COMPUTATIONAL MECHANICS 2021
36th conference with international participation
Srní
November 8 - 10, 2021

Comparison of analytical methods and FE models to calculate the stiffness of a wound composite beam

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1. Introduction

The aim of this work is the search for a new analytical method of calculating the stiffness of the wound composite beams with the circular cross-section. The FE models and several analytical calculations were performed for chosen geometry of the composite beam and all results are compared to experimental data. The experiment of three-point bending on fibre-reinforced composite beams with four different composite lay-ups was done and the stiffness was calculated. The comparison of a new semi-analytical method with the known analytical methods of stiffness calculation and FE models are introduced in this paper.

Composite beams are very variable not only in their shape or cross-section, but also in the layup of the composite material from which they are made. This creates a number of variables that we must take into account when calculating their deformation. It is well known that the available methods for calculating the deformation of composite materials do not provide relevant results for all possible types and shapes of composite beams. It turns out that these methods differ in the results for the same case of a composite beam or are too complex for the initial design of the part. Also, the results of these methods differ from a possible experiment. Analytical, semi-analytical, and numerical methods are known for calculating the deformation of composite beams. The comparison of Timoshenko's and Bernoulli's method of bending calculation, the method of calculation using ABD matrices, the use of statistical mechanics, and the numerical method of finite elements were chosen as the basis for this work. All these methods were applied to an embedded composite beam with an inter-circular cross-section. The aim of this work is to find out in which specific cases the mentioned methods of calculating the effective stiffness of composite beams are valid for the general composition of the composite material.

2. Analytical methods to reach the equivalent stiffness modulus of the composite beam

2.1 The stiffness matrix and the compliance matrix

The Hooke's law contains the stiffness matrix $\boldsymbol{S}$.

$$\boldsymbol{\sigma} = \boldsymbol{S} \cdot \boldsymbol{\varepsilon} \quad , \tag{1}$$

The modulus of elasticity E is expressed for each layer separately by means of the stiffness matrix in the main coordinate system of the composite material $O(L, T, T')$. An orthotropic material is considered. [1] To express the equivalent modulus of elasticity in the main coordinate system of the whole beam $O(x, y, z)$, it is possible to use the stiffness matrix $\boldsymbol{S}_{xy}$ or an inverse matrix the compliance matrix $\boldsymbol{C}_{xy}$. The stiffness matrix $\boldsymbol{S}_{xy}$ is expressed by the following transformation (3) to the coordinate system $O(x, y, z)$ and the compliance matrix $\boldsymbol{C_{xy}}$ is inverse to it (2).

$$\boldsymbol{C'_{xy}} = \boldsymbol{S'_{xy}}^{-1}, \tag{2}$$

$$\boldsymbol{S'_{xy}} = \boldsymbol{T_{xy}} \cdot \boldsymbol{S} \cdot \boldsymbol{T'_{xy}}. \tag{3}$$

The modulus of elasticity in the direction of the beam axis E_x can be obtained from the compliance matrix $\boldsymbol{C'_{xy}}$, and also from the stiffness matrix $\boldsymbol{S'_{xy}}$. The element S'_{11} is used from stiffness matrix $\boldsymbol{S'_{xy}}$ and element C'_{11} is used from compliance matrix $\boldsymbol{C'_{xy}}$. The usage of the stiffness matrix $\boldsymbol{S'_{xy}}$ represents the upper estimate of the equivalent stiffness, the use of the compliance matrix $\boldsymbol{C'_{xy}}$ the lower estimate of the stiffness of the composite beam. In the results section, their arithmetic mean is also used.

2.2 Calculation of equivalent elasticity modulus E_{eq} by Classical Laminate Theory

To calculate the deflection of a composite beam with a circular cross-section using method from [1] the following equation is used

$$\begin{bmatrix} \boldsymbol{N} \\ \cdots \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \vdots & \boldsymbol{B} \\ \cdots & \vdots & \cdots \\ \boldsymbol{B} & \vdots & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon^{\circ}{}_m} \\ \cdots \\ \boldsymbol{k} \end{bmatrix}, \tag{4}$$

The force loading of the beam can be expressed using the elements of the matrix $\boldsymbol{A}$ [2]. The stress of a composite material using Hooke's law is expressed. To obtain the stress relationship, it is necessary to divide this expression by the total thickness of the composite material t.

