UNIVERSITY OF WEST BOHEMIA
FACULTY OF APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS

# MASTER'S THESIS

REACTION-DIFFUSION EQUATIONS IN DISCRETE SPACE

PILSEN, 2022

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# Declaration

I hereby declare that this Master’s Thesis is the result of my own work and that all external sources of information have been duly acknowledged.

Pilsen, On May 20, 2022

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# Abstrakt

Tato práce studuje heterogenní verze semidiskrétní Nagumovy rovnice s nekonstantní kapacitou. Tento systém je uvažován na diskrétních prostorových strukturách - grafu a nekonečné mřížce. Zatímco v homogenních systémech vždy existují netriviální stacionární řešení, pro heterogenní systémy může nastat odlišná situace. Pro heterogenní semidiskrétní systém na grafu ukážeme, že může existovat pouze jediné stacionární řešení - nulové řešení. Podobně je tomu i u heterogenní rovnice na nekonečné mřížce, kde ukazujeme existenci jednoznačného omezeného řešení. Dále je dokázána existence a studovány vlastnosti nekonečně mnoha neomezených řešení.

**Klíčová slova**: Semidiskrétní systém, Nagumova rovnice, graf, mřížka, jednoznačné stacionární řešení.

# Abstract

This thesis investigates heterogeneous versions of semi-discrete Nagumo equation with non-constant capacity. This system is assumed on discrete spatial structures - graph and infinite lattice. While there always exist non-trivial stationary solutions for homogeneous systems, heterogeneous systems may behave differently. We show that heterogeneous semi-discrete Nagumo equation on a graph may have a unique stationary solution - the trivial one. Similarly, we show that unique bounded stationary solution may exists for heterogeneous lattice Nagumo equation while there are infinitely many unbounded stationary solutions.

**Keywords**: Semi-discrete system, Nagumo equation, graph, lattice, unique stationary solution.

# Contents

# Introduction 1

In this work, we study two modified versions of Nagumo equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \lambda u(1-u)(u-a), \quad x \in \mathbb{R},\, t > 0, \tag{1.1}$$

which is the focus of many studies, since it is one of the simplest model where solutions in a special form can occur, e.g., traveling waves [9]. It is used as a model for the spread of genetic traits [2] and for the propagation of nerve pulses in a nerve axon [10, 21].

Our work is focused on two periodic semi-discrete Nagumo equation with different capacities on lattice

$$\begin{cases} \dfrac{dx_i}{dt} &= D\left(x_{i-1} - 2x_i + x_{i+1}\right) + \lambda_1 x_i \left(1 - \frac{x_i}{k_1}\right)\left(\frac{x_i}{k_1} - a\right), \quad i = 2k,\, k \in \mathbb{Z}, \\ \dfrac{dx_i}{dt} &= D\left(x_{i-1} - 2x_i + x_{i+1}\right) + \lambda_2 x_i \left(1 - \frac{x_i}{k_2}\right)\left(\frac{x_i}{k_2} - a\right), \quad i = 2k-1,\, k \in \mathbb{Z} \end{cases} \tag{1.2}$$

and semi-discrete Nagumo equation with capacities on a complete bipartite graph $G = (V_1 \cup V_2, E)$, where $|V_1| = N_1$ and $|V_2| = N_2$

$$\begin{cases} \dfrac{dx_i}{dt} &= D\sum\limits_{j=N_1+1}^{N_1+N_2}\left(x_j - x_i\right) + \lambda_1 x_i \left(1 - \frac{x_i}{k_1}\right)\left(\frac{x_i}{k_1} - a\right), \quad i = 1, 2 \dots N_1, \\ \dfrac{dx_i}{dt} &= D\sum\limits_{j=1}^{N_1}\left(x_j - x_i\right) + \lambda_2 x_i \left(1 - \frac{x_i}{k_2}\right)\left(\frac{x_i}{k_2} - a\right), \quad i = N_1+1,\, N_1+2, \dots N_1+N_2. \end{cases} \tag{1.3}$$

In particular, a significant part of Chapter 2 is dedicated to the model (1.3) in a special case where $|V_1| = |V_2| = 1$, i.e.,

$$\begin{cases} \dfrac{dx_1}{dt} &= D\left(x_2 - x_1\right) + \lambda_1 x_1 \left(1 - \frac{x_1}{k_1}\right)\left(\frac{x_1}{k_1} - a\right), \\ \dfrac{dx_2}{dt} &= D\left(x_1 - x_2\right) + \lambda_2 x_2 \left(1 - \frac{x_2}{k_2}\right)\left(\frac{x_2}{k_2} - a\right), \end{cases} \tag{1.4}$$

which can be interpreted as a two patch Nagumo equation.

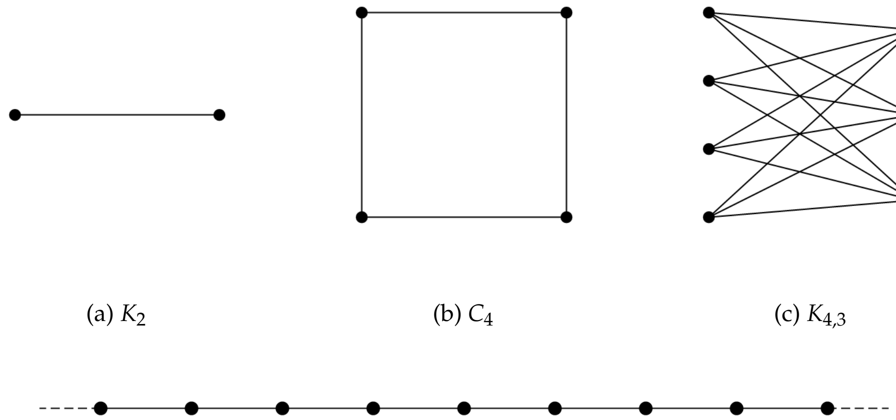
In the systems (1.2), (1.3) and (1.4), $D > 0$ represent the strength of diffusion, $a \in (0,1)$ is a viability parameter, $\lambda_1, \lambda_2 > 0$ stand for reaction functions and $k_1, k_2 > 0$ are capacities of territories.

