University of West Bohemia
Faculty of Applied Sciences
Department of Mathematics

UNIVERSITY OF WEST BOHEMIA

FACULTY OF APPLIED SCIENCES UNIVERSITY OF WEST BOHEMIA

BACHELOR THESIS

Multi-Parameter Nonlinear Boundary Value Problems

Pilsen 2021

Martin Kudláč

Prohlášení

Prohlašuji, že jsem bakalářskou práci vypracoval pod vedením vedoucí bakalářské práce samostatně za použití uvedených pramenů a literatury.

V Plzni, dne 17.5.2021

....

Poděkování

Velký vděk patří vedoucí mé práce paní docentce Holubové nejen za odborný dohled při psaní práce a ochotu mě v matematice posunout dále, ale i za rady, připomínky, milý přístup při konzultacích a velké množství času, který vedení mé práce obětovala.

Abstrakt

Práce formuluje víceparametrickou okrajovou úlohu s jedním parametrem v rovnici a druhým parametrem v okrajové podmínce s cílem vyšetřit, pro které dvojice parametrů získáme netriviální řešení. Lineární verze úlohy je vyřešena zcela zvlášť a její výsledky se následně promítají do nelineárních úloh. U nelineárních úloh nás zajímají především vlastnosti řešení a jejich existence pro různé dvojice parametrů. Práce a výsledky v ní použité jsou ilustrovány pomocí diagramů, vytvořených v softwaru Matlab.

Klíčová slova: okrajová úloha, parametr, vlastní čísla, Sturm-Liouvillova úloha, Steklovova úloha na vlastní čísla, bifurkační diagram

Abstract

This work formulates a multi-parametric boundary value problem with one parameter included in an equation and the other parameter in a boundary condition. The task is to examine the pairs of the parameter which the problem has a non-trivial solution for. A linear version of the problem is solved separately and the obtained results are furthermore used in a nonlinear versions of the problem. For the nonlinear problems, we are interested especially in properties and existence of the solution for different pairs of the parameters. The work and the included results are illustrated with diagrams created in Matlab.

Keywords: boundary problem, parameter, eigenvalues, Sturm-Liouville problem, Steklov eigenvalue problem, bifurcation diagram

Table of Contents

Preface

This work formulates a nonlinear ordinary differential equation

$$u'' = -\lambda g(u),$$

where g is any continous function and $\lambda \in \mathbb{R}$, and specifies the task to finding all the non-trivial solutions satisfying associated boundary conditions.
However, one of the boundary conditions is always supplemented with a real parameter μ yielding a multi-parameter boundary value problem

$$\begin{cases} u'' = -\lambda g(u), \\ u'(1) = \mu u(1), \\ u(0) = 0. \end{cases} \tag{$*$}$$

Including the second parameter μ makes the problem a lot more interesting.
There are several important notes to point out.
Firstly, by adding a parameter to a boundary condition, we create a Steklov eigenvalue problem. This problem, however, is usually interesting only in multi-dimensional spaces. For this reason, the problem includes one additional parameter in the equation itself.
Secondly, if we fix μ as an arbitrary real number, we get a standard Robin's boundary condition.
Thirdly, if $g(u) = u$, the problem simplifies to a linear eigenvalue problem with a parameter in one of the boundary conditions, and, for a particular value of μ, we get a Sturm-Liouville eigenvalue problem to be solved. Multi-parametric forms of Sturm-Liouville BVPs were described in [1].
After solving the problem, we obtain all the pairs $(\lambda, \mu) \in \mathbb{R}^2$ which provide a non-trivial solution of $(*)$. These pairs form so called "eigencurves", and since the problem is linear, we should be able to determine these eigencurves very precisely. This point of view on a multi-parameter boundary problem is analyzed for instance in [2].

Nevertheless, it is impossible to suggest analytic results for any g. For this reason, the work is divided into two chapters.
The first chapter analyzes the linear case of $(*)$ and thoroughly describes the set of all pairs (λ, μ) which $(*)$ has a non-trivial solution for.
The second chapter, unsurprisingly, examines nonlinear cases. In this chapter, numerical experiments become more important to give us ability to visually describe the results. The results are introduced in two separated subchapters. Since we cannot find the solution analytically, the first subchapter shows properties of the solution. Another question, answered in the second subchapter, is which pairs of (λ, μ) provide a solution, or, respectively, which pairs certainly do not provide any solution or solution of the expected properties.
The second chapter also shows a connection of the nonlinear and the linear case of the problem.

This work and the multi-parameter problem have a potential to be extended in several ways. One of them is using a jumping nonlinearity to involve the theory of Fučík spektrum with suspension bridges as a real-life application, as it was introduced in [5].

Chapter 1

Linear problem

Let us consider a homogeneous second order differential equation

$$-u'' = \lambda u, \tag{1.1}$$

where λ is a real parameter, together with boundary conditions

$$\begin{cases} u'(1) = \mu u(1), \\ u(0) = 0, \end{cases} \tag{1.2}$$

with a real parameter μ. The task is to find all pairs of λ and μ which the problem has a non-trivial solution for.

Theorem 1.1 (Necessary condition No. 1 for existence of non-trivial solution)**.** *Let there exist a non-trivial solution of the boundary value problem (1.1), (1.2). Then* $(\lambda, \mu) \notin (-\infty, 0] \times (-\infty, 1] \setminus \{(0, 1)\}$.

Proof. The proof will be carried out by contradiction. Let $\lambda \leq 0$, $\mu \leq 0$ and u be a non-trivial solution of (1.1), (1.2). Since u is non-trivial and from (1.2) we know that $u(0) = 0$, u' is obviously a non-trivial function on the interval $(0, 1)$.
Multiplying (1.1) by u and integrating both sides over the interval $(0, t)$ with $t \in (0, 1]$, we get

$$\int_0^t -u''\, u \,\mathrm{d}x = \int_0^t \lambda u^2 \,\mathrm{d}x.$$

After integrating by parts we obtain

$$-\left(u(t)u'(t) - u(0)u'(0) - \int_0^t (u')^2 \,\mathrm{d}x \right) = \lambda \int_0^t u^2 \,\mathrm{d}x.$$

Applying (1.2) and using a trivial operation yields

$$\int_0^t (u')^2 \mathrm{d}x - \lambda \int_0^t u^2 \mathrm{d}x = u(t)u'(t).$$

Since $\lambda \leq 0$ and an integral of non-negative function is positive, the left-hand side of the above equation is positive. Thus

$$u(t)\, u'(t) > 0. \tag{1.3}$$

Using (1.3) for $t \in (0,1]$ we know:

$$u(t) \neq 0, \;\; u'(t) \neq 0, \tag{1.4}$$
$$\operatorname{sgn} u(t) = \operatorname{sgn} u'(t). \tag{1.5}$$

Relations (1.1), (1.5) for $t \in (0,1]$ imply:

$$\text{for } \lambda < 0: \quad \operatorname{sgn} u(t) = \operatorname{sgn} u'(t) = \operatorname{sgn} u''(t), \tag{1.6}$$
$$\text{for } \lambda = 0: \quad u''(t) = 0. \tag{1.7}$$

For $\lambda \leq 0$ the following holds:

$$\text{for } u'(0) > 0: \quad u \text{ is positive and strictly increasing on } (0,1), \tag{1.8}$$
$$\text{for } u'(0) < 0: \quad u \text{ is negative and strictly decreasing on } (0,1). \tag{1.9}$$

Let us consider $u'(0) > 0$.
First, let $\mu \leq 0$. Applying (1.2) yields

$$u(1)\, u'(1) \leq 0,$$

which gives us a contradiction with (1.3).
Next, let $\mu \in (0,1]$. From (1.2), (1.8) we have:

$$0 < u'(1) \leq u(1), \tag{1.10}$$

with the equality only for $\mu = 1$.
Now we apply Lagrange's finite-increment theorem to u on $[0,1]$ (see [7, p. 216, Th. 1]). Thus there exists $\xi \in (0,1)$ such that

$$u'(\xi) = u(1). \tag{1.11}$$

If $\lambda < 0$, then u is convex on $(0,1)$, thus u' is strictly increasing on $(0,1)$. Applying (1.11) we get

$$u'(1) > u'(\xi) = u(1)$$

as a contradiction with (1.10).
For $\lambda = 0$, (1.7), (1.11) yields

$$u'(1) = u'(\xi) = u(1). \tag{1.12}$$

Thus, applying (1.10) we get a contradiction for $\mu \in (0,1)$ and $(\lambda, \mu) = (0,1)$ is the only pair admissible.
For $u'(0) < 0$, the proof will be carried out analogically. □

Theorem 1.2 (Necessary condition No. 2 for existence of non-trivial solution)**.** *Let there exist a non-trivial solution of the boundary value problem (1.1), (1.2). Then $\mu^2 > -\lambda$, i.e.,*

$$(\lambda, \mu) \notin \{(\lambda, \mu) \in \mathbb{R}^2, \mu^2 \leq -\lambda\}.$$

Proof. For (1.1), (1.2), let us have a non-trivial solution u. Multiplying the equation (1.1) by u' and integrating both sides over the interval $(0,1)$, we obtain

$$-\int_0^1 u''\, u' \,\mathrm{d}x = \lambda \int_0^1 u\, u' \,\mathrm{d}x.$$

We know that

$$\begin{aligned} \frac{1}{2}\left(u^2\right)' &= u\,u' && (1.13)\\ \frac{1}{2}\left((u')^2\right)' &= u'\,u'' && (1.14)\end{aligned}$$

Using the relations (1.13), (1.14), we get

$$-[u'(1)]^2 + [u'(0)]^2 = \lambda u^2(1) - \lambda u^2(0).$$

Applying boundary conditions (1.2) yields

$$[u'(0)]^2 = (\lambda + \mu^2)\, u^2(1)$$

and, since u is non-trivial and $u'(0) \neq 0$, we have the following condition for u, λ, μ:

$$\lambda + \mu^2 > 0 \quad \wedge \quad u(1) \neq 0.$$

Thus $\mu^2 > -\lambda$. □

Let $C([0,1])$ be the space of continuous functions on the interval $[0,1]$ with the standard norm $\|u\|_C := \max_{x\in[0,1]} |u(x)|$.

Let $L^2(0,1)$ be the space of square integrable functions on the inverval $[0,1]$ with the standard norm $\|u\|_{L^2} := \sqrt{\int_0^1 u^2(x)\mathrm{d}x}$.

Lemma 1.3. *Let u be a non-trivial continuously differentiable function on the interval $[0,1]$ and $u(0) = 0$. Then*

$$\|u\|_{L^2} < \|u\|_C \leq \|u'\|_{L^2}.$$

Proof. Obviously, since u is a non-trivial function and $u(0) = 0$, we know

$$\|u\|_{L^2}^2 = \int_0^1 u^2(x)\mathrm{d}x < \|u\|_C^2,$$

hence

$$\|u\|_{L^2} < \|u\|_C. \tag{1.15}$$

Now let us consider $x_m, x_n \in [0,1]$ such that for all $x \in [0,1]$, $u_m := u(x_m) \geq u(x)$ and $u_n := u(x_n) \leq u(x)$. Then applying Cauchy-Schwartz inequality gives

$$u_m = u(x_m) - u(0) = \int_0^{x_m} u'(x)\mathrm{d}x \leq \sqrt{\int_0^{x_m} [u'(x)]^2\mathrm{d}x}\,\sqrt{\int_0^{x_m} \mathrm{d}x},$$

thus

$$u_m \leq \|u'\|_{L^2}\sqrt{x_m} \leq \|u'\|_{L^2}. \tag{1.16}$$

Since $u(0) = 0$, we know that $u_n \leq 0$. Thus, analogically, we have

$$-u_n \leq \|u'\|_{L^2}\sqrt{x_n} \leq \|u'\|_{L^2}. \tag{1.17}$$

Obviously, $\|u\|_C = \max\{u_m, -u_n\}$ and using relations (1.16), (1.17) yields

$$\|u\|_C \leq \|u'\|_{L^2}. \tag{1.18}$$

□

Lemma 1.3 now allows us to formulate the necessary condition for existence of non-trivial solution of (1.1), (1.2) in a more accurate way.

Theorem 1.4 (Necessary condition No. 3 for existence of non-trivial solution)**.** *Let there exist a non-trivial solution of the boundary value problem (1.1), (1.2). Then:*

$$(\lambda, \mu) \notin \{(\lambda, \mu) \in \mathbb{R}^2;\ \mu \le 1 - \lambda,\ \mu \le 1,\ \lambda \le 1\} \setminus \{(0,1)\}.$$

Proof. Let us multiply the equation (1.1) by u and integrate both sides over $(0,1)$. Integration the left-hand side by parts and using (1.2) yields

$$-\mu u^2(1) + \|u'\|^2_{L^2} = \lambda \|u\|^2_{L^2}. \tag{1.19}$$

Now, let us distinguish the following four cases:

1. $\lambda \le 0$, $\mu \le 0$
   Since u, u' are non-trivial functions, we know that the left-hand side of the equation (1.19) is positive and the right-hand side is non-positive, which gives us a contradiction. Thus, $\lambda \le 0$, $\mu \le 0$ is not an admissible case.

2. $\lambda > 0$, $\mu > 0$
   Using Lemma 1.3 for (1.19) yields

   $$-\mu u^2(1) + \|u\|^2_C < \lambda \|u\|^2_C.$$

   We know that

   $$u^2(1) \le \|u\|^2_C, \tag{1.20}$$

   thus

   $$-\mu \|u\|^2_C + \|u\|^2_C < \lambda \|u\|^2_C$$

   and trivially

   $$\mu > 1 - \lambda.$$

3. $\lambda \le 0$, $\mu > 0$
   Applying (1.18) and (1.20) gives $-\mu u^2(1) \ge -\mu \|u'\|^2_{L^2}$. We know that

   $$\lambda \|u\|^2_{L^2} \le 0$$

   with the equality only for $\lambda = 0$. Hence, the equation (1.19) gives

   $$\begin{aligned} -\mu \|u'\|^2_{L^2} + \|u'\|^2_{L^2} &< 0 \text{ for } \lambda < 0, \\ -\mu \|u'\|^2_{L^2} + \|u'\|^2_{L^2} &= 0 \text{ for } \lambda = 0, \end{aligned}$$

   which yields

   $$\begin{aligned} \mu &> 1 \text{ for } \lambda < 0, \\ \mu &= 1 \text{ for } \lambda = 0. \end{aligned}$$

4. $\lambda > 0,\ \mu \leq 0$
   Obviously $-\mu u^2(1) + \|u'\|^2_{L^2} \geq \|u'\|^2_{L^2}$. Hence, using Lemma 1.3 for (1.19) gives
$$\|u'\|^2_{L^2} \leq \|u'\|^2_{L^2} - \mu u^2(1) = \lambda \|u\|_{L^2} < \lambda \|u'\|^2_{L^2}$$
   and thus
$$\lambda > 1.$$

□

Corollary 1.5 (Necessary conditions for existence of non-trivial solution)**.** *Let there exist a non-trivial solution of the boundary value problem (1.1), (1.2). Then necessarily:*
$$(\lambda, \mu) \notin \{(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0,1)\};\ \mu \leq 1 - \lambda,\ \mu \leq 1,\ \lambda \leq 1\} \cup \{(\lambda, \mu) \in \mathbb{R}^2;\ \mu^2 \leq -\lambda\}.$$

Proof. Theorems 1.1 - 1.4 give directly the statement. □

In the Figure 1.1, there are the pairs $(\lambda, \mu) \in \mathbb{R}^2$ which, according to the Corollary 1.5, do not provide a non-trivial solution.

1.1 General solution

The general solution of (1.1) depends on a particular choice of the parameter λ. More specifically, we have to distinguish three cases for positive, negative and zero parameters.

1. **for** $\lambda = 0$**:**
   After substitution and a few trivial operations we get the following equation
$$u'' = 0$$
   and the corresponding characteristic equation
$$\gamma^2 = 0,$$
   with double root $\gamma_{1,2} = 0$. We obtain a fundamental system $\{1, x\}$. The general solution of (1.1) is an arbitrary linear combination of the elements in the fundamental system.

2. **for** $\lambda > 0$**:**
   The characteristic equation corresponding to (1.1) takes the form
$$\gamma^2 = -\lambda$$
   with roots
$$\gamma_{1,2} = \pm i\sqrt{\lambda}.$$
   Using the Euler identity we obtain a fundamental system $\{\cos\sqrt{\lambda}x, \sin\sqrt{\lambda}x\}$. The general solution of (1.1) is an arbitrary linear combination of the elements in the fundamental system.

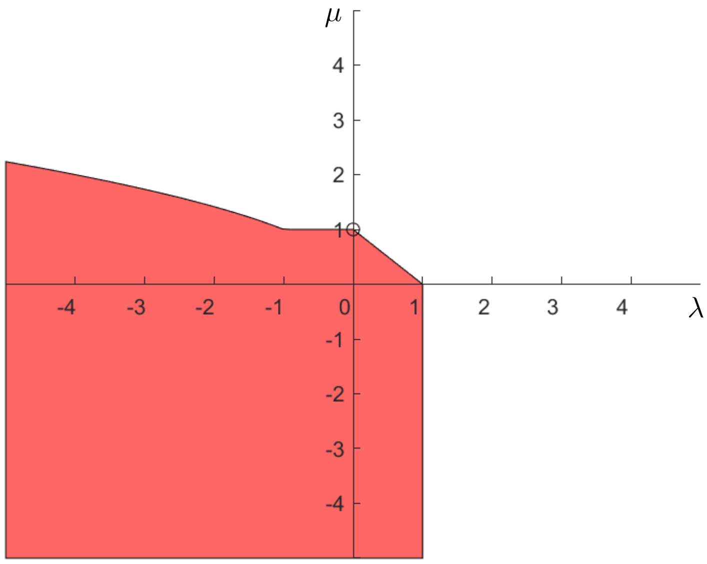


Fig. 1.1: The set of pairs $(\lambda, \mu) \in \mathbb{R}^2$ which certainly do not give a non-trivial solution of the boundary value problem (1.1), (1.2).

3. **for** $\lambda < 0$**:**
   We solve the same characteristic equation as in the previous case. However, applying the condition $\lambda < 0$ we get two real roots

$$\gamma_{1,2} = \pm\sqrt{-\lambda}.$$

   Thus the fundamental system is $\{\cosh\sqrt{-\lambda}x, \sinh\sqrt{-\lambda}x\}$. The general solution of (1.1) is an arbitrary linear combination of the elements in the fundamental system.

