

**Theorem 5.2.** *Let us assume* $\gamma > 0$,

$$\gamma + \frac{\delta}{p-1} > \frac{1}{p-1}, \tag{5.10}$$

$$\gamma < 1. \tag{5.11}$$

*Then* $z_0 > -\infty$. *Let us assume* $\gamma > 0$, (5.10) *and*

$$\gamma \ge 1. \tag{5.12}$$

*Then* $z_0 = -\infty$.

**Remark 5.3.** To visualize conditions (5.6)–(5.8) and (5.10)–(5.12), we introduce the following sets:

$$\mathcal{M}_1^1 := \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, -1 < \gamma + \frac{\delta}{p-1} \le \frac{1}{p-1}, \gamma - \delta + 1 < p\},$$

$$\mathcal{M}_1^2 := \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, -1 < \gamma + \frac{\delta}{p-1} \le \frac{1}{p-1}, \gamma - \delta + 1 \ge p\},$$

$$\mathcal{M}_1^3 := \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, \gamma + \frac{\delta}{p-1} > \frac{1}{p-1}, \gamma < 1\},$$

$$\mathcal{M}_1^4 := \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, \gamma + \frac{\delta}{p-1} > \frac{1}{p-1}, \gamma \ge 1\}.$$

Then $z_0 > -\infty$ if and only if $(\gamma,\delta) \in \mathcal{M}_1^1 \cup \mathcal{M}_1^3$ and $z_0 = -\infty$ if and only if $(\gamma,\delta) \in \mathcal{M}_1^2 \cup \mathcal{M}_1^4$. See Figure 1 for geometric interpretation. Our results generalize those from [5, Section 6].

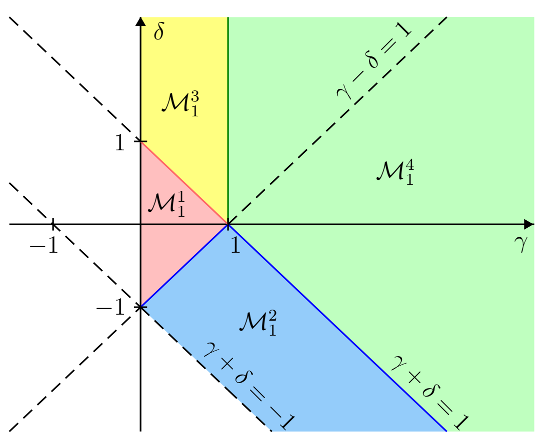


FIGURE 1. Visualization of the sets $\mathcal{M}_1^1$, $\mathcal{M}_1^2$, $\mathcal{M}_1^3$ and $\mathcal{M}_1^4$ for $p = 2$

5.2. **Asymptotics near 0.** Let us assume that $g(t) \sim -t^\alpha$ and $d(t) \sim t^\beta$ as $t \to 0+$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$. Then, formally, $f(t) \sim -t^{\alpha+\frac{\beta}{p-1}}$ as $t \to 0+$. The assumption $f \in L^1(0,1)$ yields necessary condition for parameters $\alpha$ and $\beta$:

$$\alpha + \frac{\beta}{p-1} > -1. \tag{5.13}$$

The main idea to find the asymptotics of $U = U(z)$ as $z \to +\infty$ is now based on the investigation of the asymptotics of its inverse $z = z(U)$ as $U \to 0+$. For this purpose we employ the formula (5.2) and, in particular, its limit for $U \to 0+$:

$$z_1 = \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t. \tag{5.14}$$

The main difference between this and previous case (asymptotics near 1) consists in the fact that we cannot use the comparison argument based on Lemma 4.6 due to the lack of uniqueness for the forward initial value problem (4.2). However, special form of our equation allows for the uniqueness result for this problem if we restrict on the set of positive solutions in a neighbourhood of 0. We will explain this idea below.