Semi-discrete Nagumo equation on lattice is often assumed in a form

$$\frac{dx_i}{dt} = D(x_{i-1} - 2x_i + x_{i+1}) + \lambda x_i\left(1 - x_i\right)\left(x_i - a\right), \quad i \in \mathbb{Z}, \tag{1.5}$$

and discrete Nagumo equation on graph $G = (V, E)$, where $|V| = N$ in a form

$$\frac{dx_i}{dt} = D\sum_{j \in \mathcal{N}(v_i)}\left(x_j - x_i\right) + \lambda x_i(1-x_i)(x_i - a), \quad i = 1, 2, \dots N, \tag{1.6}$$



(a) $K_2$ (b) $C_4$ (c) $K_{4,3}$

(d) $P_\infty$

Figure 1.1: Examples of four discrete structures considered in this work. The semi-discrete version of Nagumo equation (1.1) is studied with continuous time and one of these discrete spatial structures. Complete bipartite graphs (a)-(c) lead to the system in the form (1.3). In particular, for the case (a) the system simplifies to (1.4). The lattice Nagumo eqation (1.2) can be interpreted as the semi-discrete system (1.3) for a special type of (non-complete) bipartite graph - infinite path, which is illustrated in (d).

where $\mathcal{N}(v_i)$ stands for neighbors of vertex $v_i$. The interpretation of parameters (except the capacities) corresponds to these of systems (1.2) and (1.3).

The system (1.5) is an example of lattice differential equation (LDE). It naturally occurs in many scientific disciplines, e.g., biology [6], but can be also obtained by Euler discretization of spatial derivative in (1.1). As in the case of the continuous Nagumo equation (1.1), the semi-discrete Nagumo equation on lattice is the subject of many studies focusing on traveling waves, e.g., in scalar [14, 24] and vector (higher-dimensional) [7] cases.

In a case, where the number of territories, neurons, etc. is finite then it is natural to use a finite dimensional model, i.e., the system (1.6). For such a system, a number of stationary solutions based analytic properties and graph characteristics has been studies in [26].

## 1.1 Deriving models

In this section, we derive the model (1.1). For this purpose, we briefly introduce single-species model - the bistable dynamic (sometimes called Allee effect). Then we derive the reaction-diffusion equation from conservative and Fick's laws. Further, we use the method of lines to obtain discrete Nagumo equation (1.2). Last, we discuss how to derive the 2-periodic system (1.3).

### 1.1.1 Bistable dynamics

Single-species models are the cornerstone of mathematical biology and other areas of mathematical modeling. The main idea of these model is a simplicity (low number of parameters) based on the neglecting of external influences, etc. On the other hand, these models should realistically describe the species dynamic. Basic introduction of these models can be found in [19].

We concisely describe only one of these models, which is important for this work - the bistable dynamic. We discuss this model from biological perspective as in [8, 19, 20]. Biology is not the only subject which widely uses this dynamic, e.g., see [8] for ecological point of view, [1] crystallography and [2] for genetics.

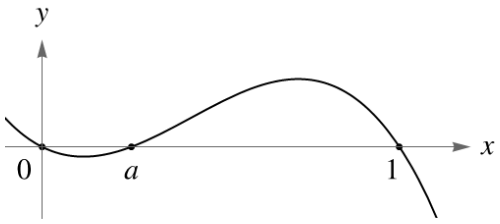



Figure 1.2: Bistable function (1.9) corresponding to the right hand side of system (1.8).

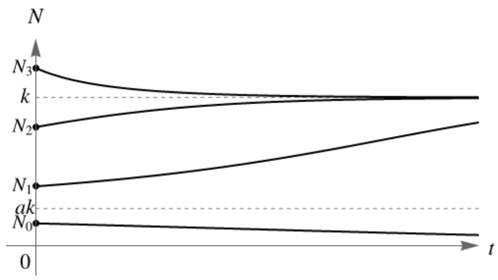



Figure 1.3: Four solutions of system (1.7) for different initial conditions $N_0$, $N_1$, $N_2$, $N_3$.

Let $m = m(t)$ be the population of the species at time $t$, then the rate of change of this population in time is given by bistable dynamic as follows

$$\frac{dm}{dt} = \lambda m \left(1 - \frac{m}{k}\right) \left(\frac{m}{k} - a\right), \tag{1.7}$$

where $\lambda > 0$ is a coefficient of the growth rate, $a \in (0, 1)$ is so called critical value and $k > 0$ is the capacity of a territory.

The bistable dynamics (1.7), can be normalized to

$$\frac{dn}{dt} = \lambda n \left(1 - n\right) \left(n - a\right), \tag{1.8}$$

by substituting $n = m/k$.

For a basic analysis of the normalized bistable dynamics (1.8), we assume the function

$$f(x) = x(1 - x)\left(x - a\right). \tag{1.9}$$

Graph of this function can be seen at Fig. 1.2.

The roots of (1.9) are trivially

$$x_1^* = 0, \quad x_2^* = a, \quad x_3^* = 1.$$

Further, we have

$$f'(0) = -a < 0, \quad f'(a) = a(1 - a) > 0, \quad f'(1) = a - 1 < 0$$

and

$$f(x) < 0,\, x \in (0, a) \cup (a, +\infty), \quad f(x) > 0,\, x \in (a, 1). \tag{1.10}$$

Applying these results to the original model (1.7), we obtain

1. $m = 0$ and $m = k$ are asymptotically stable stationary solutions of (1.7).