$$\sigma_1 = \frac{N_1}{t} = \frac{1}{t}\left(A_{11} - \begin{bmatrix} A_{12} & A_{16} \end{bmatrix} \cdot \begin{bmatrix} A_{22} & A_{26} \\ A_{62} & A_{66} \end{bmatrix}^{-1} \cdot \begin{bmatrix} A_{21} \\ A_{61} \end{bmatrix}\right) \cdot \boldsymbol{\varepsilon^{\circ}{}_1} \ . \tag{5}$$

The equivalent modulus of elasticity is expressed by the following relation.

$$E_{eq} = \left(A_{11} - \begin{bmatrix} A_{12} & A_{13} \end{bmatrix} \cdot \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{-1} \cdot \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}\right) \cdot \frac{1}{t} \ , \tag{6}$$

2.3 Statistic method for bending stiffness calculation

This method is based on an analogy to statistical mechanics. According to statistical mechanics, the (probable) energy of the system $\langle E \rangle$ is equal to the negatively taken derivative of the logarithm of the partition function Z, according to a parameter called thermodynamic beta (β)

$$\langle E \rangle = -\frac{\partial \ lnZ}{\partial \ \beta} \ , \tag{7}$$

where the partition function can be obtained by a relation

$$Z = \sum_i e^{-\beta E_i} \ . \tag{8}$$

The energy of the system is obtained by following formulas.

$$\langle E \rangle = -\frac{\partial kZ}{\beta} \qquad \beta = \frac{1}{k_B T} \tag{9}$$

In this analogy leading to the determination of bending (and similarly shear) stiffness, we consider a micro-state as a set of stiffnesses of individual layers calculated according to any mutually different method. The total bending stiffness, *EJi*, corresponding to a given micro-state is given by the sum of the stiffnesses of all layers. Each micro-state has an energy equal to the total potential energy of the beam. For example, when considering a symmetrical three-point bending

$$E_i = \Pi = U - F \cdot u = -\frac{1}{2} F \cdot u = -\frac{1}{2} F \cdot \frac{Fl^3}{48(EJ)_i} \tag{10}$$

The partition function Z expressed in this way is subjected to a numerical derivative according to equation (9) and we obtain the (assumed, average) energy of the beam $\langle E \rangle$. Backwards, from the relation for the total potential energy of the beam (10) we calculate the effective bending stiffness of the beam

$$(EJ)_{effective} = -\frac{F^2\, l^3}{96\langle E \rangle} \tag{11}$$

2.4 A new semi-analytical method

A new semi-analytical approach is based also on the Classical Laminate Theory and tries to calculate the equivalent stiffness of the beam with the combination of the tensile and bending stiffness matrix elements. The assumption for this theory is that the geometry of the composite beam with circular cross-section combines the tensile and bending loading of the material of composite beam. The combination of the elements of matrix ***A*** and matrix ***D*** (4) is used in a superposition of the stiffnesses.

$$(EJ)_{equivalent} = (EJ)_{A_{11}} + (EJ)_{D_{11}} \tag{12}$$

3. Experiment

Composite wound tubes with an inner diameter of 26 mm and 50 mm were selected as specimens. The tests were performed for four composite layups and two lengths of the composite beams. Three types of fibers were used for all composite layups. Three-point bending tests were performed on a FPZ 100/1 machine. Supports with a span of 200 mm and 400 mm for tubes with a diameter of 26 mm were used for the tests. The spans of the supports for tubes with an inner diameter of 50 mm were 400 mm and 600 mm. The beams were loaded with force through the strap. The deflection was measured with an extensometer and strain with a strain gauge. The sensors were placed in the centre of the beam under the load member.

Groups of six pieces from each combination of fibers, composite layup, and support span were tested. The average value of the equivalent stiffness EJ_{eq} was evaluated. All sample types were modelled by available FEM methods and the equivalent stiffness was calculated. All the mentioned analytical methods were also used to calculate the equivalent stiffness. A comparison of these values is shown in the following section.

4. Results

The results show the deviations of the individual methods from the experimental data in percentages. The equivalent stiffness of EJ_{eq} beams is compared. The results of two lay-ups Typical [90 °, 0 °, ± 30 °] in Fig. 1 and Diagonal [90 °, ± 45 °] in Fig. 2 of a composite beam with diameters of 26 mm and 50 mm made in several different fibre types are given here. The average stiffness (EJ_mean in the figures) obtained from the matrix S' and C' shows a good agreement with the experiment, but this method predicts higher stiffness compared with experimental data. The Statistical Mechanics has good agreement with the experimental data for the diameter of 50 mm in both lay-ups. It shows that this method fits good in cases of thin-walled beams. The new semi-analytical approach shows the constant deviations less than 25% from experimental data in all cases of specimens. These results are on the safety side of the calculation in most cases compared to the experimental data.