To sum up, the general solution of the equation (1.1) can be written as:

$$\begin{aligned}
\text{for } \lambda &= 0 : u = Ax + B;\ \ A, B \in \mathbb{R}, \\
\text{for } \lambda &> 0 : u = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x;\ \ A, B \in \mathbb{R}, \\
\text{for } \lambda &< 0 : u = A\cosh\sqrt{-\lambda}x + B\sinh\sqrt{-\lambda}x;\ \ A, B \in \mathbb{R}.
\end{aligned}$$

1.2 Boundary value problem

We apply the boundary conditions (1.2):

Case $\lambda = 0$**:**
Differentiating the general equation and substituting $x = 1$ we obtain $A = \mu u(1)$. Hence,

$$A = \mu(A + B) = \mu A$$

and thus obviously

$$\mu = 1. \tag{1.21}$$

Case $\lambda > 0$:
The second part of the condition (1.2) yields

$$u(0) = A\cos 0 + B\sin 0 = A = 0. \tag{1.22}$$

Differentiating the function u a substituting to the first part of the boundary conditions we get

$$\mu \sin\sqrt{\lambda} = \sqrt{\lambda}\cos\sqrt{\lambda}. \tag{1.23}$$

Since the functions sine and cosine are linearly independent, the last equation can be rewritten to

$$\mu = \sqrt{\lambda}\operatorname{cotg}\sqrt{\lambda}. \tag{1.24}$$

Case $\lambda < 0$:
The second part of the boundary condition trivially gives

$$A = 0. \tag{1.25}$$

After differentiating of the general solution we get

$$u' = A\sqrt{-\lambda}\sinh\sqrt{-\lambda}x + B\sqrt{-\lambda}\cosh\sqrt{-\lambda}x.$$

Substituting to the first part of the boundary equation (1.2) and using (1.25) we obtain

$$\sqrt{-\lambda}\cosh\sqrt{-\lambda} = \mu\sinh\sqrt{-\lambda}, \tag{1.26}$$

or

$$\mu = \sqrt{-\lambda}\operatorname{cotgh}\sqrt{-\lambda}, \tag{1.27}$$

respectively.

1.2.1 Set of non-trivial solutions

Theorem 1.6. *Let Σ be a set of all the ordered pairs of parameters (λ, μ) which give a non-trivial solution of the boundary value problem (1.1), (1.2). Then*

$$\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3,$$

where Σ_1, Σ_2 and Σ_3 have the following properties:
$\Sigma_1 = \{(0, 1)\}$,
$\Sigma_2 = \left\{\left(\lambda, \sqrt{\lambda}\operatorname{cotg}\sqrt{\lambda}\right) ; \lambda \in \mathbb{R}^+\right\}$,
$\Sigma_3 = \left\{\left(\lambda, \sqrt{-\lambda}\operatorname{cotgh}\sqrt{-\lambda}\right) ; \lambda \in \mathbb{R}^-\right\}$.

Proof. The statement can be proved using relations (1.21), (1.24) and (1.27). □

Having λ given, the Theorem 1.6 gives us μ such that $(\lambda, \mu) \in \Sigma$. However, for a particular μ, the BVP (1.1) becomes a BVP in Sturm-Liouville form and finding corresponding λ to keep $(\lambda, \mu) \in \Sigma$ means to find all the eigenvalues of the Sturm-Liouville BVP. Therefore it would be undoubtedly helpful to find a relation characterizing a dependance of λ on μ.
Unfortunately, the relations of λ and μ given by the Theorem 1.6 are not invertible. Thus we introduce a few Lemmas, Propositions and Theorems describing the properties of Σ for a particular μ.

Lemma 1.7. *Let $\mu \neq 0$ be an arbitrary constant. Let us consider an equation*

$$\operatorname{tgh} x = \frac{x}{\mu}. \tag{1.28}$$

For roots $x \in \mathbb{R}$ of (1.28) the following holds:
If $\mu > 1$, then there are exactly 3 roots x_1, x_2, x_3 of (1.28), where $x_1 < 0$, $x_2 = 0$, $x_3 > 0$.
If $\mu \in (-\infty, 1] \setminus \{0\}$, then there is exactly one root $x = 0$ of (1.28).

Proof. We define a function $G(x) := \frac{x}{\mu} - \operatorname{tgh} x$. Obviously, for an arbitrary $\mu \neq 0$ we have:

$$G(0) = 0. \tag{1.29}$$

Thus (1.28) has a root $x = 0$.
Let us show that the root $x = 0$ is the only root of (1.28) for $\mu \in (-\infty, 1] \setminus 0$.
G is continuously differentiable on $\mathbb{R}$. For all $x \in \mathbb{R}$ it holds that:

If $\mu > 0$:

$$\lim_{x \to -\infty} G(x) = -\infty, \tag{1.30}$$
$$\lim_{x \to +\infty} G(x) = +\infty, \tag{1.31}$$

in particular, for $\mu \in (0, 1)$:

$$G'(x) > 0 \tag{1.32}$$

and for $\mu = 1$:

$$G'(x) \geq 0 \wedge (G'(x) = 0 \Leftrightarrow x = 0). \tag{1.33}$$

If $\mu < 0$, then:

$$\lim_{x \to -\infty} G(x) = +\infty, \tag{1.34}$$
$$\lim_{x \to +\infty} G(x) = -\infty, \tag{1.35}$$
$$G'(x) < 0. \tag{1.36}$$

Since G is continuous, using (1.30)-(1.33), there exists exactly one root of (1.28) for $\mu \in (0, 1]$. Applying (1.34)-(1.36) leads to the existence of exactly one root of (1.28) for $\mu < 0$.

Let us prove the statement for $\mu > 1$. Let us define $\alpha := \operatorname{argcosh}(\sqrt{\mu})$.
For $\mu > 1$, the following holds:

$$G'(x) = 0 \Leftrightarrow x = \pm\alpha, \tag{1.37}$$
$$G'(0) = \frac{1}{\mu} - 1 < 0. \tag{1.38}$$

Relations (1.37), (1.38) imply that G is strictly decreasing on $(-\alpha, \alpha)$. Obviously, $G(-\alpha) > 0$ and $G(\alpha) < 0$ due to (1.29). Using (1.30), (1.31) gives the claim for $\mu > 1$. □

Lemma 1.8. *Let μ be an arbitrary real constant. Let us consider the following equations*

$$\operatorname{tg} \sqrt{\lambda} = \frac{\sqrt{\lambda}}{\mu}, \quad \lambda \geq 0, \tag{1.39}$$

$$\operatorname{tgh} \sqrt{-\lambda} = \frac{\sqrt{-\lambda}}{\mu}, \quad \lambda < 0. \tag{1.40}$$

Then:

- *If* $\mu \in (-\infty, 1] \setminus \{0\}$, *then* $(\lambda, \mu) \in \Sigma \quad \Leftrightarrow \quad \lambda$ *is a root of (1.39).*
- *If* $\mu = 0$, *then* $(\lambda, 0) \in \Sigma \quad \Leftrightarrow \quad \lambda = \left(\frac{\pi}{2} + k\pi\right)^2$, $k \in \mathbb{N}_0$.
- *If* $\mu > 1$, *then* $(\lambda, \mu) \in \Sigma \quad \Leftrightarrow \quad \lambda$ *is a root of either (1.39), or (1.40).*

Proof. Obivously, $(0, 1) \in \Sigma_1 \subset \Sigma$ (see (1.21)). For $\mu = 1$, $\lambda = 0$ is a root of (1.39). For equation (1.23) and $\lambda > 0$ we distinguish the two following cases:

1. $\mu = 0$
   After substitution we get
   $$\cos\sqrt{\lambda} = 0,$$
   thus
   $$\sqrt{\lambda_k} = \frac{\pi}{2} + k\pi, \ k \in \mathbb{N}_0. \tag{1.41}$$
   Hence, the problem (1.1), (1.2) has a non-trivial solution for $(\lambda, \mu) = (\lambda_{k,0}, 0)$, where $\lambda_{k,0} = \left(\frac{\pi}{2} + k\pi\right)^2$, $k \in \mathbb{N}_0$.

2. $\mu \neq 0$
   The equation (1.23) can be rewritten to
   $$\operatorname{tg}\sqrt{\lambda} = \frac{\sqrt{\lambda}}{\mu}.$$
   and the corresponding roots λ satisfy $(\lambda, \mu) \in \Sigma_2 \subset \Sigma, \ \mu \neq 0$.

For equation (1.26) and $\lambda < 0$ we distinguish the two following cases:

1. $\mu = 0$
   The equation (1.26) is reduced to
   $$\sqrt{-\lambda}\cosh\sqrt{-\lambda} = 0.$$
   There is the only solution of the above equation $\lambda = 0$, which is not admissible due to the condition $\lambda < 0$.

2. $\mu \neq 0$
   The equation (1.26) can be rewritten to
   $$\operatorname{tgh}\sqrt{-\lambda} = \frac{\sqrt{-\lambda}}{\mu}$$
   and for its roots λ we can state: $(\lambda, \mu) \in \Sigma_3 \subset \Sigma, \ \mu \neq 0$.

Hence, for $\mu \neq 0$ and $(\lambda, \mu) \in \Sigma$, λ must be a root of either (1.39), or (1.40).
The lemma 1.7 implies that (1.40) has the only root $\lambda < 0$ for $\mu > 1$ and no root for $\mu \in (\infty, 1] \setminus \{0\}$.
Thus, for $\mu \in (-\infty, 1] \setminus \{0\}$, λ is a root of the equation (1.39), and, for $\mu > 1$, λ is a root of either (1.39), or (1.40). $\square$

Lemma 1.9. *Let* $\mu \neq 0$ *be an arbitrary real constant. Then the equation* $\operatorname{tg} x = \dfrac{x}{\mu}$ *has exactly one solution on the interval* $I_k := (k\pi, (k+1)\pi)$ *for any* $k \in \mathbb{N}$.

Proof. For any $k \in \mathbb{N}$, we define intervals $I_k^{'} := (k\pi, \frac{\pi}{2} + k\pi)$ and $I_k^{''} := (\frac{\pi}{2} + k\pi, (k+1)\pi)$.
Let us prove the existence of the solution of $\operatorname{tg} x = \dfrac{x}{\mu}$ on I_k.
Apparently, $x = \frac{\pi}{2} + k\pi$ is not a solution for any $k \in \mathbb{N}$. Let us prove that there is a solution on $I_k^{'} \subset I_k$ for $\mu > 0$.
For $x \in I_k^{'}$, we define a function $F(x) := \dfrac{x}{\mu} - \operatorname{tg} x$. Then:

$$\lim_{x \to k\pi+} F(x) > 0, \tag{1.42}$$

$$\lim_{x \to (\frac{\pi}{2}+k\pi)-} F(x) = \lim_{x \to (\frac{\pi}{2}+k\pi)-} \left(\frac{x}{\mu} - \operatorname{tg} x \right) = -\infty. \tag{1.43}$$

Relations (1.42), (1.43) and continuity of F on $I_k^{'}$ yield that there exist real numbers $a, b \in I_k^{'}$ such that $a < b$, $F(a) > 0$ and $F(b) < 0$.
Now we apply the Bolzano-Cauchy intermediate-value theorem to F on $[a, b]$ (see [7, p. 160, Th. 2]). Thus, there exists ξ on (a, b) such that $F(\xi) = 0$.
Hence, for $\mu > 0$, there exists a solution of $\operatorname{tg} x = \dfrac{x}{\mu}$ na $I_k^{'}$.
The existence of a solution for $\mu < 0$ on $I_k^{''} \subset I_k$ can be proved analogically.

Let us prove that the solution on I_k is unique.
Firstly, let us show that there exists exactly one solution for $\mu > 0$ and no solution for $\mu < 0$ on the interval $I_k^{'}$.
The function F is continuously differentiable on $I_k^{'}$. The first derivative of F is negative for $\mu \geq 1$ and an arbitrary $x \in I_k^{'}$. Therefore, F is strictly decreasing (thus one-to-one) function on $I_k^{'}$. (1.42), (1.43) imply, that there exists exactly one $x \in I_k^{'}$ such that $F(x) = 0$.
For $\mu \in (0, 1)$, we have:

$$\lim_{x \to k\pi+} F'(x) = \lim_{x \to k\pi+} \left(\frac{1}{\mu} - \frac{1}{\cos^2 x} \right) > 0. \tag{1.44}$$

The first derivative of F is zero wherever

$$\cos x = \pm\sqrt{\mu}.$$

But there is exactly one solution of the above equation on $I_k^{'}$. Thus F can change its monotonocity in at most one point. Relations (1.42), (1.43), (1.44) imply that there exists exactly one element $x \in I_k^{'}$ such that $F(x) = 0$.
For $\mu < 0$, we have:

$$\begin{aligned} \lim_{x \to k\pi+} F(x) &< 0, \\ \lim_{x \to (\frac{\pi}{2}+k\pi)-} F(x) &= \lim_{x \to (\frac{\pi}{2}+k\pi)-} \left(\frac{x}{\mu} - \operatorname{tg} x \right) = -\infty, \\ \lim_{x \to k\pi+} F'(x) &= \lim_{x \to k\pi+} \left(\frac{1}{\mu} - \frac{1}{\cos^2 x} \right) < 0. \end{aligned}$$

Obviously, for $\mu < 0$, $F \neq 0$ on $I_k^{'}$, therefore there is no solution of $\operatorname{tg} x = \dfrac{x}{\mu}$ on $I_k^{'}$.
Analogically, we can prove that there exists exactly one solution for $\mu < 0$ and no solution for $\mu > 0$ on the interval $I_k^{''}$. □

Lemma 1.10. *Let $\mu \neq 0$ be an arbitrary constant and let us define intervals $I_a := \left(0, \frac{\pi}{2}\right)$, $I_b := \left(\frac{\pi}{2}, \pi\right)$. Then, for the equation $\operatorname{tg} x = \dfrac{x}{\mu}$ we have:*

- *a solution on I_a if and only if $\mu \in (0, 1)$,*
- *a solution on I_b if and only if $\mu < 0$.*

Moreover, if there exists a solution on $I_0 := I_a \cup I_b$, then the solution is unique.

Proof. Let us prove the claim for I_a.
Let us consider the function F on I_a. F is continuously differentiable on I_a and

$$\lim_{x \to 0+} F(x) = 0. \tag{1.45}$$

For $\mu \in \mathbb{R} \setminus [0, 1)$ and $x \in I_a$, the first derivative F' is negative, thus F is strictly decreasing and $F \neq 0$ for any $x \in I_0$. Therefore, for $\mu \in \mathbb{R} \setminus [0, 1)$, the equation $\operatorname{tg} x = \dfrac{x}{\mu}$ has no solution on I_a.
For $\mu \in (0, 1)$ and I_a, the proof is analogical as for Lemma 1.9.

For the interval I_b, the prove can be carried out analogically as for I_a. □

Proposition 1.11. *Let $\mu_1, \mu_2 \in \mathbb{R}$. Then, for any $k \in \mathbb{N}$, there exists exactly one pair $x_1, x_2 \in I_k$ such that*

$$\operatorname{tg} x_i = \frac{x_i}{\mu_i}, \quad i \in \{1, 2\}.$$

Moreover:
If $\mu_1 > \mu_2 > 0$, then $k\pi < x_1 < x_2 < \frac{\pi}{2} + k\pi$.
If $\mu_2 < \mu_1 < 0$, then $\frac{\pi}{2} + k\pi < x_1 < x_2 < (k+1)\pi$.

Proof. Let us define functions $f_1(x) := \dfrac{x}{\mu_1}$ and $f_2(x) := \dfrac{x}{\mu_2}$ on I_k. Lemma 1.9 implies, that both f_1, f_2 equal $g := \operatorname{tg} x$ in the exactly one point.
Now, let us consider arbitrary constants $\mu_1 > \mu_2 > 0$. We have:

$$\forall x \in I_k' : f_1(x) < f_2(x). \tag{1.46}$$

Let us assume that f_1 equals g in x_1 and f_2 equals g in x_2. Since g is strictly increasing, the relation (1.46) gives the claim.
For any constants $\mu_2 < \mu_1 < 0$, the claim can be proved analogically. □

Proposition 1.12. *Let μ_1, μ_2 be arbitrary constants and $x_1 \in I_0$, $x_2 \in I_0$ be roots of*

$$\operatorname{tg} x_i = \frac{x_i}{\mu_i}, \quad i \in \{1, 2\}.$$

Then:
the pair x_1, x_2 exists if and only if $\mu_1, \mu_2 \in (-\infty, 1) \setminus \{0\}$.
Moreover, if the pair x_1, x_2 exists, then it is unique and the following holds:
If $1 > \mu_1 > \mu_2 > 0$, then $0 < x_1 < x_2 < \dfrac{\pi}{2}$.
If $\mu_2 < \mu_1 < 0$, then $\dfrac{\pi}{2} < x_1 < x_2 < \pi$.

Proof. Let us consider f_1, f_2 on I_0. Lemma 1.10 implies that f_1 equals g and f_2 equals g if and only if $\mu \in (-\infty, 1) \setminus \{0\}$.
First, let us consider $1 > \mu_1 > \mu_2 > 0$. For I_a we have:

$$\forall x \in I_a : f_1(x) < f_2(x). \tag{1.47}$$

Let us assume that f_1 equals g in x_1 and f_2 equals g in x_2. Since g is strictly increasing, the relation (1.47) gives the claim.
For $\mu_2 < \mu_1 < 0$ we can prove the claim analogically. □

Proposition 1.13. *Let $\mu \neq 0$ be an arbitrary constant. Then there exists a sequence $(x_{k,\mu})_{k=0}^{+\infty}$ such that $x_{k,\mu} > 0$ are solutions of $\operatorname{tg} x = \dfrac{x}{\mu}$ in I_k and the sequence has the following properties:*

1. *Let $k \in \mathbb{N}_0$:*
   - (a) *for $\mu \to 0+$: $x_{k,\mu} \to \left(\frac{\pi}{2} + k\pi\right)-$,*
   - (b) *for $\mu \to 0-$: $x_{k,\mu} \to \left(\frac{\pi}{2} + k\pi\right)+$,*
   - (c) *for $\mu \to -\infty$: $x_{k,\mu} \to (k+1)\pi-$.*
2. *Let $k \in \mathbb{N}$:*
   - (a) *$x_{k,\mu} \in I_k$,*
   - (b) *for $\mu \to +\infty$: $x_{k,\mu} \to k\pi+$,*
3. *Let $k = 0$:*
   - (a) *$x_{k,\mu} \in \begin{cases} I_b \text{ for } \mu < 0, \\ I_a \text{ for } \mu \in (0,1), \end{cases}$*
     $x_{k,\mu}$ does not exist for $\mu \geq 1$,
   - (b) *for $\mu \to 1-$: $x_{k,\mu} \to 0+$.*

Proof. For $k \in \mathbb{N}_0$ we know:

$$\lim_{x_{k,\mu} \to (\frac{\pi}{2} + k\pi)-} \operatorname{tg} x = +\infty, \tag{1.48}$$

$$\lim_{x_{k,\mu} \to k\pi} \operatorname{tg} x = 0, \tag{1.49}$$

$$\lim_{x \to (\frac{\pi}{2} + k\pi)+} \operatorname{tg} x = -\infty. \tag{1.50}$$

Let $k \in \mathbb{N}$. Lemma 1.9 implies (2a). Proposition 1.11 and relations (1.48)–(1.50) yield (1a)–(1c), (2b) for $k \in \mathbb{N}$.
Let $k = 0$. Lemma 1.10 implies (3a). Proposition 1.12 and relations (1.48)–(1.50) yield (1a)–(1c) and (3b) for $k = 0$. □

Theorem 1.14. *Let μ be an arbitrary constant. Then there exists a sequence $(\lambda_{k,\mu})_{k=0}^{+\infty}$ such that for $k \in \mathbb{N}$, all the $\lambda_{k,\mu}$ satisfy $(\lambda_{k,\mu}, \mu) \in \Sigma$ and the sequence has the following properties:*

- *for $\mu > 1 :$ $\lambda_{0,\mu} \in \mathbb{R}^-$,*
- *for $\mu = 1 :$ $\lambda_{0,\mu} = 0$,*
- *for $\mu < 1 :$ $\lambda_{0,\mu} \in \left(0, \pi^2\right)$,*

and for $k \in \mathbb{N}$ we have:

- $\lambda_{k,\mu} \in \left((k\pi)^2, (k+1)^2\,\pi^2\right)$,
- *for* $\mu \to +\infty : \lambda_{k,\mu} \to (k\pi)^2+$,
- *for* $\mu \to -\infty : \lambda_{k,\mu} \to ((k+1)\pi)^2-$,
- *for* $\mu \to 0 : \lambda_{k,\mu} \to \left(\frac{\pi}{2} + k\pi\right)^2$.