2. $m = ak$ is an unstable stationary solution of (1.7).

To be able uniquely determine the solution of (1.7) we assume an initial condition.

$$m(0) = m_0.$$

Four different solutions for different initial conditions can be seen at Fig. 1.3.

### 1.1.2 RDE

Reaction–diffusion equations (RDE) plays very important role in many scientific fields. See, e.g., chemistry [22], in mathematical biology [19, 20]. To derive the reaction diffusion equation, we start with the conservative law, which state that the rate of change of the amount of material in volume $V$ is equal to the rate of flow of material across the surface of $V$ plus the material created in $V$.

$$\frac{\partial}{\partial t}\int_V u(x,t)\,dV = -\int_S J\cdot dS + \int_V f\,dV,$$

where $J$ is the flux and $f$ is the source of material. In general, the source function is $f = f(t,\, x,\, u)$. We apply divergence theorem [18, Theorem 15.12] on the surface integral of the flux to obtain

$$\int_V \left(\frac{\partial u}{\partial t} + \nabla J - f\right) dV = 0.$$

Since the volume integral has to be equal to zero for arbitrary volume, we have

$$\frac{\partial u}{\partial t} + \nabla J - f = 0. \tag{1.11}$$

Fick's law state that the flux is proportional to the gradient of the concentration of the material.

$$J = -D\nabla u. \tag{1.12}$$

Using (1.12) in the equation (1.11), we obtain

$$\frac{\partial u}{\partial t} = \nabla\cdot(D\nabla u) + f.$$

If $D$ is spatially independent, this can be simplified to

$$\frac{\partial u}{\partial t} = D\Delta u + f. \tag{1.13}$$

In our case, we assume the spatially independent $D > 0$, one dimensional space and the function $f$ given by (1.9) multiplied by reaction parameter $\lambda$. Thus, the equation (1.13) has a form (1.1).

### 1.1.3 Spatial discretization

In this work, we are not directly interested in the equation (1.1), but its spatially discrete version (1.5) and (1.6), or more precisely systems (1.2) and (1.3). The discrete Nagumo equation can be obtained by application of semi-discrete numerical scheme - method of lines. This numerical scheme is based on replacing the spatial derivatives with algebraic approximation. This way, the original partial differential equation is transformed to a system of ordinary differential equations. For an introduction to this method see [12].

The equation (1.1) contains only the second spatial partial derivation for which we use the central second difference approximation. More precisely, the second spatial derivative in a point $x_i \in \mathbb{R}$ is approximated by

$$\frac{\partial^2 u(x_i,\, t)}{\partial x^2} \approx \frac{u(x_{i-1},\, t) - 2u(x_i,\, t) + u(x_{i+1},\, t)}{\Delta x^2} + O(\Delta x^2), \tag{1.14}$$

where $x_{i\pm 1} = x_i \pm \Delta x$.

Substituting (1.14) to (1.1), we obtain

$$\frac{\partial u(x_i,\, t)}{\partial t} = D\frac{u(x_{i-1},\, t) - 2u(x_i,\, t) + u(x_{i+1},\, t)}{\Delta x^2} + \lambda u(x_i, t)\left(1 - u(x_i, t)\right)\left(u(x_i, t) - a\right).$$

We can denote $\bar{D} = D/\Delta x^2$ and omit the bar or (since we have no interest in returning to the original RDE equation) we can set $\Delta x = 1$ to obtain a system

$$\frac{\partial u(x_i, t)}{\partial t} = D(u(x_{i-1}, t) - 2u(x_i, t) + u(x_{i+1}, t) + \lambda u(x_i, t)\,(1 - u(x_i, t))\,(u(x_i, t) - a)\,, \quad i \in \mathbb{Z}. \tag{1.15}$$

By denoting $x_i = x_i(t) = u(x_i, t)$, (1.15) reduces to the system (1.5).

The spatial variable in lattice differential equation (1.5) can be viewed as the vertices of an infinite path, i.e., a graph. Each equation in LDE then represents the rate of change of population on a single vertex (territory) given by population growth rate, i.e., reaction term and diffusion to its neighbors. The diffusion in (1.15) can be split to obtain the diffusion along an edge. The diffusion rate between vertices $i$ and $i + 1$ is given by $D(x_{i+1} - x_i)$ for vertex $i$ and $D(x_i - x_{i+1})$ for vertex $i + 1$. There is no reason to limit ourselves to the infinite path (lattice).

We assume a graph $G = (V, E)$, where $V$ represent the set of vertices and $E$ the set of edges. Analogous system to (1.5) on $G$ then has a form (1.6).

We derived the LDE system (1.5), same as the RDE system (1.6), for the function $f$ given by (1.9). In this work, our main focus is on modified versions of these systems, where the reaction function is given by right hand side of (1.7) such that the capacities of different territories are not all the same. More specifically, we assume a two periodical LDE system (1.5) and bipartite graphs for the system (1.6), such that odd and even territories in LDE or the sets of vertices from bipartite graphs carry the same capacity, not equal to the territories from the other set, i.e., systems (1.2) and (1.3).

## 1.2 Preliminaries

In this section, we state some basic properties of systems (1.2) and (1.3). A separate subsection is devoted to each of these models. For the finite-dimensional system (1.3), we use the well known Picard-Lindelöf theorem for existence and uniqueness of solution, then we show that the solution with initial condition located in a $N-$dimensional cube does not leave it. For lattice differential equations (infinite dimensional system of ODE) the Picard-Lindelöf theorem does not hold. Thus, for the infinite-dimensional system (1.2) another theory has to be used. Invariant of the $\infty-$dimensional cube for solution for an initial condition within it is proven analogically to the finite system. First of all, we present different points of view for the heterogeneous systems (1.2) and (1.3).