**Lemma 5.4.** *Let $f$ be as in Theorem 4.1. Then the forward initial value problem (4.2) with $c > 0$ has a unique positive solution in $(0, s_*)$.*

*Proof.* Let $y = y(t)$, $t \in (0, s_*)$, be a solution of the forward initial value problem (4.2) with $c > 0$, cf. Lemma 4.2. Then

$$y'(t) = p'\big[c\,\big(y^+(t)\big)^{\frac{1}{p}} - f(t)\big] \ge 0, \quad t \in (0, s_*)$$

and therefore

$$y(t) = y(0) + \int_0^t y'(\sigma)\,\mathrm{d}\sigma \ge 0, \quad t \in (0, s_*).$$

Assume that there are two positive solutions $y_1 = y_1(t)$, $y_2 = y_2(t)$, $t \in (0, s_*)$ of (4.2). Then $z_1 = (y_1)^{1/p'} > 0$, $z_2 = (y_2)^{1/p'} > 0$ solve the forward initial value problem

$$\begin{gathered} z_i'(t) = c - \frac{f(t)}{(z_i(t))^{\frac{1}{p-1}}} \quad \text{for a.e. } t \in (0, s_*), \\ z_i(0) = 0 \end{gathered}$$

for $i = 1, 2$. It then follows that

$$\begin{gathered} (z_1(t) - z_2(t))' = -f(t)\Big(\frac{1}{(z_1(t))^{\frac{1}{p-1}}} - \frac{1}{(z_2(t))^{\frac{1}{p-1}}}\Big), \\ (z_1(t) - z_2(t))^+\,(z_1(t) - z_2(t))' = -f(t)\Big(\frac{1}{(z_1(t))^{\frac{1}{p-1}}} - \frac{1}{(z_2(t))^{\frac{1}{p-1}}}\Big)\,(z_1(t) - z_2(t))^+ \end{gathered}$$

for a.e. $t \in (0, s_*)$. Since $f(t) \le 0$, $t \in (0, s_*)$, it follows from here that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big[\,(z_1(t) - z_2(t))^+\,\big]^2 \le 0, \quad \text{a.e. in } (0, s_*). \tag{5.15}$$

But $z_1(0) = z_2(0) = 0$ and (5.15) imply $z_1(t) \le z_2(t)$. Similarly, we prove that $z_2(t) \le z_1(t)$. Therefore, $y_1(t) = y_2(t)$ for $t \in (0, s_*)$. $\square$

**Remark 5.5.** It follows from Lemma 5.4 that the restriction of the unique positive solution $y_{c_*} = y_{c_*}(t)$, $t \in [0,1]$, of the boundary value problem (3.7), (3.8) to the interval $(0, s_*)$ is also the unique solution of the forward initial value problem (4.2) with $c = c_*$ on $(0, s_*)$.

With the uniqueness result from Lemma 5.4 in hand, we can use the following comparison argument which is our tool for the asymptotic analysis near 0.

**Lemma 5.6.** *Let $f \in L^1(0,1)$ be as in Theorem 4.1, $0 < \theta < s_*$, $\varphi = \varphi(t)$, $\psi = \psi(t)$, $t \in [0,\theta]$ satisfy $\varphi(0) = \psi(0) = 0$, $\varphi'(t) \le h(t, \varphi(t), c_*)$, $\psi'(t) \ge h(t, \psi(t), c_*)$ for a.e. $t \in [0,\theta]$, and let $y_{c_*} = y_{c_*}(t)$, $t \in [0,1]$, be the unique solution of* (3.7), (3.8)*. Then*

$$\varphi(t) \le y_{c_*}(t) \le \psi(t), \quad t \in [0,\theta].$$

*Proof.* The proof follows directly from [14, §10.XXII] combined with the uniqueness result in Lemma 5.4 and Remark 5.5. □

The assumptions on $d$ and $g$ imply that for $\theta$ such that $0 < \theta < \min\{s_*, s_1\}$ the function $f = f(t)$ is continuous in $(0,\theta)$ and $f(t) \sim -t^{\alpha + \frac{\beta}{p-1}}$ is equivalent to

$$f(t) = -\eta(t)(1-t)^{\alpha + \frac{\beta}{p-1}}, \quad t \in (0,\theta),$$

where $\eta = \eta(t)$ is a continuous function in $(0,\theta)$, $\lim_{t\to 0+} \eta(t) \in (0,+\infty)$.