Proof. Let us consider $\lambda < 0$.
Lemma 1.8 implies $(\lambda, \mu) \in \Sigma$ if $\mu > 1$ and λ is a solution of $\operatorname{tgh} \sqrt{-\lambda} = \frac{\sqrt{-\lambda}}{\mu}$. Lemma 1.7 states that this solution is unique. Let $\lambda_{0,\mu}$ denote this solution for the corresponding μ.

Let us consider $\lambda = 0$.
Lemma 1.8 shows that $\mu = 1$ is the only μ which satisfies $(0, \mu) \in \Sigma$. Therefore $\lambda_{0,1} = 0$.

Now, let us consider $\lambda > 0$ and $\mu = 0$.
According to Lemma 1.8, $(\lambda, 0) \in \Sigma$ if $\lambda = (\frac{\pi}{2} + k\pi)^2 =: \lambda_{k,0}$, $k \in \mathbb{N}_0$.
Finally, let us consider $\lambda > 0$ and $\mu \neq 0$.
Using a substitution $x = \sqrt{\lambda}$ and Proposition 1.13 we get that positive solutions of the equation $\operatorname{tg} \sqrt{\lambda} = \frac{\sqrt{\lambda}}{\mu}$ give a sequence $(\lambda_{k,\mu})_{k=0}^{+\infty}$ with the listed properties and that $\lambda_{0,\mu} \in (0, \pi^2)$ is unique for any $\mu \in (-\infty, 0) \cup (0, 1)$.
Thus, obviously, we have exactly one $\lambda_{0,\mu}$ and a sequence $(\lambda_{k,\mu})_{k=1}^{+\infty}$ for any $\mu \in \mathbb{R}$. $\square$

The set of all pairs $(\lambda, \mu) \in \Sigma$ is described in Figure 1.2.

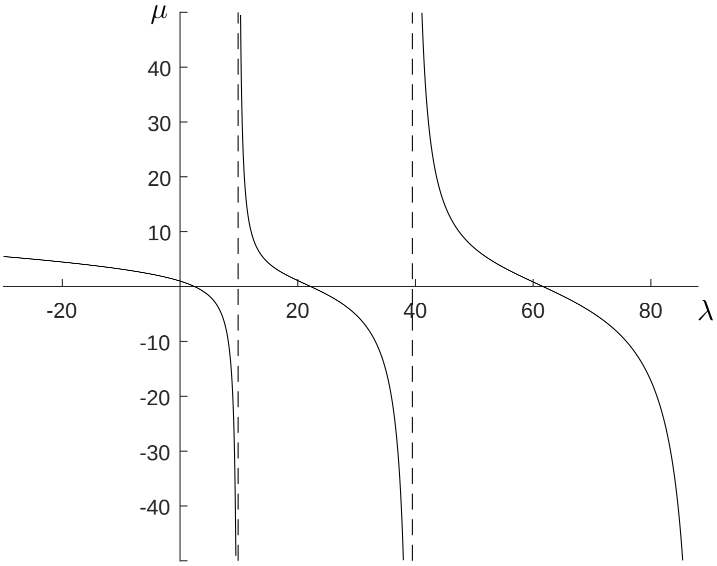
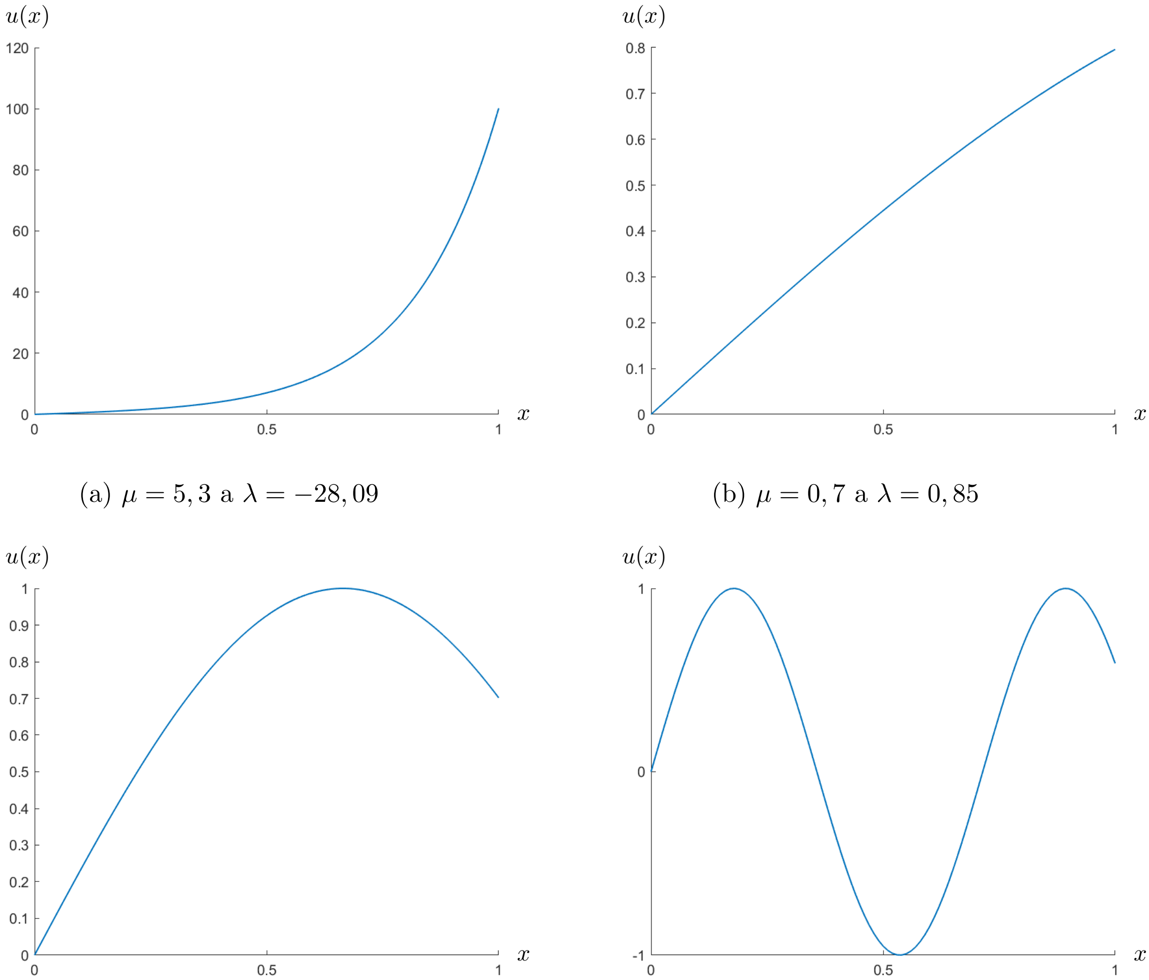


Fig. 1.2: The set of all the pairs $(\lambda, \mu) \in \mathbb{R}^2$ which give a non-trivial solution of the boundary value problem (1.1), (1.2).

In Figure 1.3, there are a few examples of non-trivial solutions u of the boundary value problem (1.1), (1.2) for particular choices of μ.



(a) $\mu = 5,3$ a $\lambda = -28,09$

(b) $\mu = 0,7$ a $\lambda = 0,85$

(c) $\mu = -2,4$ a $\lambda = 5,59$

(d) $\mu = -12$ a $\lambda = 77,307$

Fig. 1.3: Examples of non-trivial solutions u of the boundary value problem (1.1), (1.2) for particular choices of μ.

Chapter 2

Nonlinear problem

Let us consider a nonlinear BVP:

$$\begin{cases} -u'' = \lambda g(u), \\ u'(1) = \mu u(1), \\ u(0) = 0, \end{cases} \tag{2.1}$$

where $g(u)$ represents an arbitrary nonlinear continuous function. Our task is to determine properties of a non-trivial solution u depending on the function g.
Obviously, the results are highly dependent on properties of the function g. Thus we introduce several assumptions we will further use in this thesis to make clear which properties of g are important for a particular result.

Assumption A1. $\operatorname{sgn} g(u) = \operatorname{sgn} u$,

Assumption A2. g is globally Lipschitz continuous,

Assumption A3. g is odd,

Assumption A4. $|g(u)| \leq |u|$ for any u.

2.1 Bifurcation diagram

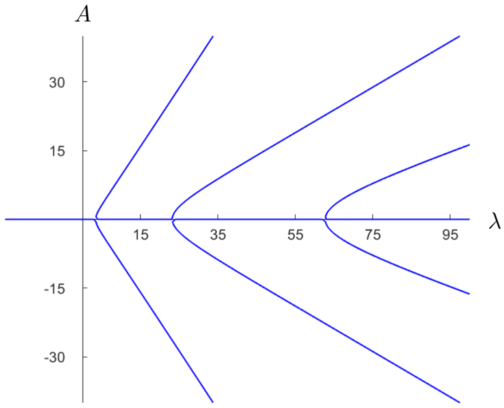
Generally speaking, since g is nonlinear, we cannot get an exact solution. Trying to visualize the results, we are the most of the time limited to numerical experiments.
Among other properties, we are interested in an existence and multiplicity of solutions or, to be more specific, in the pairs of $(\lambda, \mu) \in \mathbb{R}^2$ which provide a non-trivial solution of BVP (2.1) with a particular property. That makes the visualization even more complicated.
To begin with, a bifurcation diagram is a simple visual description of (2.1). Let us choose arbitrary fixed μ and then let us examine the corresponding BVP (2.1) with respect to λ. For the purpose of the visualization, we will consider an auxiliary initial value problem

$$\begin{cases} -u'' = \lambda g(u), \\ u(0) = 0, \\ u'(0) = A \end{cases} \tag{2.2}$$

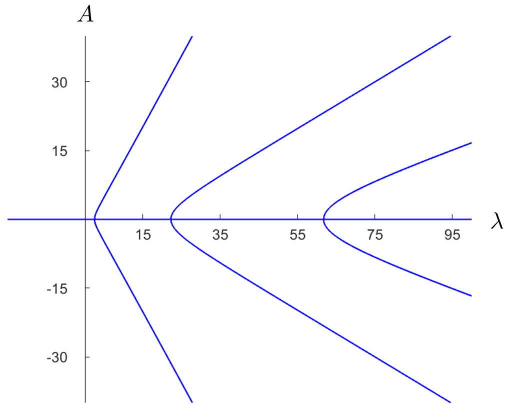
with $g(u)$ denoting an arbitrary nonlinear continuous function and $A \in \mathbb{R}$.
For any pair (λ, A), we can get a numerical solution of (2.2) easily. To determine whether the solution solves also BVP (2.1), we have to check the other boundary condition $u'(1) = \mu u(1)$. If this condition is satisfied, the pair (λ, A) provides a solution of BVP (2.1).

As we mentioned before, the results highly depend on properties of g. Initially, if g satisfies $g(0) = 0$ (or even (A1)), for $A = 0$, any $\lambda \in \mathbb{R}$ provides a solution of (2.1) and this solution is always trivial. On contrary, for $A \neq 0$, if any pair (λ, A) provides a solution of (2.1), then the solution is non-trivial.

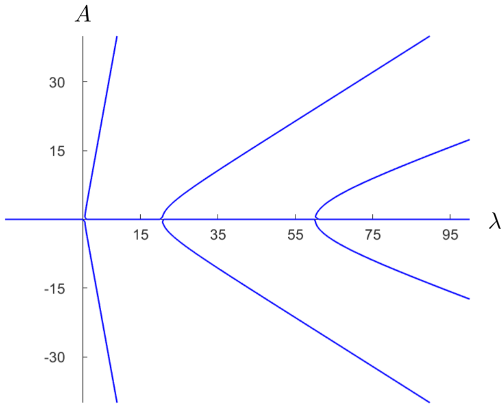
Consequently, all the pairs (λ, A) providing any solution of (2.1) can be drawn into a λA bifurcation diagram. In Figure 2.1, there can be seen examples of such diagrams for the particular choices of μ and for $g(u) = \arctan u$.



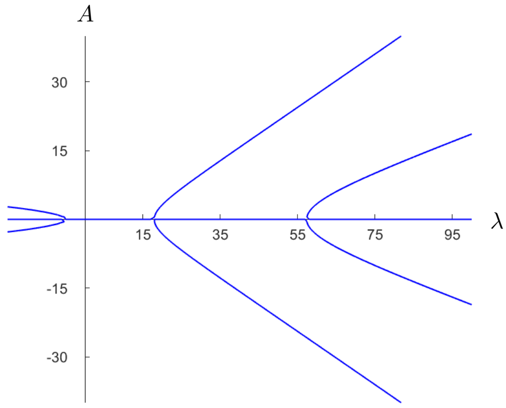
(a) $\mu = -0.5$



(b) $\mu = 0$



(c) $\mu = 0.8$



(d) $\mu = 2.2$

Fig. 2.1: Bifurcation diagram for $g(u) = \arctan u$ and the particular choices of μ.

For $g(u) = \arctan u$, it has been experimentally verified that the values of λ such that for $(\lambda, 0)$ a bifurcation occurs match with the values λ such that $(\lambda, \mu) \in \Sigma$ (see Theorem 1.6 in the Linear problem chapter). In other words, for this particular choice of g and an arbitrary value of μ, there occurs a bifurcation in λA diagram if and only if λ is an eigenvalue of linear BVP (1.1), (1.2) for the particular μ.

Bifurcation λA diagram for a general g does not have this connection with BVP (1.1), (1.2). The diagrams for a few other functions g are shown in Appendix A.

2.2 Solution properties

In the following part of the text, we will introduce statements providing properties of the solution of (2.1) based on knowledge of g or its key properties and the parameters λ, μ.

2.2.1 An auxiliary initial value problem

For simplicity, instead of (2.1), we will temporarily consider an auxiliary initial value problem (2.2) with $g(u)$ denoting an arbitrary nonlinear continuous function and $A \in \mathbb{R} \setminus \{0\}$.
Let I be an intersection of $\mathbb{R}$ and the interval of existence of the maximal solution. Moreover we assume that $[0, 1] \subset I$. Since we will look for properties of the solution of (2.2), considering only an interval $[0, 1]$, all the results will be valid also for (2.1).

If $\lambda = 0$, regardless of the properties of g, we get $u = Ax$, $I = \mathbb{R}$, in other words:

- $u'(x) = A$, $u''(x) = 0$ for any $x \in I$ and $A \in \mathbb{R} \setminus \{0\}$,
- if $A > 0$, u is strictly increasing, positive on I,
- if $A < 0$, u is strictly decreasing, positive on I.

In this case, any property of g does not affect the function u. Thus, unless stated otherwise, we will consider $\lambda \neq 0$ anywhere in this subchapter.
Now let us show how properties of nonlinear function g affect the function u.

Lemma 2.1. *For a nonlinear IVP (2.2) with g satisfying (A1), we can claim:*

1. *For $\lambda \neq 0$, $u''(x) = 0$ wherever $u(x) = 0$ on I.*
2. *For $\lambda < 0$ and $u'(0) < 0$, both u and u'' are negative on right neighbourhood of 0. Consequently, $u(x)$ and $u'(x)$ are negative and strictly decreasing for any $x \in I$.*
3. *Analogically, for $\lambda < 0$ and $u'(0) > 0$, $u(x)$ and $u'(x)$ are positive and strictly increasing for any $x \in I$.*
4. *For $\lambda > 0$, u is concave (convex) wherever u is positive (negative).*
5. *For $\lambda < 0$, u is concave (convex) wherever u is negative (positive).*

Proof. Let $g(u)$ be a function satisfying (A1), then the equation (2.1) directly gives

$$\operatorname{sgn} u'' = -\operatorname{sgn}(\lambda u)$$

implying the claims of this Lemma. □

To make sure that some necessary properties of u will be preserved, we will further consider g satisfying (A2). Moreover, this assumption automatically garantuees that $[0, 1] \subset I = \mathbb{R}$ (see [6, p. 39, Corollary 2.6]).

Notation 2.2. By X^p we donote a set containing all $x \in \mathbb{R}_0^+$, such that for any $x^p \in X^p$, $u(x^p) = 0$. Since u is continuous and non-trivial and g satisfies (A2), elements of the set X^p are at most countable. Let us denote each element x_n^p where $n \in M \subset \mathbb{N}_0$, such that $x_0^p < x_1^p < x_2^p < \dots$.

Remark 2.3. Trivially, $|X^p| \geq 1$ and $x_0^p = 0$.

Properties of a solution are further developed by the following Proposition.

Proposition 2.4. *Let us consider a nonlinear IVP (2.2), where $g(u)$ is any function satisfying (A1), (A2). Then we have:*

1. *For $\lambda < 0$ and $u'(0) > 0$, u is positive, strictly increasing, and convex on I, u' is positive and strictly increasing on I.*

2. *For $\lambda < 0$ and $u'(0) < 0$, u is negative, strictly decreasing, and concave on I, u' is negative and strictly decreasing on I.*

3. *For any $x_n^p \in X^p$, it holds that*

$$u'(x_n^p) = \begin{cases} -u'(0) & \text{for } n \in \mathbb{N} \text{ odd,} \\ \ \ u'(0) & \text{for } n \in \mathbb{N} \text{ even.} \end{cases}$$

Moreover, if we suppose that $|X^p| > 1$, then we get:

4. *For $\lambda > 0$ and $u'(0) > 0$, u is positive and concave on $(0, x_1^p)$, u' is strictly decreasing on $(0, x_1^p)$.*

5. *For $\lambda > 0$ and $u'(0) < 0$, u is negative and convex on $(0, x_1^p)$, u' is strictly increasing on $(0, x_1^p)$.*

Proof. Lemma 2.1 gives directly the statement in the points 1 and 2.