### 1.2.1 Comparison of homogeneous and heterogeneous systems

Lattice Nagumo equation (1.5) comes directly from spatial discretization of the continuous Nagumo equation (1.1) as shown in Section 1.1.3. Clearly, the heterogeneous lattice Nagumo equation (1.2) can be obtained by spatial discretization of continuous model (1.1) with periodical reaction function. Here we want to present a different point of view, where a interpretation of the bistable dynamic from Section 1.1.1 plays main role.

Assume the normalized bistable dynamics 1.8. It is clear, that function $n = n(t)$ in here, represents a concentration of population, chemical, etc. This dynamic was obtain by substitution $n = m/k$ into the system (1.7), where $k$ is a capacity of territory. There are two ways how to present the function $m = m(t)$ from original system.

1. Function $m$ represents a concentration of population, chemical, etc. for which the upper bound (capacity) of positive reaction given by (1.7) is less or equal to one, i.e., $k \leq 1$.

2. Function $m$ represents an absolute frequency of population. In such case, the capacity of positive reaction, i.e., the parameter $k$, may have relatively large value, e.g., one thousand.

Assuming the second case from previous paragraph, we can interpret the difference of the homogeneous system (1.5) and heterogeneous system (1.2) as follows. While the diffusion in homogeneous system take place based on the difference of concentrations, in heterogeneous system it be held based on differences of absolute frequencies of populations. The same way we can interpret the difference between the discrete Nagumo equations on a graph (1.3) and (1.6).

### 1.2.2 Discrete Nagumo equation on a graph

The system (1.3) is a finite dimensional system of first order ordinary differential equations. For such a system, the existence and uniqueness of solution is given by Picard-Lindelöf theorem. In general, this theorem does not give a good estimate of interval on which the solution exists, but it can be extended by following theorems. Here we mention just the brief summary of this theory, which is used to prove the existence of unique solution of (1.3) for any $t > 0$. For a complete theory of existence and uniqueness of solutions of finite dimensional system see [16].

As mentioned above, the local existence of unique solution of (1.3) follows from the Picard-Lindelöf theorem [16, Theorem 8.13].

THEOREM 1.1 (Picard-Lindelöf theorem). *Assume that $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous vector function on rectangle*

$$Q := \{(t, \mathbf{x}) : t_0 \leq t \leq t_0 + a, \ ||\mathbf{x} - \mathbf{x}_0|| \leq b\}$$

*and assume that $f(t, \mathbf{x})$ satisfies a uniform Lipschitz condition with respect to $\mathbf{x}$ on $Q$. Let*

$$M := \max \{||f(t, \mathbf{x})|| : (t, \mathbf{x}) \in Q\}$$

*and*

$$\alpha := \min \left\{ a, \frac{b}{M} \right\}.$$

*Then the initial value problem*

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = f(t, \mathbf{x}), \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

*has a unique solution $\mathbf{x}$ on $[t_0, t_0 + \alpha.]$*

*Futhermore,*

$$||\mathbf{x}(t) - \mathbf{x}_0|| \leq b,$$

*for $t \in [t_0, t_0 + \alpha]$*

Since the right hand side of (1.3) contains the polynomials of $\mathbf{x}$ the uniform Lipschitz condition is satisfied for arbitrary rectangle $Q$, defined in this theorem. It gives us only local existence of solution of $\mathbf{x}(t)$ for $t$ from some interval $I \subset \mathbb{R}$. We say that a solution $\mathbf{y}(t)$ is an extension of $\mathbf{x}(t)$ provided $J \supset I$ and $\mathbf{y}(t) = \mathbf{x}(t)$ for $t \in I$. We say that $J$ is a maximal interval of existence for $\mathbf{x}$ provided there does not exist the extension of $\mathbf{x}(t)$, see [16, Definitions 8.29 and 8.31]. The following theorem gives us a sufficient condition for the maximal interval of existence.

THEOREM 1.2 (Extension theorem). *Assume $D$ is an open subset of $\mathbb{R} \times \mathbb{R}^N$, $f : D \to \mathbb{R}^N$ is continuous and $\mathbf{x}$ is a solution of*

$$\frac{d\mathbf{x}}{dt} = f(t, \mathbf{x})$$

*on $(\alpha, \beta)$, $-\infty \leq \alpha < \beta \leq +\infty$. Then $\mathbf{x}$ can be extended to a maximal interval of existence $(\omega_1, \omega_2)$, $-\infty \leq \omega_1 < \omega_2 \leq +\infty$.*

*Futhermore, $\mathbf{x}(t) \to \partial D$ as $t \to \omega_1-$ and $t \to \omega_2+$, where $\partial D$ is a boundary of $D$.*

COROLLARY 1.3. *Assume that $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous and bounded. Then every solution of*

$$\frac{d\mathbf{x}}{dt} = f(t, \mathbf{x})$$

*has the maximal interval of existence $(-\infty, +\infty)$.*

The polynomial right hand side of (1.3), is not a bounded vector function of $\mathbf{x}$. Instead of direct application of the last corollary, we have to show that (1.3) with appropriate initial condition leads to bounded solution, i.e., the right hand side stays a bounded function.

LEMMA 1.4. *Assume that* $\mathbf{x}$ *is a solution to the system* (1.3) *with initial condition* $\mathbf{x}(0) = \mathbf{x}_0$ *such that* $x_i \in [0, \max\{k_1, k_2\}]$ *for* $i = 1, 2, \ldots N$ *on interval of existence* $(\alpha, \beta)$, *then the solution* $\mathbf{x}$ *satisfy* $x_i(t) \in [0, \max\{k_1, k_2\}]$ *for all* $t \in (\alpha, \beta)$.