In what follows we discuss different cases with respect to parameters $\alpha$, $\beta$ and $p$.

A. Let $\alpha + \frac{\beta}{p-1} \le \frac{1}{p-1}$. For $\kappa > 0$ we set $y_\kappa(t) = \kappa t^{\alpha + \frac{\beta}{p-1} + 1}$, $t \in [0,\theta]$. Then

$$\begin{aligned}
&y_\kappa' - p'[c_* (y_\kappa)^{\frac{1}{p}} - f(t)] \\
&= \kappa\Big(\alpha + \frac{\beta}{p-1} + 1\Big) t^{\alpha + \frac{\beta}{p-1}} - p'\big[c_* \kappa^{\frac{1}{p}} t^{\frac{\alpha + \frac{\beta}{p-1} + 1}{p}} + \eta(t) t^{\alpha + \frac{\beta}{p-1}}\big] \\
&= t^{\alpha + \frac{\beta}{p-1}} \big[\kappa\Big(\alpha + \frac{\beta}{p-1} + 1\Big) - p'\eta(t)\big] - t^{\frac{\alpha + \frac{\beta}{p-1} + 1}{p}} p' c_* \kappa^{\frac{1}{p}},
\end{aligned} \tag{5.16}$$

for a.e. $t \in [0,\theta]$. The assumption $\alpha + \frac{\beta}{p-1} \le \frac{1}{p-1}$ implies

$$\alpha + \frac{\beta}{p-1} \le \frac{\alpha + \frac{\beta}{p-1} + 1}{p}$$

and therefore the power $t^{\alpha + \frac{\beta}{p-1}}$ dominates the power $t^{\frac{\alpha + \frac{\beta}{p-1} + 1}{p}}$ near 0.

- A1. There exists $\underline{\kappa} \ll 1$ so small that $y_{\underline{\kappa}}'(t) \le p'[c_*(y_{\underline{\kappa}}(t)^{\frac{1}{p}}) - f(t)]$ for a.e. $t \in [0,\theta]$.
- A2. There exists $\bar{\kappa} \gg 1$ so large that $y_{\bar{\kappa}}'(t) \ge p'[c_*(y_{\bar{\kappa}}(t)^{\frac{1}{p}}) - f(t)]$ for a.e. $t \in [0,\theta]$.

It follows from Lemma 5.6 that solution $y_{c_*} = y_{c_*}(t)$ of the BVP (3.7), (3.8) must satisfy

$$y_{\underline{\kappa}}(t) \le y_{c_*}(t) \le y_{\bar{\kappa}}(t), \quad t \in [0,\theta].$$

**Case A1.** Let $\frac{\alpha - \beta + 1}{p} < 1$. Then there exists $c_1 > 0$ such that

$$z_1 = \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \,\mathrm{d}t \le \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{\underline{\kappa}}(t))^{\frac{1}{p}}} \,\mathrm{d}t$$

$$\le c_1 \int_0^{s_*} \frac{t^{\frac{\beta}{p-1}}}{t^{\frac{\alpha+\frac{\beta}{p-1}+1}{p}}}\,\mathrm{d}t = c_1 \int_0^{s_*} \frac{\mathrm{d}t}{t^{\frac{\alpha-\beta+1}{p}}} < +\infty.$$

**Case A2.** Let $\frac{\alpha-\beta+1}{p} \ge 1$. Then there exists $c_2 > 0$ such that

$$\begin{aligned} z_1 &= \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t \ge \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{\bar\kappa}(t))^{\frac{1}{p}}}\,\mathrm{d}t \\ &\ge c_2 \int_0^{s_*} \frac{t^{\frac{\beta}{p-1}}}{t^{\frac{\alpha+\frac{\beta}{p-1}+1}{p}}}\,\mathrm{d}t = c_2 \int_0^{s_*} \frac{\mathrm{d}t}{t^{\frac{\alpha-\beta+1}{p}}} = +\infty. \end{aligned}$$

We can summarize these two cases as follows.

