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Travelling waves for generalized Fisher–Kolmogorov equation with density-dependent degenerate and singular diffusion

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1 Introduction

We are concerned with the quasilinear reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(d(u) \left| \frac{\partial u}{\partial x} \right|^{p-2} \frac{\partial u}{\partial x} \right) + g(u), \quad (x,t) \in \mathbb{R} \times [0, +\infty), \quad p > 1, \tag{1}$$

and its travelling wave solutions $u(x,t) = U(x - ct)$, where c denotes the (unknown) speed of propagation. For $p = 2$, equation (1) has been widely studied. It is used to model various phenomena in population dynamics, such as the spread of a favoured gene, proposed by Fisher (1937), or density-dependent dispersal, cf. Murray (2002).

In contrast to classical models, we assume weaker conditions on the reaction and diffusion terms, which have not yet appeared in literature in this generality. Our aim is to provide a broad theoretical background for the mathematical treatment of rather general models and to discuss conditions for the existence of solutions even under very weak assumptions.

2 Problem setting

We consider the reaction term $g = g(s)$ to be a continuous, possibly non-Lipschitz function such that $g(0) = g(1) = 0$, $g > 0$ in $(0,1)$. The diffusion coefficient $d : [0,1] \to \mathbb{R}$, $d > 0$ in $(0,1)$, need not be continuous in $[0,1]$ or even in $(0,1)$. In particular, d may vanish or be singular at one or both endpoints and it may also have discontinuities of the first kind at a finite number of points in $(0,1)$.

Substituting $u(x,t) = U(x - ct)$ into (1), we obtain that the travelling wave profile $U(z)$, $z = x - ct$, satisfies

$$\left(d(U(z)) \left| U'(z) \right|^{p-2} U'(z) \right)' + cU'(z) + g(U(z)) = 0, \quad z \in \mathbb{R}. \tag{2}$$

Assumptions on $d = d(s)$ imply that we cannot expect (2) to be satisfied in a classical sense. Therefore, our definition of solution is based on the first integral of (2). We look for a continuous function $U = U(z)$ with values in $[0,1]$, $U(-\infty) = 1$, $U(+\infty) = 0$ and such that $U'(z)$ exists whenever $U(z) \in (0,1) \setminus \bigcup_{i=1}^{n} \{s_i\}$ where s_i are the points of discontinuity of d. At s_i, the following transition condition must hold

$$|U'(\xi_i-)|^{p-2} U'(\xi_i-) \lim_{s \to s_i+} d(s) = |U'(\xi_i+)|^{p-2} U'(\xi_i+) \lim_{s \to s_i-} d(s),$$

so that the jumps of d are properly compensated by the jumps in the derivative of U.

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3 Equivalent first order ODE

From the definition of solution and assumptions on g, we prove that $U = U(z)$ satisfying $U(-\infty) = 1$, $U(+\infty) = 0$ is nonincreasing in $\mathbb{R}$ and strictly decreasing whenever $U(z) \in (0,1)$. These properties allow us to use the phase plane transformation introduced in Enguiça et al. (2013) and show that the second order problem is equivalent to

$$\begin{cases} y'(t) = p' \left[c \left(y^+(t) \right)^{\frac{1}{p}} - f(t) \right], & t \in (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{3}$$

where $f(t) = (d(t))^{\frac{1}{p-1}} g(t)$ and $p > 1$, $p' > 1$ are conjugate numbers.

4 Results

We obtain existence results for (2) with boundary conditions $U(-\infty) = 1, U(+\infty) = 0$ by investigating the equivalent first order problem (3). We look for $c \in \mathbb{R}$ and an absolutely continuous function $y_c = y_c(t)$, $y_c > 0$ in $(0,1)$. Our method is based on comparison results for ODEs in the sense of Carathéodory.

Let

$$0 < \mu := \sup_{t \in (0,1)} \frac{(d(t))^{\frac{1}{p-1}} g(t)}{t^{p'-1}} < +\infty, \quad 0 < \nu := \liminf_{t \to 0+} \frac{(d(t))^{\frac{1}{p-1}} g(t)}{t^{p'-1}} .$$

We prove that there exists a number $c^* \in (0, (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \mu^{\frac{1}{p'}}]$ such that the boundary value problem (3) possesses a unique positive solution $y_c = y_c(t)$ if and only if $c \geq c^*$. On the other hand, if $0 < c < (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \nu^{\frac{1}{p'}}$, then (3) has no positive solution. The equivalence with the second order boundary value problem then yields that there exists a continuum of admissible wave speeds and corresponding nonincreasing travelling wave solutions to (1) which are unique up to translation. For μ and ν as above we conclude that the minimal wave speed c^* must satisfy

$$(p')^{\frac{1}{p'}} p^{\frac{1}{p}} \nu^{\frac{1}{p'}} \leq c^* \leq (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \mu^{\frac{1}{p'}} .$$

We can see that it is the mutual interaction between d and g that plays a key role in the existence result, not their individual properties. Compared to more classical settings, however, we have no information about whether the solutions reach equilibria 0 and 1. Such classification is possible only by assuming power-type behaviour of the reaction and diffusion terms near 0 and 1.

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