For $\lambda \in \mathbb{R}$ and $t \in (0, T)$, $T \in I$, let us multiply the equation in (2.2) by u' and integrate over $(0, t)$. Since $G(u) := \int_0^{u(t)} g(s)\,\mathrm{d}s$ is a primitive function to $g(s)$ and

$$\left(\frac{(u')^2}{2}\right)' = u''u',$$

$$(G(u))' = g(u)u',$$

we obtain

$$\frac{(u'(t))^2}{2} - \frac{(u'(0))^2}{2} = -\lambda G(u(t)).$$

For $t = x_n^p \in X^p$, we get

$$\frac{(u'(x_n^p))^2}{2} - \frac{(u'(0))^2}{2} = -\lambda G(0) = 0,$$

hence

$$|u'(x_n^p)| = |u'(0)|. \tag{2.3}$$

Now let us assume that $u'(x_n^p) > 0, n \in \mathbb{N}_0$. The function u is continuous and $u(x_n^p) = 0$, therefore $u > 0$ on (x_n^p, x_{n+1}^p) and using (2.3) yields that $u'(x_n^p) = -u'(x_{n+1}^p)$. For $u'(x_n^p) < 0, n \in \mathbb{N}_0$, this would be carried out analogically.
Thus the point 3 holds.

Now let us assume $\lambda > 0$ and $t \in [0, x_1^p]$, x_1^p is supposed to exist.

Integrating the equation in (2.2) on $(0, t)$, we get

$$u'(t) - u'(0) = -\lambda \int_0^t g(u(x))\mathrm{d}x.$$

If $u'(0) > 0$, we have an integral of a positive function on the right side of the above equation, hence

$$u'(t) = u'(0) - \lambda \int_0^t g(u(x))\mathrm{d}x < u'(0). \tag{2.4}$$

Analogically, assuming $u'(0) < 0$ yields

$$u'(0) = u'(t) + \lambda \int_0^t g(u(x))\mathrm{d}x < u'(t). \tag{2.5}$$

The equations (2.4),(2.5) imply that

$$u'(0) = \begin{cases} \max\limits_{x\in[0,x_1^p]} u'(x) \text{ if } u'(0) > 0, \\ \min\limits_{x\in[0,x_1^p]} u'(x) \text{ if } u'(0) < 0. \end{cases} \tag{2.6}$$

Using (2.6) and Lemma 6 gives the claim for the points 4, 5. □

Remark 2.5. The first two points of Proposition 2.4 imply, that if $|X^p| > 1$, then $\lambda > 0$.

* * *

Furthermore, we will consider g with properties (A1), (A2), (A3).
In the following statements we will work with oddness of function u. We know that with (A2) satisfied we have $I = \mathbb{R}$, which allows us to formulate the statements correctly.

Proposition 2.6. *Let us consider a nonlinear IVP (2.2), where $g(u)$ satisfies (A1), (A2), (A3). Then u is an odd function.*

Proof. We know that

$$g(u) = -g(-u). \tag{2.7}$$

Let us define function $v(x) := -u(-x)$. Then $v''(x) = -u''(-x) = \lambda g(-v(x))$. Using (2.7) yields

$$v''(x) = -\lambda g(v(x)). \tag{2.8}$$

Trivially

$$v(0) = -u(0) = 0, \tag{2.9}$$
$$v'(0) = u'(0) = A. \tag{2.10}$$

Formulating an auxiliary IVP (2.8)-(2.10), we get an IVP equivalent to (2.2). Since g is Lipschitz continuous, there exists exactly one solution on $\mathbb{R}$ and therefore we get a relation

$$u(x) = v(x) = -u(-x)$$

giving directly the claim. □

Theorem 2.7 (Symmetry of the solution)**.** *Let u be a non-trivial solution of IVP (2.2), where $g(u)$ is any function satisfying (A1), (A2), (A3).*
If there exists $x_1^p \in X^p$, then for any $n \in \mathbb{N}_0$, there exists $x_n^p = n\, x_1^p \in X^p$, and

$$u(x) = u(x - x_{2n}^p) = -u(x - x_{2n+1}^p) = u(x_{2n+1}^p - x),$$

more specifically for $n \in \mathbb{N}_0$:

- *we call the part of u on $[x_n^p, x_{n+1}^p]$ an "arc",*
- *every arc for $x = [x_n^p, x_{n+1}^p]$, $n \in \mathbb{N}_0$ is axially symmetric with respect to an axis $x_{n+1}^e := \frac{x_n^p + x_{n+1}^p}{2}$,*
- *every arc is strictly increasing on $[x_n^p, x_{n+1}^e]$ and strictly decreasing on $[x_{n+1}^e, x_{n+1}^p]$, or vice versa,*
- *arcs for $x \in [x_{2n}^p, x_{2n+1}^p]$ and $x \in [x_{2n+1}^p, x_{2(n+1)}^p]$ are centrally symmetric with respect to a point $x = x_{2n+1}^p$,*
- *arcs for $x \in [x_{2n}^p, x_{2n+1}^p]$ and $x \in [x_{2(n+1)}^p, x_{2n+3}^p]$ are axially symmetric with respect to an axis $x = \frac{x_{2n}^p + x_{2n+3}^p}{2}$.*

Proof. Firstly, let us define $v_1(x) := -u(x - x_1^p)$. Then

$$v_1''(x) = -u''(x - x_1^p) = \lambda g(-v_1(x)).$$

Since g is an odd function, the above relation yields

$$v_1''(x) = -\lambda g(v_1(x)). \tag{2.11}$$

Using the Proposition (2.6), we get

$$v_1(0) = -u(-x_1^p) = u(x_1^p) = 0, \tag{2.12}$$
$$v_1'(0) = -u'(-x_1^p) = -u'(x_1^p) = A. \tag{2.13}$$

Now let us define $v_2(x) := u(x_1^p - x)$. Then

$$v_2''(x) = u''(x_1^p - x) = -\lambda g(v_2(x)), \tag{2.14}$$
$$v_2(0) = u(x_1^p) = 0, \tag{2.15}$$
$$v_2'(0) = -u'(x_1^p) = A. \tag{2.16}$$

The relations (2.11)-(2.13) and (2.14)-(2.16) give us two IVPs equivalent to (2.2). Since g is Lipschitz continuous, using Picard-Lindelöf Theorem (see [3, p. 350, Th. 8.13]), there exists exactly one solution of (2.2) on I and therefore

$$v_1(x) \;=\; u(x) = -u(x - x_1^p), \tag{2.17}$$
$$v_2(x) \;=\; u(x) = u(x_1^p - x). \tag{2.18}$$

Now we show that there exists $x_n^p \in X^p$ for any $n \in \mathbb{N}$.
Using the identity (2.17) for $x = 2\, x_1^p$, we get $u(2\, x_1^p) = 0$.

Identity (2.17) also yields that there is no other $x \in (x_1^p, 2\,x_1^p)$, such that $u(x) = 0$. Thus $x_2^p := 2\,x_1^p \in X^p$.
The last step we can repeat step by step for any $x = n\,x_1^p$, $n \in \mathbb{N}$, hence

$$x_n^p \in X^p, n \in \mathbb{N}_0.$$

That allows us to define $v_3(x) := u(x - x_{2n}^p), n \in \mathbb{N}$. Then

$$v_3''(x) = u''(x - x_{2n}^p) = -\lambda g(v_1(x)). \tag{2.19}$$

Since u is, according to the Proposition 2.6, odd, u' is even. Thus

$$\begin{aligned} v_3(0) &= u(-x_{2n}^p) = -u(x_{2n}^p) = 0, && (2.20)\\ v_3'(0) &= u'(-x_{2n}^p) = u'(x_{2n}^p) = A. && (2.21) \end{aligned}$$

Using the relations (2.19)-(2.21) and Picard-Lindelöf Theorem once again yields the following identity:

$$v_3(x) = u(x) = u(x - x_{2n}^p), n \in \mathbb{N},$$

which allows us to formulate (2.17) and (2.18) generally:

$$\begin{aligned} v_1(x) &= u(x) = -u(x - x_{2n+1}^p), && (2.22)\\ v_2(x) &= u(x) = u(x_{2n+1}^p - x). && (2.23) \end{aligned}$$

The above relations, among other properties, show that every arc for $x = [x_n^p, x_{n+1}^p]$, $n \in \mathbb{N}_0$ is axially symmetric with respect to an axis $x_{n+1}^e = \frac{x_n^p + x_{n+1}^p}{2}$. In the last part of the proof we will show a course of a single arc.
Without loss of generality, we assume that $u'(0) > 0$. According to the Remark 2.5, $\lambda > 0$. Using Proposition 4, we know that u is positive and concave on $(0, x_1^p)$ and u' is strictly decreasing on $(0, x_1^p)$. Since u is continuous on $[0, x_1^p]$, Rolle's Theorem (see [7, p. 215, Prop. 1]) gives us $\xi \in (0, x_1^p)$ such that $u'(\xi) = 0$.
Since u is concave, ξ is a point of a local maximum and even the only point of a local extremum on $(0, x_1^p)$. Because of the symmetry described above, $\xi = x_1^e$. Hence u is strictly increasing on $(0, x_1^e)$ and strictly decreasing on (x_1^e, x_1^p). Analogically, we get the statement for all the arcs. □

Corollary 2.8. *Let us consider a nonlinear IVP (2.2), where $g(u)$ is any function satisfying (A1), (A2), (A3). Then*

$$|u'(0)| = \max_{x \in I} |u'(x)|.$$

Proof. Applying the relation (2.6) to all the arcs of u using the Theorem 2.7 gives the claim. □

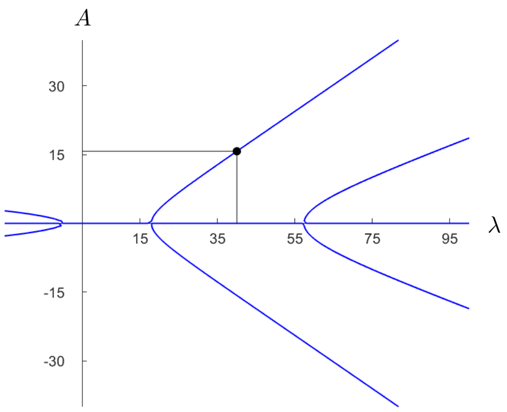
Remark 2.9. Results brought by Theorem 2.7 imply that a non-trivial solution u of IVP (2.2) is a periodic function with a period $2x_1^p$.

2.2.2 Back to the boundary value problem

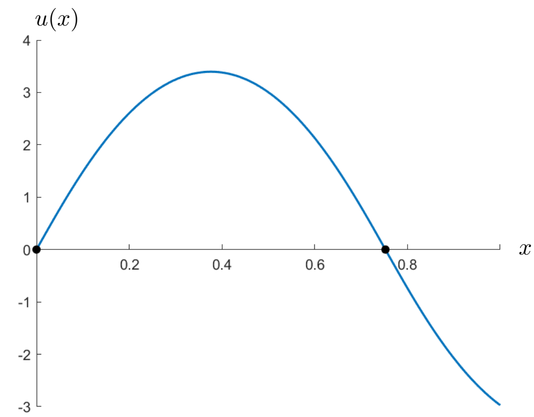
Now we will consider the original BVP (2.1). All the results gained in the previous subchapter are still valid. Involving the other boundary condition brings us more properties of the solution of (2.1).

Lemma 2.10. *Let u be a non-trivial solution of (2.1). Then $u(1) \neq 0$.*

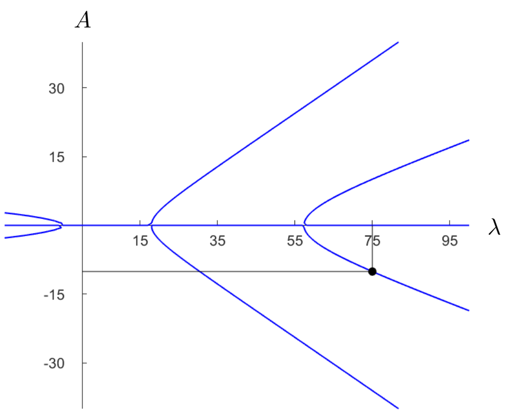
Proof. The proof will be carried out by contradiction.
Let us suppose $u(1) = 0$ and u is non-trivial. From (2.1), we know that $u(1) = u'(1) = 0$. Thus, trivially, $u(x) \equiv 0$ for any $x \in [0, 1]$. □



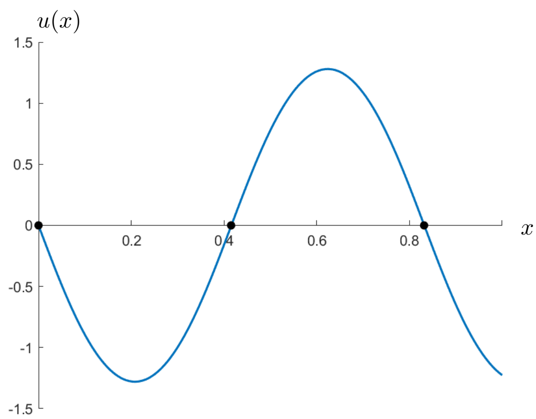
(a) Chosen point (λ, A) on the second branch of bifurcation diagram



(b) Corresponding solution of (2.1) with two nodes



(c) Chosen point (λ, A) on the third branch of bifurcation diagram



(d) Corresponding solution of (2.1) with three nodes

Fig. 2.2: Connection between m and the bifurcation diagram for (2.1) with $g(u) = \arctan u$.

Before we introduce the following notations and remarks, we remind the following:
X^p denotes a set of points $x \geq 0$ such that $u(x) = 0$. From the assumption (A2) and thanks to the non-triviality of u, X^p is at most countable and the elements of X^p can be ordered into a sequence $x_0^p < x_1^p < x_2^p < \dots$.
Theorem 2.7 brought $x_{n+1}^e = \frac{x_n^p + x_{n+1}^p}{2}$ for any $n \in \mathbb{N}_0$ and it showed that an arc on $[x_n^p, x_{n+1}^p]$ has the only point of local extremum in x_{n+1}^e.

Notation 2.11. By m we denote cardinality of $|X^p| \cap [0, 1]$.

Remark 2.12. Value $m-1$ corresponds to a count of completed arcs of u on $[0,1]$.

For an interpretation of the further results, it is important to notice another meaning of m discovered experimentally.
For $g(u) = \arctan u$, the connection of the bifurcation diagram and the linear problem has been described in the beginning of this chapter. Each point (λ, A), $A \neq 0$ in the bifurcation diagram provides exactly one non-trivial solution of (2.1). Simultaneously, every non-trivial solution is connected to one of the points. All of these points can be assigned to so called "branch". These branches can be ordered from left to right with positive integers. Experiments showed that this ordering corresponds with m. In other words, if the particular $u(x)$ solving (2.1) has $m = 2$ (there exists exactly two points between 0 and 1 such that u equals to zero), u solves also (2.2) for (λ, A) which lies on the second branch of the bifurcation diagram. At the same time, if (λ, A) lies on the third branch, then solution of (2.2) also solves (2.1) and $m = 3$.
This connection is also described in Figure 2.2.

Notation 2.13. Let X^e be a set of x_n^e such that $x_n^e \leq 1$ for any $n \in \mathbb{N}$.

Remark 2.14. Let the assumptions of Theorem 2.7 be satisfied. Then the Theorem also claims that $M := \max\limits_{x\in[0,1]} |u| = |u(x_n^e)|$ for any $n \in \mathbb{N}$.

Lemma 2.15. *Let us consider $\lambda > 0$. Let u be a non-trivial solution of (2.1), where $g(u)$ is any function satisfying (A1), (A2), (A3), and $m > 1$. Then $\mu = 0$ if and only if $M = |u(1)|$.*

Proof. Since $m > 1$, we know that u has at least one completed arc on $[0,1]$ and $M = |u(x_1^e)|$.
Firstly, let us assume that $\mu = 0$. We know that $u'(1) = \mu\, u(1) = 0$ and Lemma 2.10 shows that $u(1) \neq 0$. Thus $x = 1$ has to be a point of local extremum. Using Remark 2.14, we have that $M = |u(1)|$.
Secondly, let us suppose that $M = |u(1)|$, more specifically that $x = 1$ is a point of local extremum of u. Hence $|u(x_1^e)| = |u(1)|$ and using symmetry from Theorem 2.7 yields that $|u'(x_1^e)| = 0 = |u'(1)|$. Since $u'(1) = \mu u(1)$ and $u(1) \neq 0$ from Lemma 2.10, certainly, $\mu = 0$. $\square$

Lemma 2.16. *Let μ be an arbitrary constant and let u be a non-trivial solution of (2.1) with $g(u)$ satisfying (A1), (A2), (A3). Furthermore let $m > 1$. Then for $\lambda > 0$ and $n \in \{1, ..., |X^e|\}$ we have the following:*

- *if $\mu > 0$, then* $\frac{2n-1}{2m-1} < x_n^e < \frac{2n-1}{2(m-1)}$,
- *if $\mu < 0$, then* $\frac{2n-1}{2m} < x_n^e < \frac{2n-1}{2m-1}$,
- *if $\mu = 0$, then* $x_n^e = \frac{2n-1}{2m-1}$,

and for $n \in \{1, ..., m\}$ we can say:

- *if $\mu > 0$, then* $\frac{2n}{2m-1} < x_n^p < \frac{n}{m-1}$,
- *if $\mu < 0$, then* $\frac{n}{m} < x_n^p < \frac{2n}{2m-1}$,
- *if $\mu = 0$, then* $x_n^p = \frac{2n}{2m-1}$.

Proof. Firstly, let us prove the statement for x_1^e.
From Lemma 2.10 we know that $u(1) \neq 0$.
Let us assume that $u(1) > 0$.
For $\mu = 0$, we have $u(1) = M$. Due to symmetry given by Theorem 2.7, we get $1 = (2m-1)x_1^e$ and hence

$$x_1^e = \frac{1}{2m-1}.$$

For $\mu > 0$, $u'(1) = \mu u(1) > 0$, thus the m-th arc of u does not reach its maximum. Moreover, u completed $m-1$ arcs. Therefore, using Theorem 2.7, naturally

$$\frac{1}{2m-1} < x_1^e < \frac{1}{2(m-1)}.$$

On contrary, for $\mu < 0$, $u'(1) = \mu u(1) < 0$, which implies that the $m-th$ arc of u reaches its maximum, but it is not completed yet. In other words, after using Theorem 2.7

$$\frac{1}{2m} < x_1^e < \frac{1}{2m-1}.$$

Theorem 2.7 allows us to use the above results for any $x_n^e \in X^e$, which yields the first part of the statement.
Moreover, it holds that

$$x_n^p = 2nx_1^e$$

and that gives us the second part of the statement.
For $u(1) < 0$, the proof would be carried out analogically.
Since, according to Lemma 2.10, $u(1) \neq 0$, we have proved the statement. □

Lemma 2.17. *Let μ be an arbitrary constant and let u be a non-trivial solution of (2.1) with g satisfying (A1), (A2), (A3). Let $m > 1$ and let us denote $D := |u'(0)|$ and*

$$P := \begin{cases} 2(m-1) \text{ for } \mu > 0, \\ 2m-1 \text{ for } \mu \leq 0. \end{cases}$$

Then $PM < D$.