*Proof.* We prove this lemma only for the forward time, i.e. $t \in [0, \beta)$. The opposite side of interval can be proven analogically. This is done by indirect proof. Denote

$$t^* := \min\left\{t \in [0, \beta) : \exists i \in \{1, 2, \ldots N\},\, x_i(t) = \max\{k_1, k_2\},\, \frac{dx_i}{dt}(t) > 0\right\}, \tag{1.16}$$

i.e., $t^*$ is the very first moment when some function $x_i$, which is a solution of (1.3), is about to leave the interval $[0, \max\{k_1, k_2\}]$.

For $x_i$, where index $i$ is given by (1.16), we have

$$x_i\left(1 - \frac{x_i}{k_l}\right)\left(\frac{x_i}{k_l} - a\right) \le 0, \quad l = 1, 2.$$

Thus

$$\frac{dx_i}{dt} \le D \sum_{j: v_j \in V_k} (x_j - k), \tag{1.17}$$

where $V_k$ denotes one of the sets $V_1$ and $V_2$, such that $v_i \notin V_k$ and $k = k_1$ if $i \in \{1, 2, \ldots, N_1\}$ or $k = k_2$ if $i \in \{N_1 + 1, N_1 + 2, \ldots, N_1 + N_2\}$.

Moreover, from (1.16) we have $dx_i/dt > 0$. With (1.17), it is necessary that there exists index $j$ such that $v_j \in V_k$ and $x_j > k$. Since $x_j$ is continuous function on $[0, t^*]$ by intermediate value theorem [18, Theorem 1.13] there exists $\bar{t} \in [0, t^*]$ which satisfy (1.16). A contradiction.

Analogously we define

$$t_* := \min\left\{t \in [0\, \beta) : \exists i \in \{1, 2, \ldots N\},\, x_i(t) = 0,\, \frac{dx_i}{dt}(t) < 0\right\}. \tag{1.18}$$

For $x_i = 0$ we have

$$x_i\left(1 - \frac{x_i}{k_l}\right)\left(\frac{x_i}{k_l} - a\right) = 0, \quad l = 1, 2,$$

which leads to

$$\frac{dx_i}{dt} = D \sum_{j: v_j \in V_k} x_j \tag{1.19}$$

where $V_k$ denote one of the sets $V_1$ and $V_2$, such that $v_i \notin V_k$ and $k = k_1$ if $i \in \{1, 2, \ldots, N_1\}$ or $k = k_2$ if $i \in \{N_1 + 1, N_1 + 2, \ldots, N_1 + N_2\}$.

Equation (1.19) with (1.18) we get $x_j < 0$ for all $j$ such that $v_j \in V_k$. Same argument is used here to derive a contradiction. □

Based on Lemma 1.4, we can modify Corollary 1.3 to obtain the unique solution for $t > 0$.

COROLLARY 1.5. *For the system* (1.3) *with initial condition*

$$\mathbf{x}(0) = \mathbf{x}_0 \in [0, \max\{k_1, k_2\}]^N,$$

*there exists a unique solution with the interval of existence* $(-\infty, +\infty)$.

### 1.2.3 Lattice Nagumo equation

Here, we want to show that the lattice differential equations (1.2) with an initial condition have a solutions. For lattice differential equations, i.e., infinite system of ordinary differential equations, the classical existence theory does not hold. Many authors omit the discussion of existence of solution for differential equations on lattice in works which are mainly focused on another study of these systems. On the other hand, many authors study only the existence. In [25] complete well-posedness of the lattice Nagumo equation 1.5 and its generalisations is proven by a weak maximum principle. In [23] the existence is shown by assumption that right hand sides of lattice differential equations are given by strongly continuous functions. There are several results, based on the so called measures of noncompactness, see [4, 5]. The last mentioned results depends strongly on the choice of sequence spaces.

We apply the results, which are based on the so called degree of nondensifiability [11, Definitions 2.2 and 2.3] and is independent of a chosen Banach space. In particular, we do not use directly the degree of nondensifiability. Instead, we use the remark which follows the existence theorem in that work, which state that a condition based on this property is less restrictive than the Lipschitz one. Thus, the existence of solution of (1.2) is a consequence of the following theorem [11, Theorem 4.4].

THEOREM 1.6. *Let $(X, ||\cdot||)$ be a Banach space. Assume that following conditions hold*

1. *The map $f := (..., f_{-1}, f_0, f_1, ...) : I \times X \to X$ is continuous and the initial condition $\mathbf{x} \in X$.*
2. *There is $h : I \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $||f(t, ..., x_{-1}, x_0, x_1, ...)|| \leq h(t, R)$ for almost everywhere $t \in I$ and $\mathbf{x} \in \mathcal{C}(I, X)$, whenever $||\mathbf{x}||_\infty \leq R$. Also, there are $R_0 > 0$ and $K > 0$ such that*
$$\frac{h(t, R_0)}{R_0} \leq K, \tag{1.20}$$
*for almost everywhere $t \in I$.*
3. *The functions of $f$ satisfy the Lipschitz condition, i.e., $||f_i(t, \mathbf{x}) - f_i(t, \mathbf{y})|| \leq L||\mathbf{x} - \mathbf{x}||$ for some $L > 0$.*

*Then for $\rho \in (0 \min\{1, 1/K\})$, where $K$ defined in* (1.20)*, the system*
$$\frac{dx_i}{dt} = f_i(t, ..., x_{-1}, x_0, x_1, ...)$$
*with initial condition $\mathbf{x}(0) = \mathbf{x}_0$ has a solution $\mathbf{x}(t) \in \mathcal{C}([0, \rho], \bar{B}(\mathbf{x}_0, R_0))$.*

Before we verify assumptions of Theorem (1.6) for the system (1.2), we have to choose suitable Banach space. Simple observation shows, that there is not a very wide possible choices. Assume the system (1.5), i.e., the system (1.2) with $k_1 = k_2 = 1$. This system has a stationary solution $(..., 1, 1, 1, ...)$ which is not an element of any sequence space $\ell^p(\mathbb{Z})$, except $\ell^\infty(\mathbb{Z})$. Thus, for the system (1.2) we assume space
$$\ell^\infty(\mathbb{Z}) := \left\{\{x_i\}_{i\in\mathbb{Z}} : \sup |x_i| < +\infty\right\}.$$
Other approach can be found in [13], where the space of weighted sequences is used.