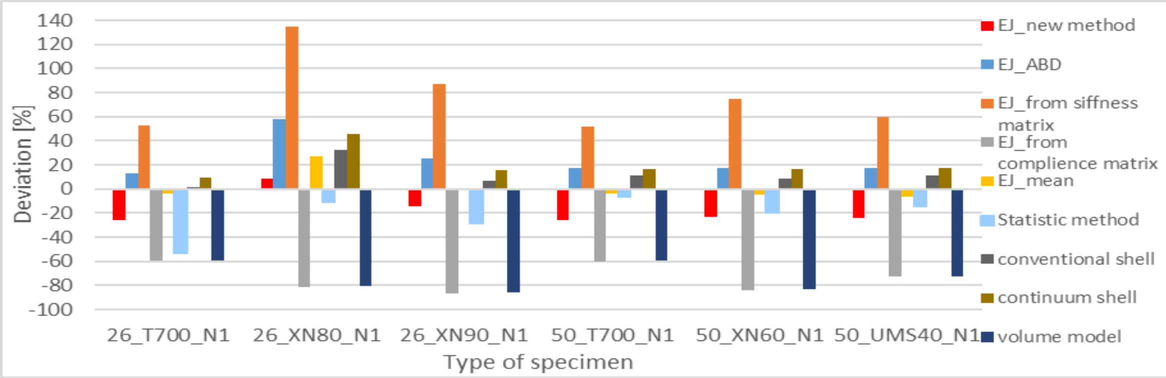

Fig. 1. The deviation from the experiment of equivalent stiffness for beams with Typical layup [90°, 0°, ±30°]

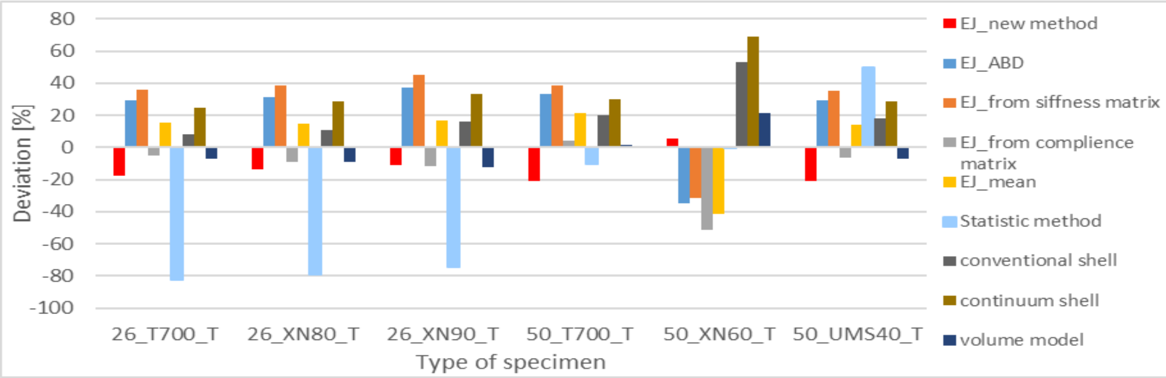

Fig. 2. The deviation from the experiment of equivalent stiffness for beams with Diagonal layup [90°, ±45°]

5. Conclusion

None of the methods described above gives sufficiently accurate predictions of the stiffness of experimentally tested beams. There is still a noticeable problem where the results of different methods approach different results of the beam equivalent stiffness.

From the performed comparison, methods with the appropriate results are selected. The new semi-analytical approach reached a good agreement with experimental data in all composite lay-ups. The method based on statistic mechanics reached a good conformity in cases with diagonal lay-ups. A method based on the mean of an upper and lower estimate of the stiffness of the composite lay-up seems to be almost equally suitable, but with the deviation that predicts the greater stiffness than the experiment. In terms of computational complexity, the proposed approaches are less demanding than the FE method and are therefore suitable for fast usage for the preliminary design. The numerical optimization of these approaches is also possible.

Acknowledgement

The work has been supported by the research project of Students grant agency CTU SGS21/151/OHK2/3T/12.

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