Proof. From Lemma 2.16, we know that $x_1^e \leq \frac{1}{P}$ and Corollary 2.8 says that $D \geq |u'(x)|$ for any $x \in [0,1]$. Since $m > 1$, certainly, $\lambda > 0$.
Lemma 2.1 implies that for any $x \in [0, x_1^p]$ it holds that $|u(x)| < D\,x$.
Therefore

$$M = |u(x_1^e)| < D\,x_1^e \leq \frac{D}{P},$$

in other words $PM < D$. □

Proposition 2.18. *Let u be a non-trivial solution of (2.1) with g satisfying (A1), (A2), (A3) and let $|g(x)| \leq c$ for any $x \in \mathbb{R}$. Furthermore, let us suppose $m > 1$. Then*

$$D \leq \frac{\lambda c}{P}.$$

Proof. From $m > 1$ we know that $\lambda > 0$ and that at least one arc of u is completed. Thus there exists x_1^e.
From the assumption that absolute value of g is bounded by c, we have the following system of inequalities:

$$-c\lambda \leq -\lambda g(u(x)) = u''(x) \leq \lambda c. \tag{2.24}$$

After integration from 0 to x_1^e with respect to x, the first inequality of (2.24) gives

$$u'(0) \leq \lambda\, c\, x_1^e$$

and using Lemma 2.16 yields

$$u'(0) \leq \frac{\lambda c}{P}.$$

Analogically, from the second inequality of (2.24), we have

$$u'(0) \geq -\frac{\lambda c}{P}$$

and hence

$$D \leq \frac{\lambda\, c}{P}.$$

□

The result of Proposition 2.18 is described in Figures 2.3 and 2.4.

2.3 Solution existence

This subchapter will present results related to existence of the solution. The main aim is to bring statements which prove that with certain properties of g and with particular values of the parameters λ and μ, BVP (2.1) does not provide any non-trivial solution.

Proposition 2.4 gave us a few elementary properties for IVP with g satisfying (A1). (A2). Now we involve the other boundary condition of (2.1), which yields the following Corollary.

Corollary 2.19. *Let there exist a non-trivial solution of the BVP (2.1) with g satisfying (A1). Then certainly:*

$$(\lambda, \mu) \notin (-\infty, 0) \times (-\infty, 0].$$

Proof. From Proposition 2.4, for $\lambda < 0$ we know that both $u(x)$ and $u'(x)$ are strictly increasing (decreasing) for $u'(0)$ positive (negative), therefore

$$\operatorname{sgn} u'(1) = \operatorname{sgn} u'(0) = \operatorname{sgn} u(0) = \operatorname{sgn} u(1).$$

The first boundary condition of (2.1) gives us

$$u'(1) = \mu u(1),$$

thus, necessarily, $\mu > 0$. □

For the purpose of the following Lemma, we will shortly remind an auxiliary notation from the previous subchapter:

$$P = \begin{cases} 2(m-1) & \text{for } \mu > 0, \\ 2m - 1 & \text{for } \mu \leq 0. \end{cases}$$

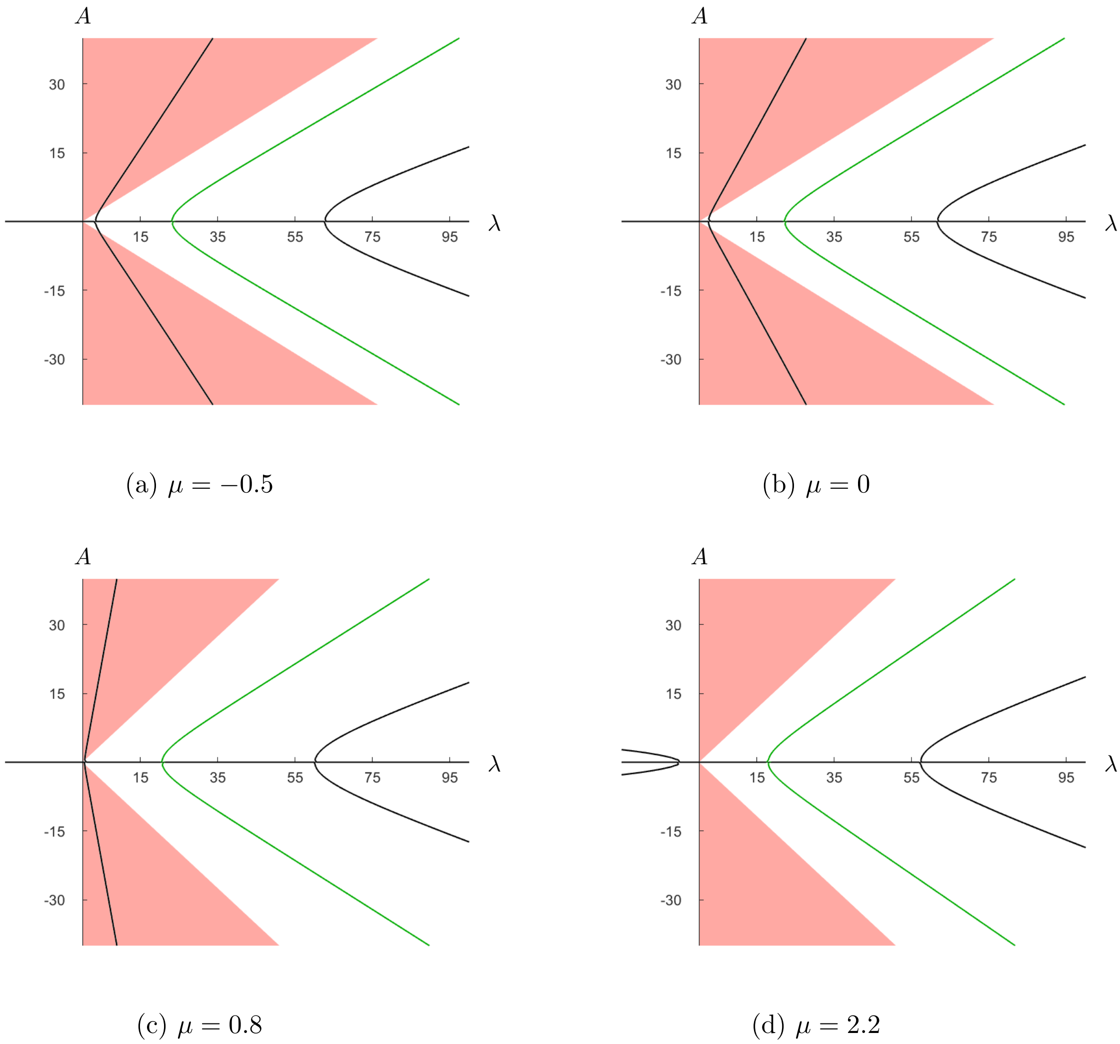


Fig. 2.3: Points (λ, A) (red) which certainly do not lie on the second branch (green) of the bifurcation diagram.

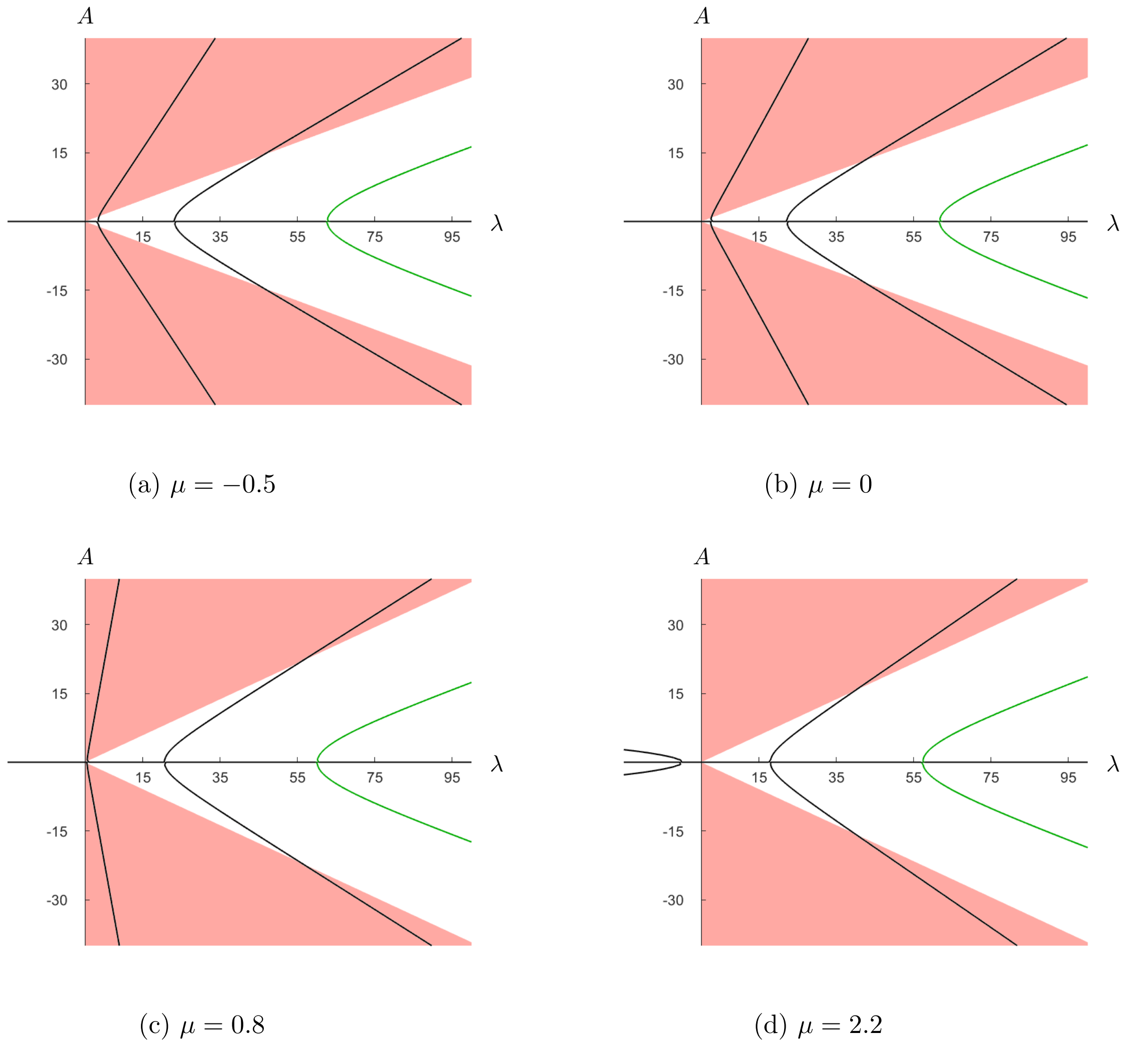
Lemma 2.20. *Let $m > 1$ and let u be a non-trivial solution of (2.1), with $g(u)$ satisfying (A1), (A2), (A3), and (A4). Then $\lambda > 2P^2$.*

Proof. As it was shown before, since $m > 1$, we have $\lambda > 0$ and there exists x_1^e on $[0, 1]$, because at least one arc is completed.
Let $u'(0) \neq 0$. By integrating the equation in (2.1) from 0 to x_1^e, we get

$$D = \lambda \left| \int_0^{x_1^e} g(u(x))\, \mathrm{d}x \right| \leq \int_0^{x_1^e} |g(u(x))|\, \mathrm{d}x.$$

Since $|u(x)| \geq |g(u(x))|$ for any x, we now have

$$D \leq \lambda \int_0^{x_1^e} |u(x)|\, \mathrm{d}x.$$



(a) $\mu = -0.5$

(b) $\mu = 0$

(c) $\mu = 0.8$

(d) $\mu = 2.2$

Fig. 2.4: Points (λ, A) (red) which certainly do not lie on the third branch (green) of the bifurcation diagram.

Since $|u(x)|$ is concave and positive on $(0, x_1^e)$ and it holds that $|u(x_1^e)| < D\, x_1^e$, we can say

$$\int_0^{x_1^e} |u(x)| \; \mathrm{d}x < \frac{D\,(x_1^e)^2}{2}.$$

Combining the last two inequalities together using Lemma 2.16, a trivial operation yields

$$2P^2 < \lambda,$$

which gives the claim. $\square$

Lemma 2.21. *Let u be a non-trivial solution of (2.1) with g satisfying (A1), (A2), (A3), and (A4). Let $\mu > 0$ and $m \geq 3$. Then the following holds:*

- *For m odd:*

$$\begin{aligned} \lambda &> \frac{(2m-1)^2\,(2m-2-\mu)}{m-1} && \textit{if} \quad \mu < 2(m-1), \\ \lambda &> (1-2m)(-2+2m-\mu) && \textit{if} \quad \mu > 2(m-1), \end{aligned}$$

- *for m even:*

$$\begin{aligned}
\lambda &> \frac{(2m-1)^2\,(2m-2-\mu)}{9(m-1)} \qquad \textit{if} \quad \mu < 2(m-1),\\
\lambda &> \frac{1}{3}(1-2m)(2m-2-\mu) \qquad \textit{if} \quad \mu > 2(m-1).
\end{aligned}$$

Proof. Without loss of generality, we assume $u'(0) > 0$.
If $m \geq 3$, we can use the periodicity of u given by Theorem 2.7 and described in Remark 2.9. Thus we can write:

$$\text{if } m \text{ is even, then} \qquad \int_0^1 g(u(x))\mathrm{d}x = \int_{x^p_{m-2}}^1 g(u(x))\mathrm{d}x,$$

$$\text{if } m \text{ is odd, then} \qquad \int_0^1 g(u(x))\mathrm{d}x = \int_{x^p_{m-1}}^1 g(u(x))\mathrm{d}x.$$

Now let us consider m odd. After integrating the equation in (2.1) from 0 to 1, we have

$$-u'(1) + u'(0) = \lambda \int_{x^p_{m-1}}^1 g(u(x))\mathrm{d}x. \tag{2.25}$$

Using (A4), we are allowed to apply a similar procedure as in the proof of Lemma 2.17. More specifically, we use

$$\int_{x^p_{m-1}}^1 g(u(x))\mathrm{d}x \leq \int_{x^p_{m-1}}^1 u(x)\mathrm{d}x < D\,\frac{(1-x^p_{m-1})^2}{2}.$$

Since $m > 1$, we get $\lambda > 0$ and the following holds

$$D < D\lambda\,\frac{(1-x^p_{m-1})^2}{2} + u'(1). \tag{2.26}$$

From Lemma 2.16, we have

$$(1-x^p_{m-1})^2 \leq \left(1 - \frac{2(m-1)}{2m-1}\right)^2 = \left(\frac{1}{2m-1}\right)^2$$

and that together with the boundary conditions in (2.1) yield

$$D\,\frac{2-\lambda\left(\frac{1}{2m-1}\right)^2}{2} < \mu u(1) < \mu M.$$

Using Lemma 2.17, we have

$$D\,\frac{2-\lambda\left(\frac{1}{2m-1}\right)^2}{2} < \mu\frac{D}{2(m-1)}.$$

Trivial operations yield

$$\frac{2-\frac{\mu}{m-1}}{\left(\frac{1}{2m-1}\right)^2} < \lambda,$$

which can be rewritten to

$$\frac{(2m-1)^2(2m-2-\mu)}{m-1} < \lambda.$$

Although this inequality is valid for any μ, the left hand side becomes non-positive for $\mu \geq 2(m-1)$ and we know that $\lambda > 0$. For this reason, we consider this inequality only for $\mu < 2(m-1)$.

For m even, equation (2.25) changes to

$$-u'(1) + u'(0) = \lambda \int_{x^p_{m-2}}^{1} g(u(x))\mathrm{d}x.$$

and as an analogy to (2.26) we have

$$D < \lambda \frac{(1 - x^p_{m-2})^2\, D}{2} + u'(1).$$

Using the same procedure as in the case with m odd yields

$$\frac{(2m-1)^2(2m-2-\mu)}{9(m-1)} < \lambda.$$

Again, this inequality comes to consideration only for $\mu < 2(m-1)$.

Now for m odd we know

$$\int_{x^p_{m-1}}^{1} g(u(x))\mathrm{d}x \leq \int_{x^p_{m-1}}^{1} u(x)\mathrm{d}x < (1 - x^p_{m-1})\, u(1),$$

which, applied on (2.25), yields

$$D < \lambda(1 - x^p_{m-1})u(1) + u'(1).$$

Using the boundary conditions, Lemma 2.16, Lemma 2.17, and relation $u(1) < M$, we get

$$2(m-1)M < \frac{\lambda}{2m-1} + \mu M,$$

which gives

$$(1-2m)(-2+2m-\mu) < \lambda,$$

which is useful only if $\mu > 2(m-1)$.

For m even and $\mu > 2(m-1)$, using the very same procedure, we obtain

$$\frac{1}{3}(1-2m)(2m-2-\mu) < \lambda.$$

□

Remark 2.22. It is possible to formulate Lemma 2.21 even for $\mu < 0$, however, the results would be always worse than the results given by Lemma 2.20.

* * *

If we apply tools of the Functional analysis, we can further develop the set of pairs $(\lambda, \mu) \in \mathbb{R}^2$ which do not provide a non-trivial solution.
Using a Green function, we will introduce the operator describing our BVP (2.1).

Definition 2.23. By the Green function corresponding to (2.1) we call any function meeting the following properties:

1. $\gamma(s, x)$ is continuous,
2. $\gamma''(s, x) = 0$ for $s \in (0, x)$ and $s \in (x, 1)$,
3. $\gamma'(x-, x) = \gamma'(x+, x) + 1$,
4. $\gamma(0, x) = 0$,
5. $\gamma'(1, x) = \mu\gamma(1, x)$.

Lemma 2.24. *Let $\mu \neq 1$ be an arbitrary constant. A function*

$$\gamma(s, x) := \begin{cases} \left(\dfrac{\mu}{1-\mu}x + 1\right) s & \textit{for } s \in [0, x], \\ \left(\dfrac{\mu}{1-\mu}s + 1\right) x & \textit{for } s \in (x, 1] \end{cases} \tag{2.27}$$

is the Green function corresponding to BVP (2.1).