Based on our choice of Banach space, the assumptions of Theorem 1.6 are easy to verify, since the functions in the right hand sides of (1.2) are polynomial. Moreover, we will see that the function $h(t, R) := \max\{k_1, k_2\}$ is sufficient for an initial condition in $[0, \max\{k_1, k_2\}]^{\mathbb{Z}}$. Thus by (1.20) we have $K = 1$. Therefore, we have existence of solution on interval $[0, \rho]$, where $\rho \in (0, 1)$.

Finally, we show that a solution (1.2) with initial condition $\mathbf{x} \in [0, \max\{k_1, k_2\}]^{\mathbb{Z}}$ stays in this $\infty-$dimensional cube.

LEMMA 1.7. *Assume that a solution of* (1.2) *exists for $t > 0$. Let the initial conditions for this system $\mathbf{x}(0) = \mathbf{x}_0 \in \ell^\infty(\mathbb{Z})$ satisfy $x_j(0) \in [0, \max\{k_1, k_2\}]$ for all $j \in \mathbb{Z}$. Then the solution $\mathbf{x}(t)$ satisfy $x_j(t) \in [0, \max\{k_1, k_2\}]$ for all $t > 0$ and $j \in \mathbb{Z}$.*

*Proof.* Proof of this lemma is the same as for the finite dimensional system, i.e., the proof of Lemma 1.4. The statement is proven by indirect proof. We define

$$t^* := \min\left\{t \in \mathbb{R}^+ : \exists j \in \mathbb{Z},\, x_j(t) = \max\{k_1, k_2\},\, \frac{dx_j}{dt}(t) > 0\right\}, \tag{1.21}$$

i.e., $t^*$ is the very first moment such that any function $x_j$ is about to leave $[0, \max\{k_1, k_2\}]$.

For both, even and odd, variant of the index $j$ in (1.21) we have

$$0 < \frac{dx_j}{dt}(t^*) \leq D\left(x_{j-1}(t^*) - 2\max\{k_1, k_2\} + x_{j+1}(t^*)\right).$$

This leads to the inequality

$$\frac{x_{j-1}(t^*) + x_{j+1}(t^*)}{2} > \max\{k_1, k_2\},$$

which can be satisfied only if $x_{j-1}(t^*) > \max\{k_1, k_2\}$ or $x_{j+1}(t^*) > \max\{k_1, k_2\}$. Since $x_j(t)$ is the first function which is about to leave interval $[0, \max\{k_1, k_2\}]$, a contradiction.

The same approach is used for

$$t_* := \min\left\{t \in \mathbb{R}^+ : \exists j \in \mathbb{Z},\, x_j(t) = 0 \;\; \& \;\; \frac{dx_j}{dt}(t) < 0\right\}.$$

This leads to the inequality

$$x_{j-1}(t_*) + x_{j+1}(t_*) < 0,$$

i.e., one of its neighbor had to leave the interval before. $\square$

# RDE in heterogeneous graphs 2

Both, the finite semi-discrete Nagumo equation (1.6) and its modified version with capacities (1.3), contain various combinations of stationary solutions. While for sufficiently weak diffusion, both of these systems have $3^N$ stationary solutions, where $N$ is the number of territories, for increasing diffusion their behavior (complicated bifurcations) may be very different. The homogeneity of the system (1.6) guarantee that, except the origin, there always exist at least two more stationary solutions. We show, that this may not be the case for the system (1.3).

The first part of this chapter is devoted to the semi-discrete Nagumo equation (1.3) on the graph $G = K_2$, that is the system (1.4). For this system, we derive the most important results, which are further used for heterogeneous lattice Nagumo equation (1.2) in Chapter 3. As an intermediate step between the two dimensional system (1.4) and the infinite dimensional system (1.2), we apply these results for complete bipartite graphs, i.e., the system (1.3).

## 2.1 Model description $G = K_2$

We start with the simplest form of graph assumed for (1.3), that is $G = K_2$. Moreover, we start with even simpler form of (1.4), where the viability parameter is fixed $a = 1/2$. The general form with viability parameter $a \in (0, 1)$ is discussed in Section 2.1.2. We assume a system

$$\begin{cases} \dfrac{dx_1}{dt} &= D(x_2 - x_1) + \lambda_1 x_1 \left(1 - \frac{x_1}{k_1}\right)\left(\frac{x_1}{k_1} - \frac{1}{2}\right), \\ \dfrac{dx_2}{dt} &= D(x_1 - x_2) + \lambda_2 x_2 \left(1 - \frac{x_2}{k_2}\right)\left(\frac{x_2}{k_2} - \frac{1}{2}\right), \end{cases} \tag{2.1}$$

where $\lambda_1, \lambda_2 > 0$ and $k_1, k_2 > 0$ stand for the population growth rates and the capacities of territory, while $D > 0$ is the diffusion coefficient.