**Theorem 5.7.** *Let us assume $\alpha > 0$,*

$$-1 < \alpha + \frac{\beta}{p-1} \le \frac{1}{p-1}, \tag{5.17}$$

$$\frac{\alpha-\beta+1}{p} < 1. \tag{5.18}$$

*Then $z_1 < +\infty$. Let us assume $\alpha > 0$, (5.17) and*

$$\frac{\alpha-\beta+1}{p} \ge 1. \tag{5.19}$$

*Then $z_1 = +\infty$.*

B. Let $\alpha + \frac{\beta}{p-1} > \frac{1}{p-1}$. For $\kappa > 0$ we set $y_\kappa(t) = \kappa t^{p'}$, $t \in [0,\theta]$. Then

$$\begin{aligned} y'_\kappa - p'[c_*\,(y_\kappa)^{\frac{1}{p}} - f(t)] &= \kappa p' t^{p'-1} - p'[c_* \kappa^{\frac{1}{p}} t^{\frac{p'}{p}} + \eta(t) t^{\alpha+\frac{\beta}{p-1}}] \\ &= \big(\kappa p' - p' c_* \kappa^{\frac{1}{p}}\big) t^{\frac{1}{p-1}} - p'\eta(t) t^{\alpha+\frac{\beta}{p-1}}, \end{aligned} \tag{5.20}$$

for a.e. $t \in [0,\theta]$. The assumption $\alpha + \frac{\beta}{p-1} > \frac{1}{p-1}$ implies that the power $t^{\frac{1}{p-1}}$ dominates $t^{\alpha+\frac{\beta}{p-1}}$ near 0.

- B1. There exists $\underline{\kappa} \ll 1$ so small that $y'_{\underline{\kappa}}(t) \le p'[c_*(y_{\underline{\kappa}}(t)^{\frac{1}{p}}) - f(t)]$ for a.e. $t \in [0,\theta]$.
- B2. There exists $\bar\kappa \gg 1$ so large that $y'_{\bar\kappa}(t) \ge p'[c_*(y_{\bar\kappa}(t)^{\frac{1}{p}}) - f(t)]$ for a.e. $t \in [0,\theta]$.

From Lemma 5.6 we conclude that solution $y_{c_*} = y_{c_*}(t)$ of the BVP (3.7), (3.8) must satisfy

$$y_{\underline{\kappa}}(t) \le y_{c_*}(t) \le y_{\bar\kappa}(t), \quad t \in [0,\theta].$$

**Case B1.** Let $\beta > p-2$. Then there exists $c_3 > 0$ such that

$$z_1 = \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t \le \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{\underline{\kappa}}(t))^{\frac{1}{p}}}\,\mathrm{d}t \le c_3 \int_0^{s_*} \frac{t^{\frac{\beta}{p-1}}}{t^{\frac{p'}{p}}}\,\mathrm{d}t = c_3 \int_0^{s_*} \frac{\mathrm{d}t}{t^{\frac{1-\beta}{p-1}}} < +\infty.$$

**Case B2.** Let $\beta \le p-2$. Then there exists $c_4 > 0$ such that

$$z_1 = \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t \ge \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{\bar\kappa}(t))^{\frac{1}{p}}}\,\mathrm{d}t \ge c_4 \int_0^{s_*} \frac{t^{\frac{\beta}{p-1}}}{t^{\frac{p'}{p}}}\,\mathrm{d}t = c_4 \int_0^{s_*} \frac{\mathrm{d}t}{t^{\frac{1-\beta}{p-1}}} = +\infty.$$

We summarize these two cases as follows.