Proof. The Green function $\gamma(s, x)$, $x \in [0, 1]$, $s \in [0, 1]$ for (2.1) must satisfy the properties 1 - 5 from Definition 2.23.
Since $\gamma''(s, x) = 0$ from property 2, the Green function can be generally written as

$$\gamma(s, x) := \begin{cases} A(x)s + B(x) & \text{for } s \in [0, x], \\ C(x)s + D(x) & \text{for } s \in (x, 1] \end{cases}$$

and naturally

$$\gamma'(s, x) := \begin{cases} A(x) & \text{for } s \in [0, x], \\ C(x) & \text{for } s \in (x, 1]. \end{cases}$$

Property 4 causes

$$B(x) = 0. \tag{2.28}$$

Relation in property 5 yields $C = \mu(C + D)$, thus for $\mu \neq 1$

$$C(x) = \frac{\mu}{1-\mu} D(x). \tag{2.29}$$

From property 3, we get

$$A(x) = C(x) + 1 \tag{2.30}$$

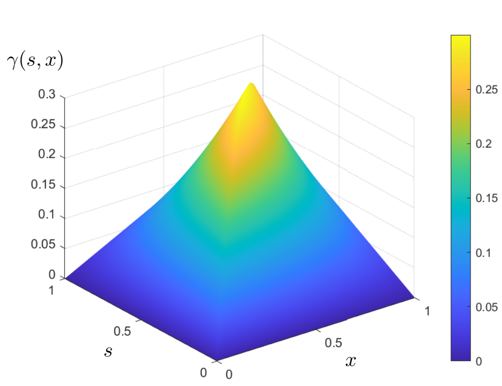
and property 1 used for $s = x$ gives

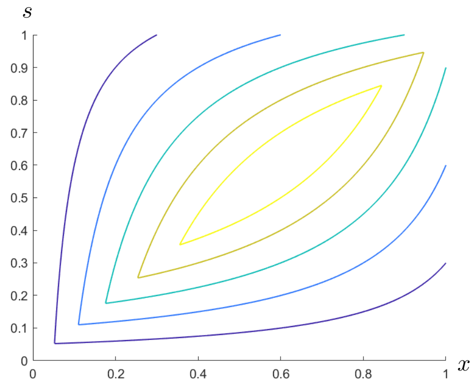
$$\begin{aligned} A(x)\, x + B(x) &= C(x)\, x + D(x), \\ \frac{\mu}{1-\mu} D(x)\, x + x &= \frac{\mu}{1-\mu} D(x)\, x + D(x), \end{aligned}$$

thus

$$D(x) = x. \tag{2.31}$$

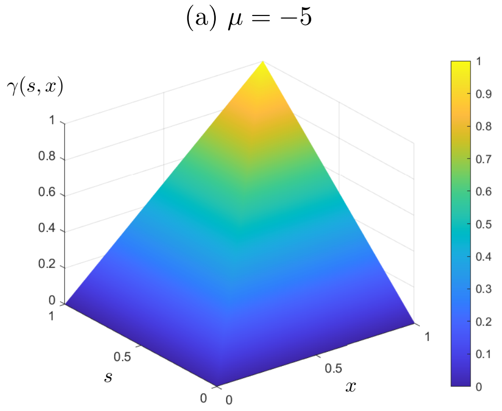
Identities (2.28)-(2.31) imply the statement. □

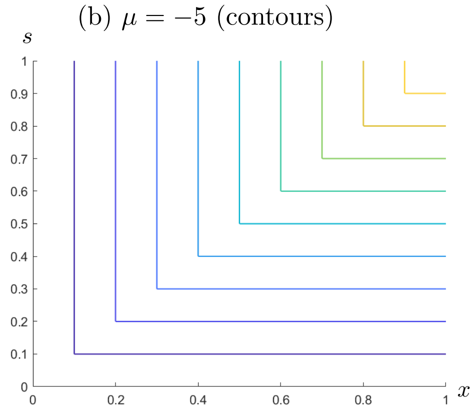




(a) $\mu = -5$

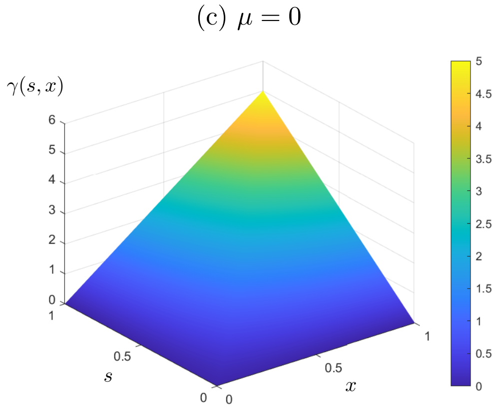
(b) $\mu = -5$ (contours)

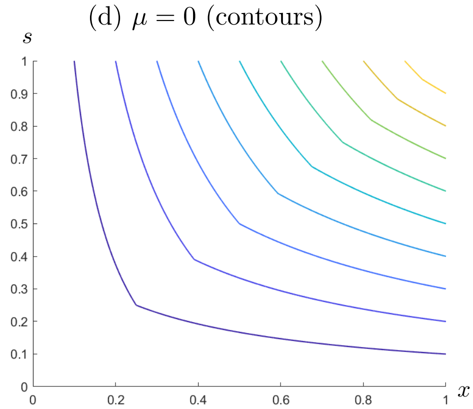




(c) $\mu = 0$

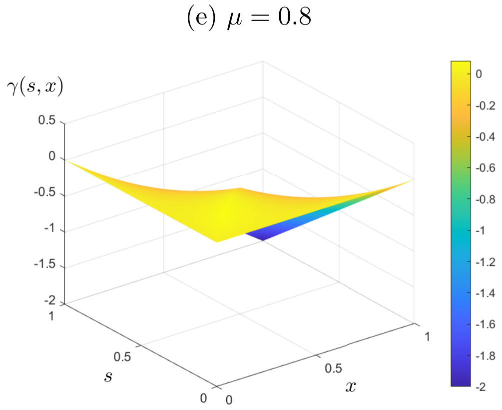
(d) $\mu = 0$ (contours)

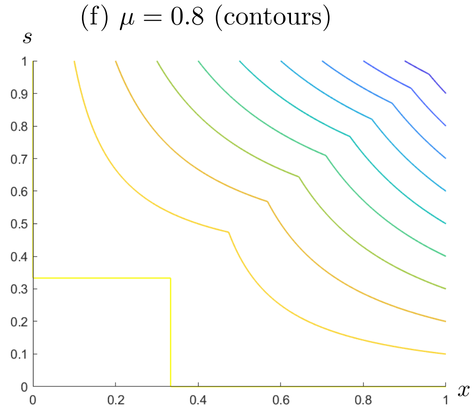




(e) $\mu = 0.8$

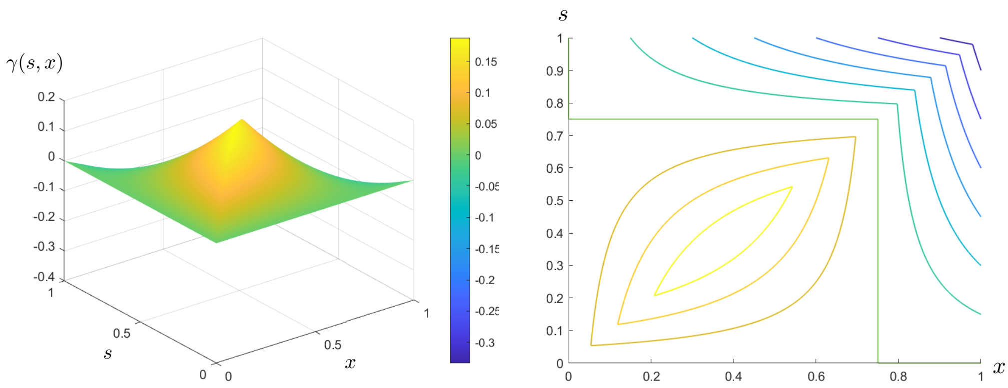
(f) $\mu = 0.8$ (contours)





(g) $\mu = 1.5$

(h) $\mu = 1.5$ (contours)



(i) $\mu = 4$ (j) $\mu = 4$ (contours)

Fig. 2.5: Green function and its contours for the particular choices of μ

The Green function (2.27) for a few values of μ is shown in the Figure 2.5.

Lemma 2.25. *Let $\mu \neq 1$ and let $T : C([0,1]) \to C([0,1])$ be an operator defined as*

$$T(u)(x) := \lambda \int_0^1 \gamma(s,x)\, g(u(s))\, ds$$

with Green function $\gamma(s,x)$. Let $u(x)$ be a solution of (2.1). Then $u(x) = T(u)(x)$.

Proof. Apparently, T is well-defined and any continuous function is mapped on a continuous function.
Let us multiply the equation in (2.1) by $\gamma(s,x)$ and then let us integrate the equation over $(0,1)$ with respect to s. That yields

$$-\int_0^t u''(s)\gamma(s,x)\mathrm{d}s - \int_t^1 u''(s)\gamma(s,x)\mathrm{d}s = \lambda \int_0^1 \gamma(s,x)g(u(s))\mathrm{d}s = T(u)(x).$$

Integrating by parts gives

$$-\big[u'(s)\gamma(s,x) - u(s)\gamma'(s,x)\big]_{s=0}^{s=t} - \big[u'(s)\gamma(s,x) - u(s)\gamma'(s,x)\big]_{s=t}^{s=1} = \lambda \int_0^1 \gamma(s,x)g(u(s))\mathrm{d}s$$

$$\begin{aligned}-u'(x)\gamma(x-,x) + u(x)\gamma'(x-,x) + u'(0)\gamma(0,x) - u(0)\gamma'(0,x) - u'(1)\gamma(1,x) + u(1)\gamma'(1,x) +\\ +u'(x)\gamma(x+,x) - u(x)\gamma'(x+,x) = \lambda \textstyle\int_0^1 \gamma(s,x)g(u(s))\mathrm{d}s.\end{aligned}$$

The boundary conditions of (2.1) and properties 4, 5 from Definition 2.23 state:

$$u(0) = \gamma(0,x) = 0,\ u'(1) = \mu u(1), \text{ and } \gamma'(1,x) = \mu\gamma(1,x).$$

With that being used, we get

$$\begin{aligned}&u(x)\left(\gamma'(x-,x) - \gamma'(x+,x)\right) + u'(x)\left(\gamma(x+,x) - \gamma(x-,x)\right) + u(1)\left(-\mu\gamma(1,x) + \mu\gamma(1,x)\right) =\\ &= \lambda \int_0^1 \gamma(s,x)g(u(s))\mathrm{d}s.\end{aligned}$$

Properties 1 and 3 from Definition 2.23 yield

$$u(x) = \lambda \int_0^1 \gamma(s,x)g(u(s))\mathrm{d}s.$$

□

Using Banach contraction principle, the operator T can be utilized in the proof of the following Theorem, which determines the pairs (λ, μ) which do not provide any solution but the trivial one.

Theorem 2.26. *Let u be a non-trivial solution of (2.1) with g satisfying $g(0) = 0$, (A2) with L as a constant of Lipschitz continuity. Moreover, let us suppose that $\mu \neq 1$. Then necessarily*

$$|\lambda| \geq \frac{1}{S\,L},$$

where

$$S := \begin{cases} \dfrac{(2-\mu)^2}{2(2-2\mu)^2} & \text{if } \mu \leq 0, \\[2ex] \dfrac{\mu}{2(1-\mu)} + \dfrac{1}{2} & \text{if } \mu \in (0,1), \\[2ex] \dfrac{(\mu^2-2\mu+2)^2}{2(2\mu^2-2\mu)^2} & \text{if } (\mu > 2) \ \wedge \ (\mu^4 - 8\mu^3 + 12\mu^2 - 8\mu + 4 > 0), \\[2ex] \dfrac{-\mu}{2(1-\mu)} - \dfrac{1}{2} & \text{otherwise.} \end{cases}$$

Before we prove Theorem 2.26, we introduce the following Lemma showing a connection of the Green function (2.27) and the constant S defined in Theorem 2.26. This Lemma will be used in the proof of Theorem 2.26.

Lemma 2.27. *Let $\mu \neq 1$ be an arbitrary constant and let $\gamma(s, x)$ be defined by (2.27). Then*

$$J(\mu, x) := \int_0^1 |\gamma(s,x)|\,ds \leq S = \max_{x\in[0,1]} \int_0^1 |\gamma(s,x)|\,ds.$$

Proof. In this proof, we will look for x such that $\int_0^1 |\gamma(s,x)| \mathrm{d}s$ is maximal for a particular μ. This value of x will be denoted by $x_m(\mu)$. Existence of such $x_m(\mu)$ on $[0,1]$ is given by Weierstrass maximum-value Theorem (see [7, p. 161, Th. 3]).
Since

$$\frac{\mu}{1-\mu} = -1 + \frac{-1}{\mu - 1},$$

for $\gamma(s,t)$, we have to distinguish the three following cases with respect to μ. These cases are also described in Figure 2.6.

1. $\mu \leq 0$:
   In this instance $\frac{\mu}{1-\mu} \in (-1, 0]$.
   Since $\gamma(s,x)$ is positive for any allowed s and x, we can write

$$J = \int_0^x \left(\frac{\mu}{1-\mu} x\,s + s\right) \mathrm{d}s + \int_x^1 \left(\frac{\mu}{1-\mu} x\,s + x\right) \mathrm{d}s. \tag{2.32}$$

   After integrating we get

$$J = \frac{-x^2}{2} + \left(\frac{\mu}{2\,(1-\mu)} + 1\right) x$$

and then

$$J' := \frac{\partial J}{\partial x} = \frac{\mu}{2\,(1-\mu)} + 1 - x.$$

Thus, for $x \in [0,1]$, we consider three suspicious points $x_1 = 0$, $x_2 = \frac{\mu}{2\,(1-\mu)} + 1$, $x_3 = 1$. Clearly, x_1 is a point of minimum.

Since $J(\mu, x_2) = \frac{(2-\mu)^2}{2(2-2\mu)^2}$ and $J(\mu, x_3) = \frac{1}{2-2\mu}$, after trivial operation we get

$$J(\mu, x_3) \leq J(\mu, x_2) \Leftrightarrow \mu^2(\mu - 1) < 0,$$

which is certainly satisfied for $\mu \leq 0$.

Hence for $\mu \leq 0$, $x_m(\mu) = \frac{\mu}{2\,(1-\mu)} + 1$ and $J(\mu, x_m(\mu)) = \frac{(2-\mu)^2}{2(2-2\mu)^2}$.

2. $0 < \mu < 1$:
   In this case, $\frac{\mu}{1-\mu} > 0$ and relation (2.32) is preserved. However, in this case we have only two suspicious points $x_1 = 0$ and $x_3 = 1$, because $x_2 \notin [0,1]$. Since x_1 is clearly a point of minimum, we directly get

$$\begin{aligned} x_m(\mu) &= 1, \\ J(\mu, x_m(\mu)) &= \frac{\mu}{2(1-\mu)} + \frac{1}{2}. \end{aligned}$$

3. $\mu > 1$:
   Now $\frac{\mu}{1-\mu} < -1$ and a curse of the functions in the arguments of the integrals in J depends on x.

   For $\frac{\mu}{1-\mu} x\, s + s$ we know:

   - it is equal to zero for $s = 0$,
   - it is strictly increasing for $x \in \left[0, 1 - \frac{1}{\mu}\right)$ and strictly decreasing for $x \in \left(1 - \frac{1}{\mu}, 1\right]$,
   - it does not change its sign for any $s \in (0,1]$.

   Similarly, for $\frac{\mu}{1-\mu} x s + x$ we have:

   - regardless of x, it is strictly decreasing,
   - it is negative if $s = 1$ and $x \in [0,1]$,
   - if it changes its sign, the change occurs for $s = 1 - \frac{1}{\mu}$.

   Considering all the properties together, we have to furthermore distinguish two cases with respect to x:

   (a) $0 \leq x < 1 - \frac{1}{\mu} < 1$:
   We get

$$J = \int_0^x \left(\frac{\mu}{1-\mu} x\, s + s \right) \mathrm{d}s + \int_x^{1-\frac{1}{\mu}} \left(\frac{\mu}{1-\mu} x\, s + x \right) \mathrm{d}s +$$

$$+ \int_{1-\frac{1}{\mu}}^{1} \left(-\frac{\mu}{1-\mu} x\, s - x \right) \mathrm{d}s$$

and

$$J' = -x + \frac{\mu^2 - 2\mu + 2}{2\mu^2 - 2\mu}.$$

Thus we have 2 suspicious points $x_1 = 0$, $x_2 = 1 - \frac{1}{\mu}$ and the third potential suspicious point $x_3 = \frac{\mu^2-2\mu+2}{2\mu^2-2\mu}$.

We have to examine a range of μ which x_3 is in interval $\left(0, 1 - \frac{1}{\mu}\right)$ for.

Clearly, $x_3 > 0$. Considering

$$x_3 < 1 - \frac{1}{\mu},$$

trivial operations simplifies the inequality to

$$\mu^2(\mu - 2) > 0.$$

Thus x_3 is a suspicious point if $\mu > 2$.

(b) $1 - \frac{1}{\mu} \leq x \leq 1$:
We have:

$$\begin{aligned} J &= \int_0^x \left(-\frac{\mu}{1-\mu} x\, s - s \right) \mathrm{d}s + \int_x^1 \left(-\frac{\mu}{1-\mu} x\, s - x \right) \mathrm{d}s, \\ J' &= x - 1 - \frac{\mu}{2(1-\mu)}. \end{aligned}$$

Since $1 + \frac{\mu}{2(1-\mu)} < 1 - \frac{1}{\mu}$, we get the only suspicious point $x_4 = 1$.

Now we have to choose the correct point of maximum.

Again, x_1 is a point of minimum.

Inequality $J(\mu, x_2) < J(\mu, x_4)$ holds if and only if $\mu^2 - \mu + 1 > 0$, which is satisfied for any μ.