First of all we shortly discuss the system (2.1) without diffusion, i.e., we assume $D = 0$. That is

$$\begin{cases} \dfrac{dx_1}{dt} &= \lambda_1 x_1 \left(1 - \frac{x_1}{k_1}\right)\left(\frac{x_1}{k_1} - \frac{1}{2}\right), \\ \dfrac{dx_2}{dt} &= \lambda_2 x_2 \left(1 - \frac{x_2}{k_2}\right)\left(\frac{x_2}{k_2} - \frac{1}{2}\right), \end{cases}$$

This system contain two independent differential equation (1.7). Thus, it can be analyzed separately. We obtain 9 stationary solutions, given by combinations of 3 stationary solutions for each bistable dynamic, see Section 1.1.1. One may ask if these solutions do exist for $D > 0$. We show that this hypothesis is true at least for $D$ very small. We can rewrite the system (2.1) to a vector form

$$G(d, v) := DAv + F(v), \tag{2.2}$$

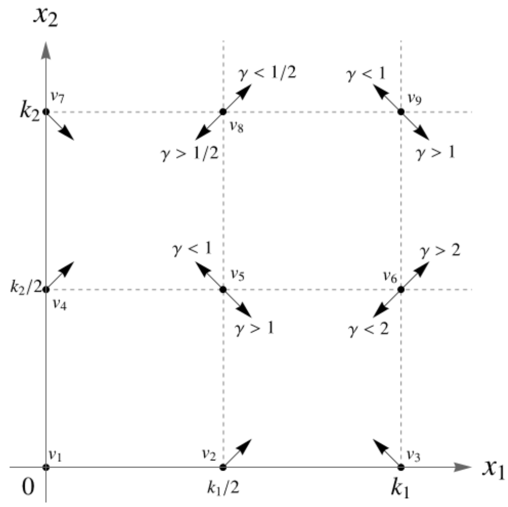



Figure 2.1: Change of position of the stationary solutions of (2.1) for small $D > 0$ based on the value of $\gamma$ given by (2.5).

where

$$v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad F(v) = \begin{bmatrix} \lambda_1 f_{k_1}(x_1) \\ \lambda_2 f_{k_2}(x_2) \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},$$

where the function $f_k(x)$ has a form

$$f_k(x) = x\left(1 - \frac{x}{k}\right)\left(\frac{x}{k} - \frac{1}{2}\right).$$

This can be used to prove the following theorem.

THEOREM 2.1. *For system* (2.2) *with sufficiently small $D > 0$, there exist nine stationary solutions. Moreover, they belong to the first quadrant of xy plane.*

*Proof.* Stationary solutions $v_{ss}$ of (2.2) satisfy $G(D, v_{ss}) = 0$. For $D = 0$, there are nine stationary solutions and they have the form $v_i = [x_1^*, x_2^*]$, where $x_1^* \in \{0, k_1/2, k_1\}$, $x_2^* \in \{0, k_2/2, k_2\}$ for $i \in 1, 2, ... 9$.

Elements of $F(v)$ from (2.2) are polynomial functions which are infinitely many times differentiable and so is the expression $DAv$, therefore $G(D, v) \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}^2)$.

We can express

$$\frac{\partial G}{\partial v} = F'(v) + DA = \begin{bmatrix} \lambda_1 f'_{k_1}(x_1) & 0 \\ 0 & \lambda_2 f'_{k_2}(x_2) \end{bmatrix} + D\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{2.3}$$

We are looking for the behaviour around zero value of diffusion (parameter $D$). In the case of $D = 0$, the expression (2.3) reduces just to the matrix with diagonal entries $\lambda_1 f'_{k_1}(x_1)$ and $\lambda_2 f'_{k_2}(x_2)$. These functions represent the bistable model (1.7). Thus, we know that the derivatives are nonzero in stationary solutions and therefore, the matrix is invertible.

We have verified assumptions of the Implicit function theorem [17, Theorem 1.3.3] and therefore we know there exist a neighborhood of $v_i$ and a function $v_i(D)$ which satisfies $G(D, v_i(D))$, $i = 1, 2, ..., 9$. This function can be expressed as

$$v_i'(D) = -D\begin{bmatrix} \frac{1}{\lambda_1 f'_{k_1}(x_1^*)} & 0 \\ 0 & \frac{1}{\lambda_2 f'_{k_2}(x_2^*)} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}. \tag{2.4}$$

From the coordinates of stationary solutions we get the sign of $v_i'(D)$, $i = 1, 2, ..., 9$, i.e., the direction of position change of a given stationary solution. Thus, we show that they all stay in the first quadrant.

Except the trivial solution, which stays at the same position, there are eight more stationary solutions whose change of positions we would like to know. For the stationary solutions that are originally located on the axes we obtain

$$v_2(0) = \begin{bmatrix} \frac{k_1}{2} \\ 0 \end{bmatrix}, \operatorname{sgn} v_2'(D) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad v_3(0) = \begin{bmatrix} k_1 \\ 0 \end{bmatrix}, \operatorname{sgn} v_3'(D) = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$
$$v_4(0) = \begin{bmatrix} 0 \\ \frac{k_2}{2} \end{bmatrix}, \operatorname{sgn} v_4'(D) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad v_7(0) = \begin{bmatrix} 0 \\ k_2 \end{bmatrix}, \operatorname{sgn} v_7'(D) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The change of position of four stationary solutions originally located inside the first quadrant depends on a ratio of capacities $k_1$ and $k_2$. For this purpose we define