**Theorem 5.8.** *Let us assume* $\alpha > 0$,

$$\alpha + \frac{\beta}{p-1} > \frac{1}{p-1}, \tag{5.21}$$

$$\beta > 2 - p. \tag{5.22}$$

*Then* $z_1 < +\infty$. *Let us assume* $\alpha > 0$, (5.21) *and*

$$\beta \leq 2 - p. \tag{5.23}$$

*Then* $z_1 = +\infty$.

**Remark 5.9.** To visualize conditions (5.17)–(5.19) and (5.21)–(5.23), we introduce the sets:

$$\mathcal{M}_0^1 := \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, -1 < \alpha + \frac{\beta}{p-1} \leq \frac{1}{p-1}, \alpha - \beta + 1 < p\},$$

$$\mathcal{M}_0^2 := \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, -1 < \alpha + \frac{\beta}{p-1} \leq \frac{1}{p-1}, \alpha - \beta + 1 \geq p\},$$

$$\mathcal{M}_0^3 := \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, \alpha + \frac{\beta}{p-1} > \frac{1}{p-1}, \beta > 2 - p\},$$

$$\mathcal{M}_0^4 := \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, \alpha + \frac{\beta}{p-1} > \frac{1}{p-1}, \beta \leq 2 - p\}.$$

Then $z_1 < +\infty$ if and only if $(\alpha,\beta) \in \mathcal{M}_0^1 \cup \mathcal{M}_0^3$ and $z_1 = +\infty$ if and only if $(\alpha,\beta) \in \mathcal{M}_0^2 \cup \mathcal{M}_0^4$. The reader is invited to see Figure 2 for geometric interpretation and compare the sets $\mathcal{M}_0^1$, $\mathcal{M}_0^2$, $\mathcal{M}_0^3$, $\mathcal{M}_0^4$ and $\mathcal{M}_1^1$, $\mathcal{M}_1^2$, $\mathcal{M}_1^3$, $\mathcal{M}_1^4$.

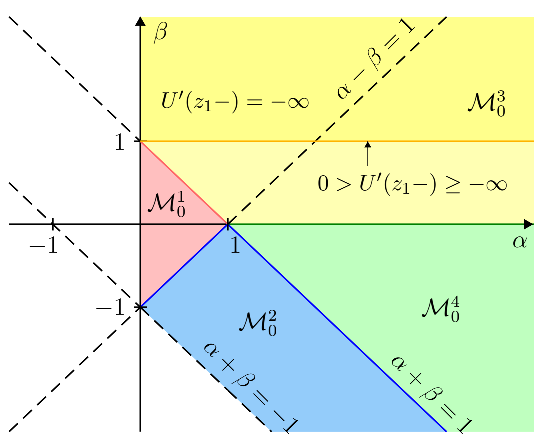


FIGURE 2. Visualization of the sets $\mathcal{M}_0^1$, $\mathcal{M}_0^2$, $\mathcal{M}_0^3$ and $\mathcal{M}_0^4$ for $p = 2$

**Remark 5.10.** Let us assume $z_0 > -\infty$, i.e., $(\gamma,\delta) \in \mathcal{M}_1^1 \cup \mathcal{M}_1^3$. Then $U'(z_0-) = 0$ and it follows from Definition 2.1 that $U'(z_0+)$ exists finite or infinite, see Remark 2.4. Since $U$ is a monotone decreasing function, we have $-\infty \leq U'(z_0+) \leq 0$. If $z_1 < +\infty$, i.e., $(\alpha,\beta) \in \mathcal{M}_0^1 \cup \mathcal{M}_0^3$ then by similar reasons $U'(z_1+) = 0$ and $-\infty \leq U'(z_1-) \leq 0$.