An inequality $J(\mu, x_3) > J(\mu, x_4)$ can be reduced to

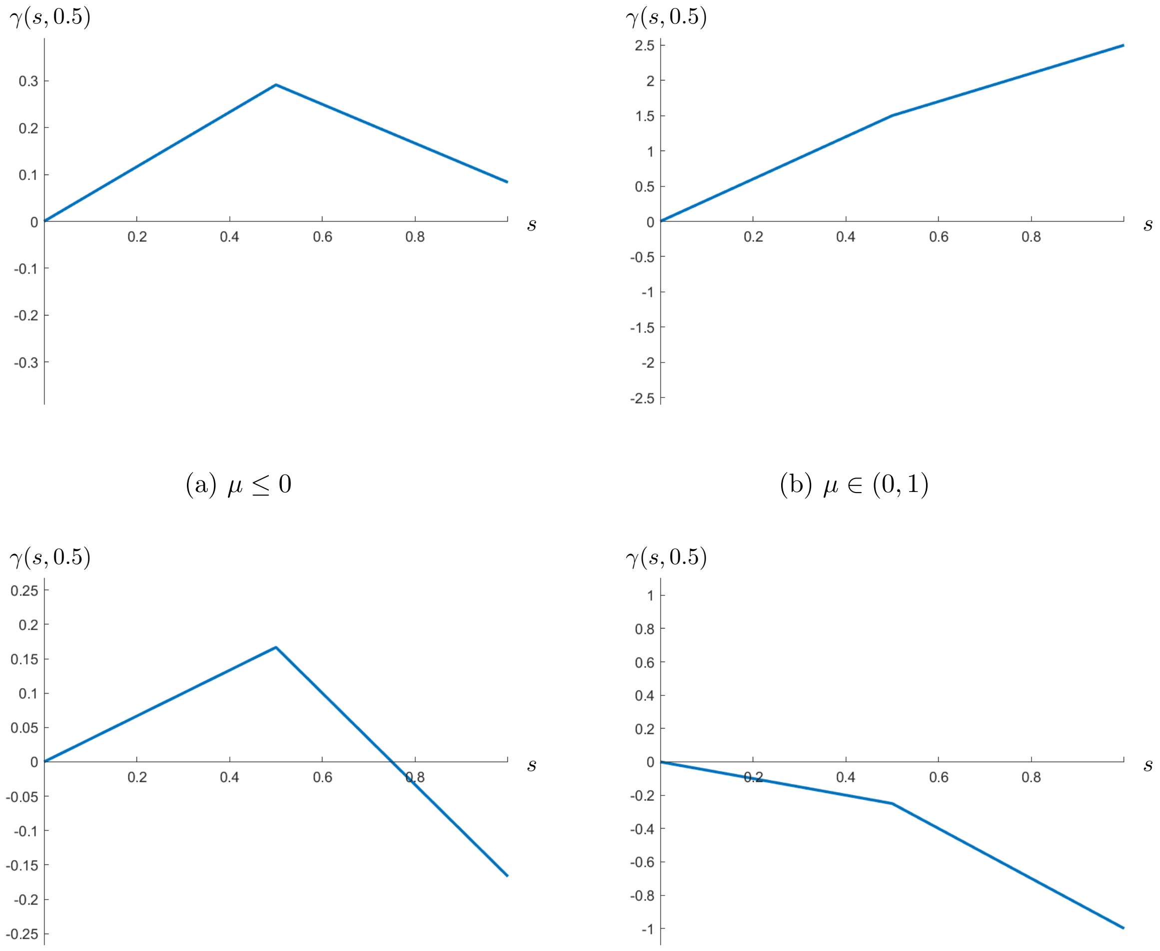
$$\mu^4 - 8\mu^3 + 12\mu^2 - 8\mu + 4 > 0.$$

Thus if $\mu > 2$ and $\mu^4 - 8\mu^3 + 12\mu^2 - 8\mu + 4 > 0$, then $x_m(\mu) = \frac{\mu^2-2\mu+2}{2\mu^2-2\mu}$ and $J(\mu, x_m(\mu)) = \frac{(\mu^2-2\mu+2)^2}{2(2\mu^2-2\mu)^2}$. Otherwise $x_m(\mu) = 1$ and $J(\mu, x_m(\mu)) = -\frac{\mu}{2(1-\mu)} - \frac{1}{2}$.

Hence

$$S = \max_{x \in [0,1]} \int_0^1 |\gamma(s, x)|\, \mathrm{d}s.$$

□



(a) $\mu \leq 0$

(b) $\mu \in (0, 1)$

(c) $\mu > 1$ and $0 \leq x < 1 - \frac{1}{\mu} \leq 1$

(d) $\mu > 1$ and $1 - \frac{1}{\mu} \leq x \leq 1$

Fig. 2.6: Green function $\gamma(s, x)$ for $x = 0.5$ and the particular μ.

Remark 2.28. It can be shown that the condition

$$(\mu > 2) \;\; \wedge \;\; (\mu^4 - 8\mu^3 + 12\mu^2 - 8\mu + 4 > 0)$$

can be reduced to $\mu > 2 + \sqrt{3} + \sqrt{3 + 2\sqrt{3}} \approx 6.2745$.

Proof (of Theorem 2.26). Let $(C([0, 1]), \rho_\infty)$ be a metric space of continuous functions on $[0, 1]$ with a metric ρ_∞ induced by a norm

$$\|f\|_\infty := \sup_{x \in [0,1]} |f(x)|.$$

Firstly, we will prove that $T(u)(x)$ brought by Lemma 2.25 is a contraction mapping. For any $u_1, u_2 \in C([0, 1])$ and any $x \in [0, 1]$, we have

$$K(x) := \left| \lambda \int_0^1 \gamma(s, x)(g(u_1(s)) - g(u_2(s)))\mathrm{d}s \right| \leq |\lambda| \int_0^1 |\gamma(s, x)(g(u_1(s)) - g(u_2(s)))|\mathrm{d}s,$$

where $\gamma(s,x)$ corresponds to (2.27).
Using the assumption (A2) with L being the constant of Lipschitz continuity for g, we get

$$K(x) \le L|\lambda| \int_0^1 |\gamma(s,t)||u_1(s) - u_2(s)|\mathrm{d}s.$$

Using Lemma 2.27 yields

$$K(x) \le SL\,|\lambda| \sup_{s\in[0,1]} |u_1(s) - u_2(s)|.$$

Since $|\lambda| < \frac{1}{SL}$, it holds that

$$K(x) \le q\,\rho_\infty(u_1,u_2), \quad q \in [0,1).$$

The last inequality is valid for any $x \in [0,1]$ and $\rho_\infty(T(u_1),T(u_2)) = \sup_{x\in[0,1]} K(x)$, thus, undoubtedly,

$$\rho_\infty(T(u_1),T(u_2)) \le q\,\rho_\infty(u_1,u_2), \quad q \in [0,1),$$

in other words T is a contraction mapping.
Since we assume $g(0) = 0$, BVP (2.1) has always a trivial solution. Moreover, since (X,ρ_∞) is a complete metric space, we can apply Banach Fixed-Point Theorem (see [4, p. 300, Th. 5.1-2]), which yields that there exists exactly one $u \in C([0,1])$ such that $T(u)(x) = u(x)$. That means that if $|\lambda| < \frac{1}{SL}$, BVP (2.1) has only the trivial solution. □

To visualize the results of Theorem 2.26 we can use the bifurcation diagram shown in Figure 2.1. In Figure 2.9, there are the pairs (λ, A) which, according to the Theorem 2.26, certainly do not provide any non-trivial solution. From Figure 2.9, it is certain that the accuracy of the limitation for λ differs with respect to μ.
In the beginning of this chapter, we described the connection of the bifurcation diagram for $g(u) = \arctan u$ with the linear problem (1.1), (1.2). Since, for the particular μ, (1.1), (1.2) becomes a Sturm-Liouvell problem, a set containing absolute values of all the eigenvalues of (1.1), (1.2) has a minimum. The minimum will be denoted by λ_{min}.
If we compare λ_{min} and the results of Theorem 2.26 (see Figure 2.10), we get a diagram of the accuracy of $\frac{1}{SL}$ with respect to μ. This diagram is shown in Figure 2.11.

2.4 Multiplicity of the solution

Since BVP (2.1) contains two parameters λ and μ, it would be probably useful to determine which pairs $(\lambda,\mu) \in \mathbb{R}^2$ provide a non-trivial solution.
For simplicity, let us stick with $g(u) = \arctan u$, which satisfies all the assumptions (A1)–(A4). Going through the values of $\lambda > 0$ from zero to the infinity, the experiments showed that once new branch comes into being, it never disappears. This observation can be partially supported by Proposition 2.18.
That means that for any fixed $\mu \in \mathbb{R}$, if we choose a pair (λ,μ) for any λ bigger than the lowest positive eigenvalue of (1.1), (1.2), we always get at least one non-trivial solution.
Therefore, it makes more sense to pose a different question: for a certain pair $(\lambda,\mu) \in \mathbb{R}^2$, how many non-trivial solution we can get from BVP (2.1)?
This thesis will not answer this question completely, but, according to the experiments, for this particular choice of g, the multiplicity of the non-trivial solution is strictly connected to the set Σ, shown in Figure 1.2.

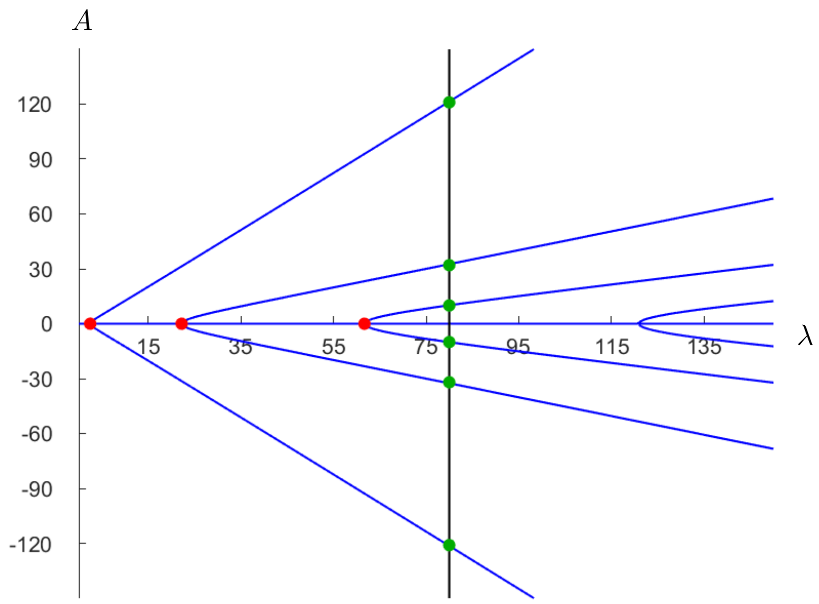


Fig. 2.7: Non-trivial solutions (green) of BVP (2.1) for $g(u) = \arctan u$ mapped in the bifurcation diagram for $\lambda = 80$ (black line) and $\mu = 0$ with eigenvalues of (1.1), (1.2) (red).

To be more specific, let us have $g = \arctan u$ and a pair $(\tilde{\lambda}, \tilde{\mu}) \in \mathbb{R}^+ \times \mathbb{R}$. Let us assume that for $\mu = \tilde{\mu}$, linear problem (1.1), (1.2) has k positive eigenvalues lower than $\tilde{\lambda}$. Then for $(\lambda, \mu) = (\tilde{\lambda}, \tilde{\mu})$, BVP (2.1) has exactly $2k$ non-trivial solutions. Moreover, since g satisfies (A3), it is certain that if (λ, μ) provides a non-trivial solution u, then $-u$ is also a non-trivial solution. In Figure 2.7, there is shown an example for $g(u) = \arctan u$ and $(\tilde{\lambda}, \tilde{\mu}) = (80, 0)$. In the bifurcation diagram, there are all the positive eigenvalues lower than 80 marked by red points and, as green points, there can be seen all six non-trivial solutions for this particular configuration.
For $\lambda < 0$, we get a completely different result. As it was proved in Corollary 2.19, any pair $(\lambda, \mu) \in \mathbb{R}^- \times \mathbb{R}^-$ do not provide any non-trivial solution. As Figures 1.2 and 2.1 suggest, a branch in the bifurcation diagram for $\lambda < 0$ appears only for $\mu > 1$. It seems that for $(\lambda, \mu) \in \mathbb{R}^- \times (1, \infty)$, there exists two non-trivial solutions u and $-u$ as long as λ is lower than the only negative eigenvalue of (1.1), (1.2) for μ given.
This is illustrated in Figure 2.8.

Additionally, using the results of Corollary 2.19, Lemmas 2.17, 2.21, and Theorem 2.26 for a certain m, we can numerically generate a set of pairs (λ, μ) which certainly do not provide a non-trivial solution with m nodes.
In Figure 2.12, there are several examples of the sets of (λ, μ) which certainly do not provide a solution with m given.

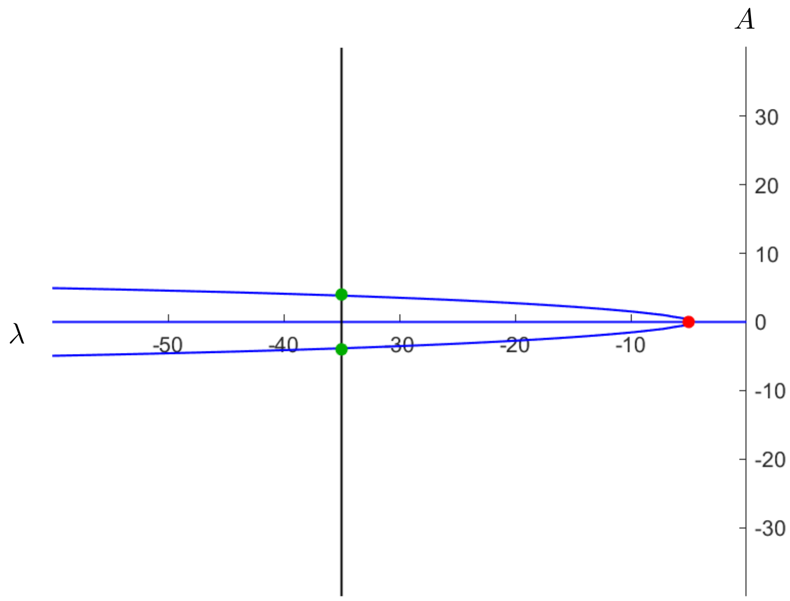


Fig. 2.8: Non-trivial solutions (green) of BVP (2.1) for $g(u) = \arctan u$ mapped in the bifurcation diagram for $\lambda = -35$ (black line) and $\mu = 2.2$ with eigenvalues of (1.1), (1.2) (red).

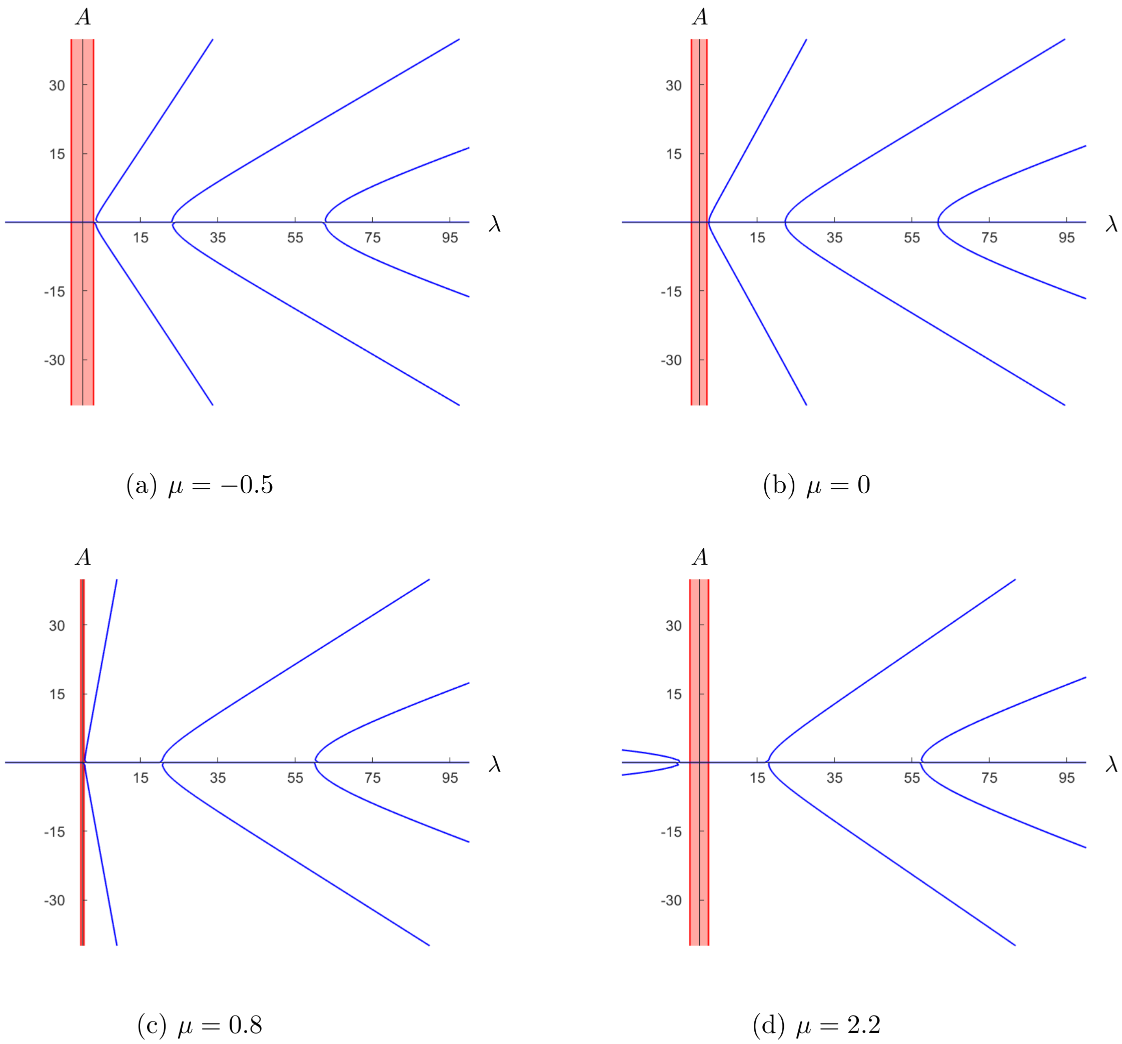


Fig. 2.9: Examples of the pairs (λ, A) (red) which, according to the Theorem 2.26, do not provide any trivial solution.

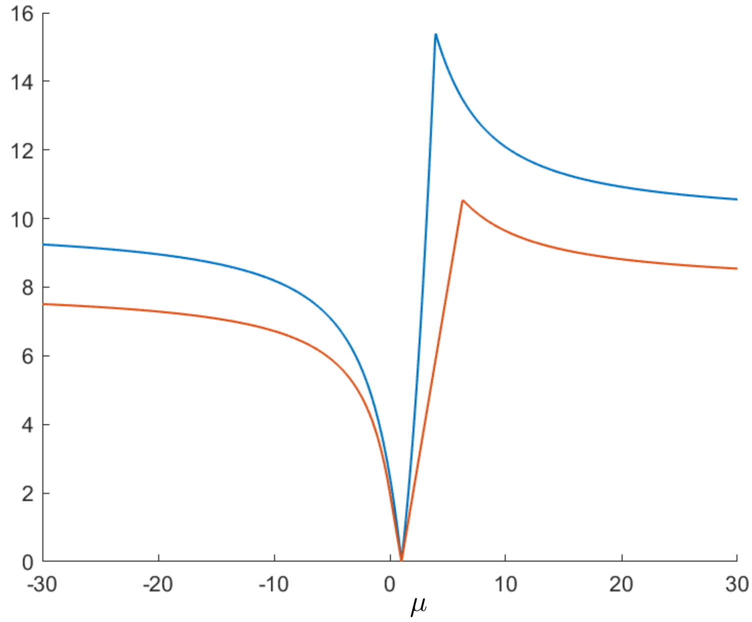


Fig. 2.10: λ_{min} (blue) and the values of $\frac{1}{SL}$ for $g(u) = \arctan u$ (orange).

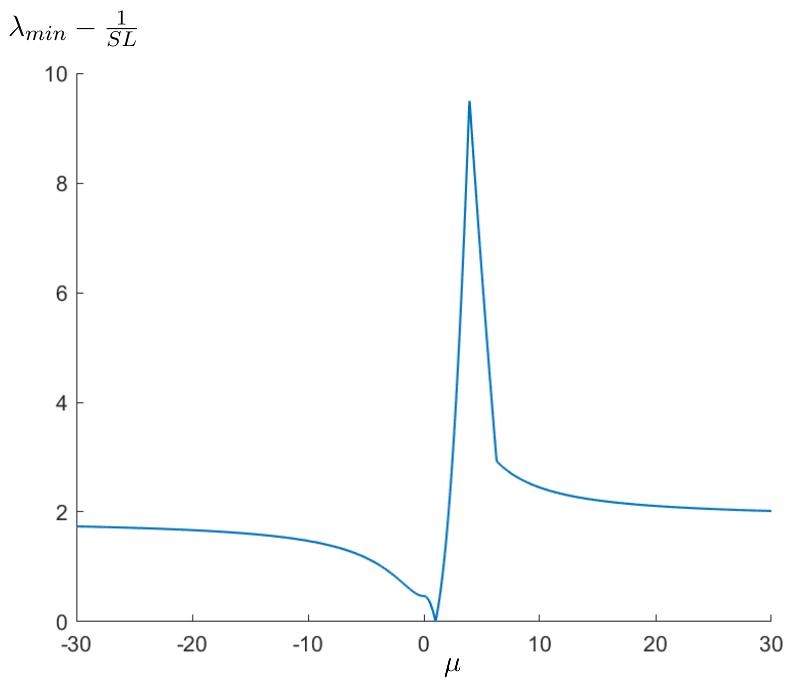
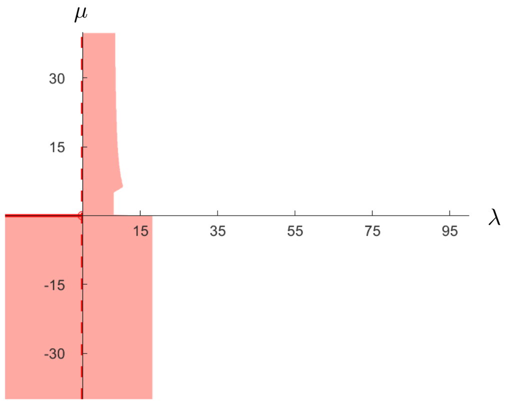
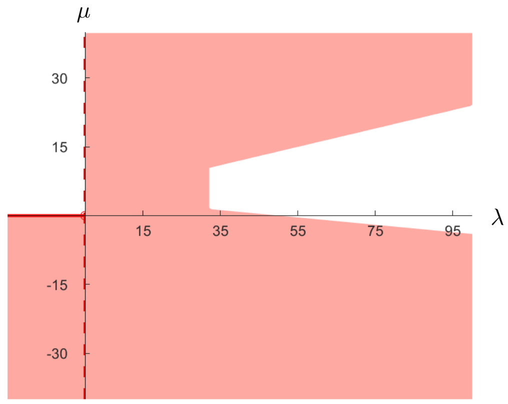


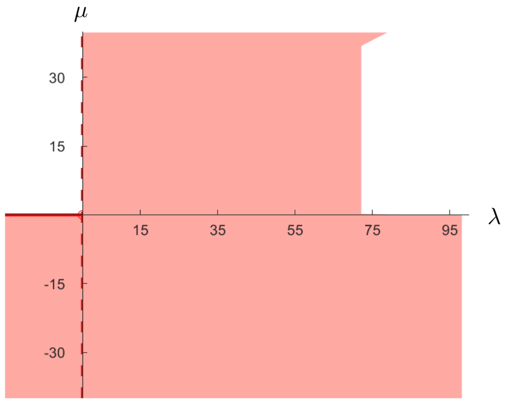
Fig. 2.11: Accuracy of the limitations brought by Theorem (2.26) for $g(u) = \arctan u$, i.e.: value of $\lambda_{min} - \frac{1}{SL}$.



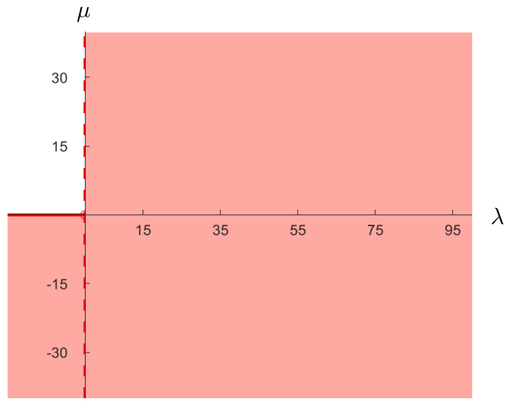
(a) $m = 2$



(b) $m = 3$



(c) $m = 4$

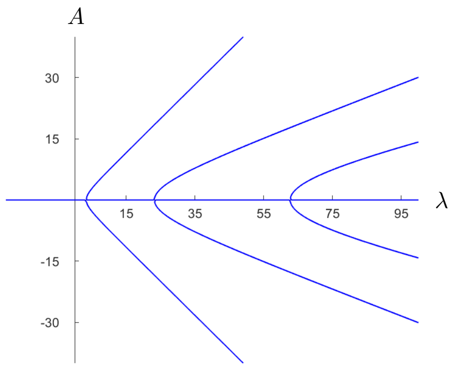


(d) $m = 5$

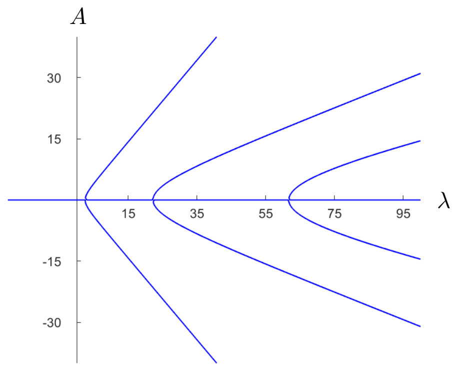
Fig. 2.12: The pairs (λ, μ) which certainly do not provide any non-trivial solution with m given (red).

Appendix A

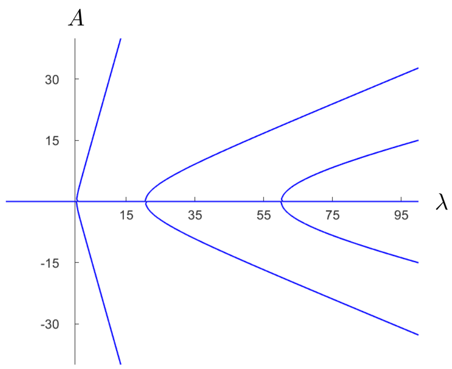
Bifurcation diagrams



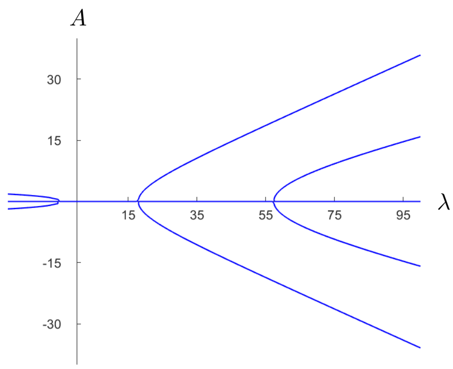
(a) $\mu = -0.5$



(b) $\mu = 0$

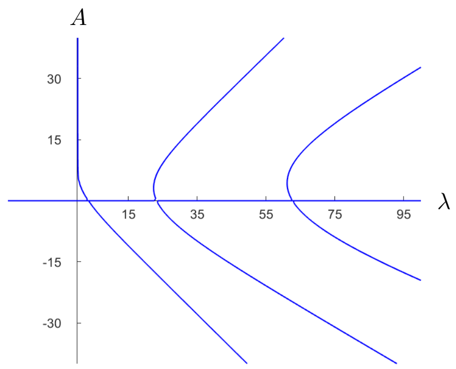


(c) $\mu = 0.8$

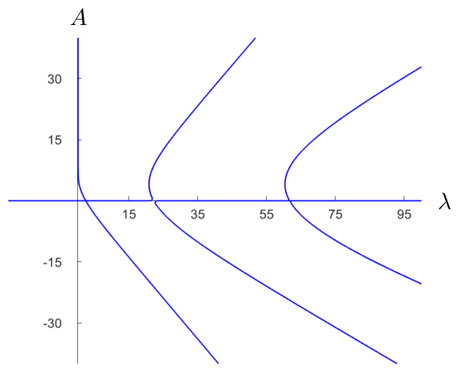


(d) $\mu = 2.2$

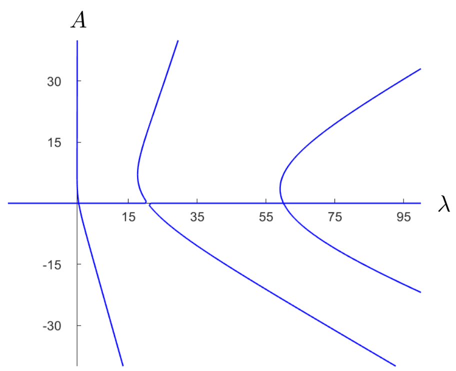
Fig. A.1: Bifurcation diagram for $g(u) = \tanh u$ and the particular choices of μ.



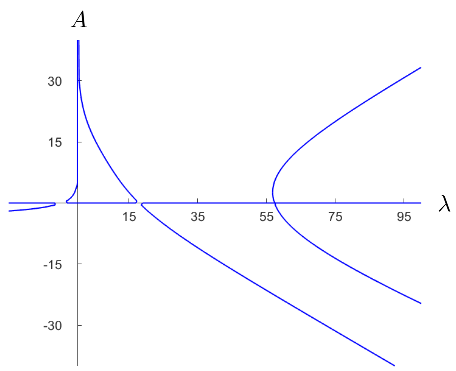
(a) $\mu = -0.5$



(b) $\mu = 0$

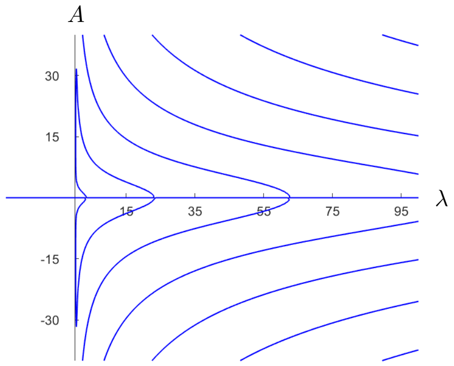


(c) $\mu = 0.8$

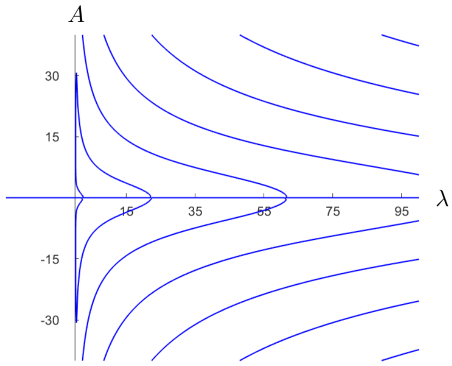


(d) $\mu = 2.2$

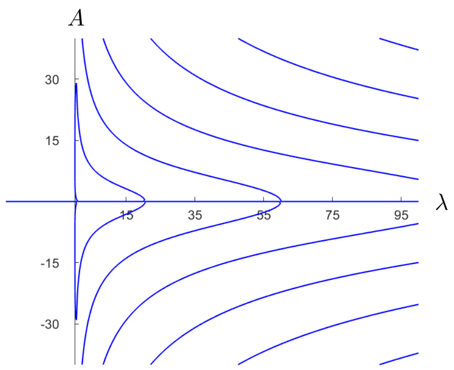
Fig. A.2: Bifurcation diagram for $g(u) = e^u - 1$ and the particular choices of μ.



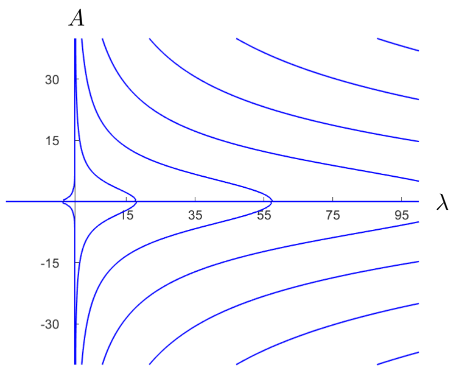
(a) $\mu = -0.5$



(b) $\mu = 0$

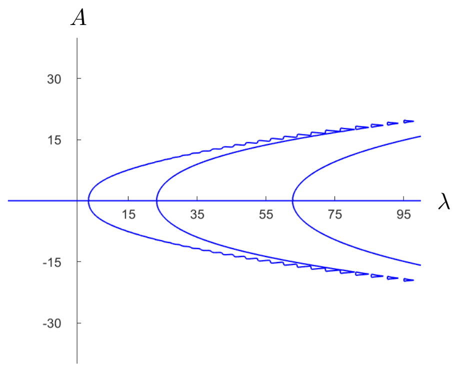


(c) $\mu = 0.8$

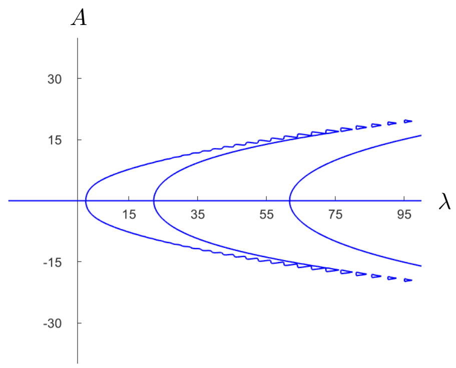


(d) $\mu = 2.2$

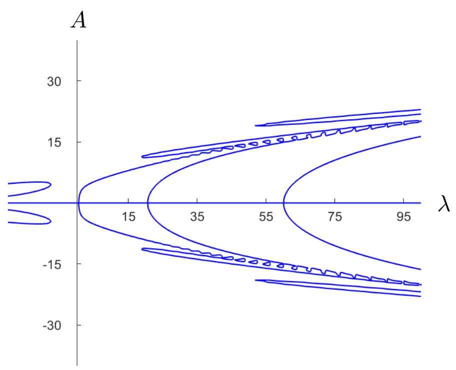
Fig. A.3: Bifurcation diagram for $g(u) = u + u^3$ and the particular choices of μ.



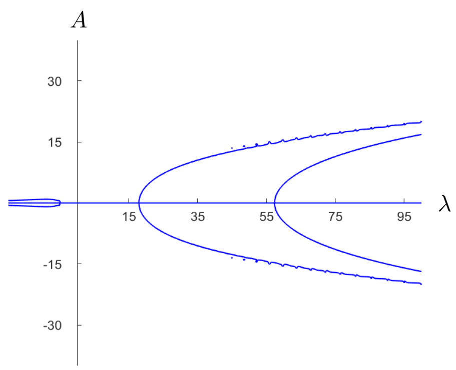
(a) $\mu = -0.5$



(b) $\mu = 0$

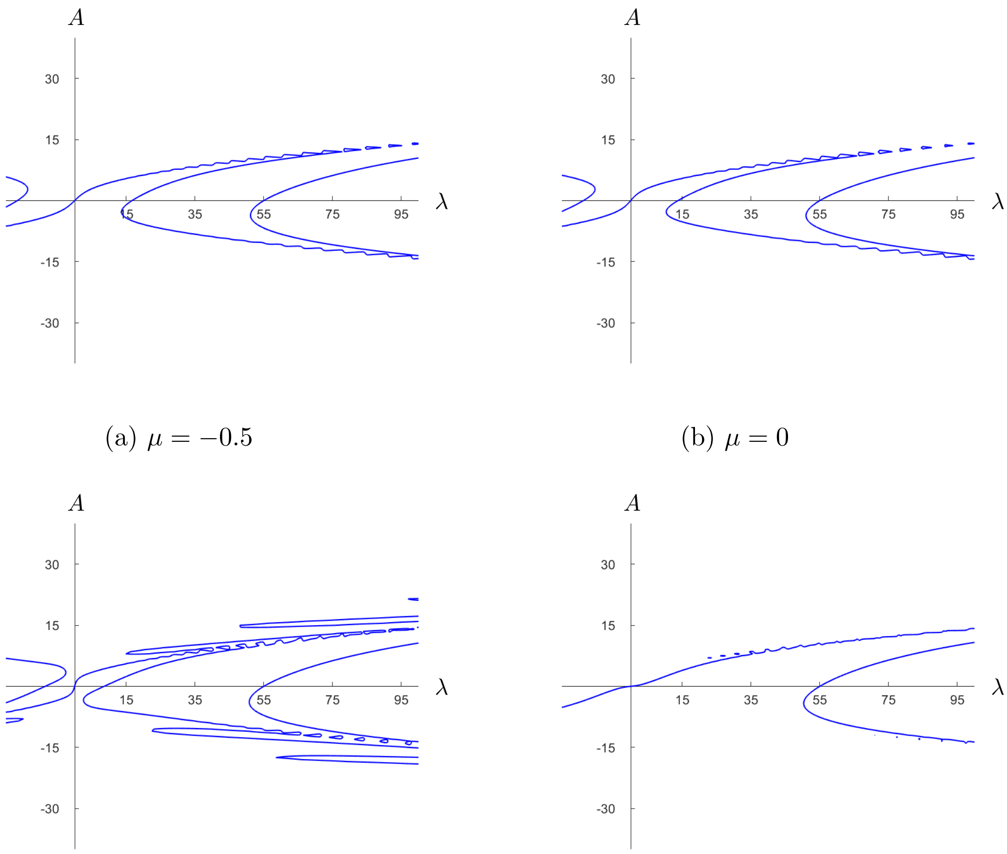


(c) $\mu = 0.8$



(d) $\mu = 2.2$

Fig. A.4: Bifurcation diagram for $g(u) = \sin u$ and the particular choices of μ.



(a) $\mu = -0.5$

(b) $\mu = 0$

(c) $\mu = 0.8$

(d) $\mu = 2.2$

Fig. A.5: Bifurcation diagram for $g(u) = \cos u$ and the particular choices of μ.

Bibliography

[1] F. V. Atkinson and A. B. Mingarelli. *Multiparameter Eigenvalue Problems: Sturm-Liouville Theory.* CRC Press, 2010.

[2] P. Binding and H. Volkmer. Eigencurves for two-parameter sturm-liouville equations. *SIAM review*, 38(1):27–48, 1996.

[3] W. G. Kelley and A. C. Peterson. *The theory of differential equations: classical and qualitative.* Springer Science & Business Media, 2010.

[4] E. Kreyszig. *Introductory Functional Analysis with Applications*, volume 1. wiley New York, 1978.

[5] F. Sadyrbaev and A. Gritsans. Nonlinear Spectra for Parameter Dependent Ordinary Differential Equations. *Nonlinear analysis: modelling and control*, 12(2):253–267, 2007.

[6] G. Teschl. *Ordinary Differential Equations and Dynamical Systems*, volume 140. American Mathematical Soc., 2012.

[7] V. A. Zorich. *Mathematical Analysis 1.* Springer, 2004.