In the case $(\alpha,\beta)\in\mathcal{M}_0^3$ our one-sided estimates on $z_1$ allow for more precise information about the smoothness of $U$ at $z_1$. Indeed, in this case we have

$$\begin{aligned} 0 \ge z'(0+) &= \lim_{U\to 0+} z'(U) \\ &= -\lim_{U\to 0+}\frac{\big(d(U)\big)^{\frac{1}{p-1}}}{\big(y_{c_*}(U)\big)^{\frac{1}{p}}} \ge -\lim_{U\to 0+}\frac{\big(d(U)\big)^{\frac{1}{p-1}}}{\big(y_{\underline{c}}(U)\big)^{\frac{1}{p}}} \\ &\ge -c_3\lim_{U\to 0+} U^{\frac{\beta-1}{p-1}}. \end{aligned} \tag{5.24}$$

We distinguish the following two cases:

1. If $\beta>1$ then from (5.24) we obtain $z'(0+)=0$ and therefore $U'(z_1-)=-\infty$.
2. If $\beta=1$ then we deduce from (5.24) that $0\ge z'(0+)\ge -c_3$ and therefore $0>U'(z_1-)\ge-\infty$.

In either case the traveling wave profile $U$ is "sharp" in the sense that $U'$ has a jump at $z_1$ (finite or infinite).

In other cases $(\alpha,\beta)\in\mathcal{M}_0^1$ and $(\gamma,\delta)\in\mathcal{M}_1^1\cup\mathcal{M}_1^3$ our one-sided estimates on $z_1$ and $z_0$, respectively, do not provide analogous information as above. This is a big difference between the traveling wave and standing wave, see [7].

**Acknowledgements.** Michaela Zahradníková was supported by the project SGS-2019-010 of the University of West Bohemia in Pilsen.

## References

[1] D. G. Aronson, H. F. Weinberger; Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. In: *Partial Differential Equations and Related Topics,* Springer Berlin Heidelberg, 5–49, 1975.

[2] D. G. Aronson, H. F. Weinberger; Multidimensional nonlinear diffusion arising in population genetics, *Advances in Mathematics* **30** (1978), 33–76.

[3] E. A. Coddington, N. Levinson; *Theory of ordinary differential equations*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[4] H. Cohen; Nonlinear diffusion problems. In: *Studies in applied mathematics, MAA Studies in Math.*, 7: 27–64, 1971.

[5] P. Drábek, P. Takáč; New patterns of travelling waves in the generalized Fisher-Kolmogorov equation, *Nonlinear Differ. Equ. Appl.* **23** (2016), no. 7, 1–19.

[6] P. Drábek, P. Takáč; Convergence to travelling waves in Fisher's population genetics model with a non-Lipschitzian reaction term, *J. Math. Biol.* **75** (2017), no. 4, 929–972.

[7] P. Drábek, M. Zahradníková; Bistable equation with discontinuous density dependent diffusion with degenerations and singularities, *Electron. J. Qual. Theory Differ. Equ.* **2021** (2021), no. 61, 1–16.

[8] R. Enguiça, A. Gavioli, L. Sanchez; A class of singular first order differential equations with applications in reaction-diffusion, *Discrete Contin. Dyn. Syst.* **33** (2013), 173–191.

[9] P. C. Fife, J. B. McLeod; The approach of solutions of nonlinear diffusion equations to travelling front solutions, *Arch. Rational Mech. Anal.* **65** (1977), no. 4, 335–361.

[10] I. M. Gel'fand; Some problems in the theory of quasi-linear equations, *Uspehi Mat. Nauk* **14** (1959), no. 2 (86), 87–158.

[11] J. Nagumo, S. Yoshizawa, S. Arimoto; Bistable transmission lines, *IEEE Transactions on Circuit Theory* **12** (1965), no. 3, 400–412.

[12] D. Strier, D. Zanette, H. S. Wio; Wave fronts in a bistable reaction-diffusion system with density-dependent diffusivity, *Physica A: Statistical Mechanics and its Applications* **226** (1996), no. 3, 310–323.

[13] A. I. Volpert, V. A. Volpert, V. A. Volpert; *Traveling wave solutions of parabolic systems*, Translations of Mathematical Monographs, Vol. 140, American Mathematical Society, Providence, RI, 1994.

[14] W. Walter; *Ordinary differential equations*, Graduate Texts in Mathematics, Vol. 182, Springer-Verlag, New York, 1998.

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