$$\gamma := k_2/k_1. \tag{2.5}$$

The change of position of such a stationary solutions is then given as follows

$$v_5(0) = \begin{bmatrix} \frac{k_1}{2} \\ \frac{k_2}{2} \end{bmatrix}, \operatorname{sgn} v_5'(D) = \begin{bmatrix} \operatorname{sgn}(1-\gamma) \\ \operatorname{sgn}(\gamma-1) \end{bmatrix}, \quad v_6(0) = \begin{bmatrix} k_1 \\ \frac{k_2}{2} \end{bmatrix}, \operatorname{sgn} v_6'(D) = \begin{bmatrix} \operatorname{sgn}(\gamma-2) \\ \operatorname{sgn}(\gamma-2) \end{bmatrix},$$
$$v_8(0) = \begin{bmatrix} \frac{k_1}{2} \\ k_2 \end{bmatrix}, \operatorname{sgn} v_8'(D) = \begin{bmatrix} \operatorname{sgn}(1-2\gamma) \\ \operatorname{sgn}(1-2\gamma) \end{bmatrix}, \quad v_9(0) = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, \operatorname{sgn} v_9'(D) = \begin{bmatrix} \operatorname{sgn}(\gamma-1) \\ \operatorname{sgn}(1-\gamma) \end{bmatrix}.$$

Even that in these cases the change of position depends on $\gamma$ it does not cause any problem. They all are located far away from axes and thus cannot leave it with little change of $d$. □

We can see indicated change of stationary solutions' positions from previous proof in Fig. 2.1. The stationary solutions originally located on axes move inside of the first quadrant and the possible changes of others stationary solution depend on the value of parameter $\gamma$ given by (2.5).

For the future analysis we would like to simplify the system (2.1). The number of parameters in this system can be reduced. Substituting

$$x = x_1/k_1, \quad y = x_2/k_2, \quad \tau = Dt \tag{2.6}$$

leads to a system

$$\begin{cases} \dfrac{dx}{d\tau} &= \gamma y - x + \alpha x\,(1-x)\left(x - \frac{1}{2}\right), \\ \dfrac{dy}{d\tau} &= \frac{1}{\gamma} x - y + \beta y\,(1-y)\left(y - \frac{1}{2}\right), \end{cases} \tag{2.7}$$

where $0 < \alpha = \lambda_1/D$ and $0 < \beta = \lambda_2/D$ stand for ratios of the growth rate of each population and the diffusion, while $0 < \gamma = k_2/k_1$ is ratio of capacities.

From now on we will only focus on model (2.7) where the diffusion coefficient is included in parameters $\alpha$ and $\beta$. Let us discuss the crucial difference between the heterogeneous model (2.7) and the homogeneous special case with $\gamma = 1$ i.e., the system (1.6) for $G = (V, E)$ with $|V| = 2$. If territories have the same capacity, there are always at least three stationary solutions (spatially homogeneous), see [26]. If the capacities are not equal we come to a different conclusion. There exist parameter values for which the trivial solution is the unique stationary solution of (2.7).

### 2.1.1 Non-existence of non-trivial stationary solutions

To prove our main claim that there may not exist any stationary solutions of (2.7) but the trivial one, we prove few auxiliary lemmas. First of all, we prove a lemma that estimates for what $\gamma$ we can not expect such state. Then we show that if a non-trivial stationary solution does exist, it has to be located in one of a specific regions of the first quadrant. In the following lemmas we estimate conditions for each region under which any stationary solutions cannot be located there. In conclusion of these lemmas we formulate Theorem 2.10. This theorem gives us sufficient conditions for existence of the unique stationary solution for system (2.7). To conclude, we compare the statement of the theorem with numerical results.

REMARK 2.2. *Without loss of generality, the system* (2.7) *can be assumed with* $\gamma \in (0, 1]$. *In the opposite case we can define* $\bar{\gamma} = 1/\gamma$ *and rewrite* $x \to y$, $y \to x$, $\alpha \to \beta$ *and* $\beta \to \alpha$.

LEMMA 2.3. *For* $\gamma \in (1/2,\ 1]$ *the system* (2.7) *has at least three stationary solutions while for* $\gamma = 1/2$ *this system has at least two stationary solutions.*

*Proof.* The case $\gamma = 1$ is analysed in [26]. Thus, we know that there are at least three stationary solutions $\mathbf{x}_1 = [0,\ 0]$, $\mathbf{x}_2 = [1/2,\ 1/2]$ and $\mathbf{x}_3 = [1,\ 1]$.

For the case $\gamma = 1/2$ we can observe that except $\mathbf{x}_1 = [0,\ 0]$ there also exists stationary solution $\mathbf{x}_2 = [1/2,\ 1]$.

Finally, the statement of this lemma for the case $\gamma \in (1/2,\ 1)$ is based on a simple observation. A stationary solutions of system (2.7) are intersections of its nullclines. These nullclines are given by

$$\begin{aligned} N_x &= \left\{ (x,\ y) \in \mathbb{R}_0^{+2} : y = \frac{x - \alpha x(1-x)\left(x - \frac{1}{2}\right)}{\gamma} \right\}, \\ N_y &= \left\{ (x,\ y) \in \mathbb{R}_0^{+2} : x = \gamma\left(y - \beta y(1-y)\left(y - \frac{1}{2}\right)\right) \right\}. \end{aligned} \tag{2.8}$$

$N_x$ does intersect the line $y = x/\gamma$ in points $\mathbf{o} = (0,\ 0)$, $A = (1/2,\ 1/(2\gamma))$ and $B = (1,\ 1/\gamma)$. $N_y$ intersect the same line in points $\mathbf{o} = (0,\ 0)$, $C = (\gamma/2,\ 1/2)$ and $D = (\gamma,\ 1)$. If $\gamma \in (1/2,\ 1)$, then the points lay on the line in order $C - A - D - B$.

For $x \in (0,\ 1/2)$, the nullcline $N_x$ is located above the line $y = x/\gamma$. The nullcline $N_y$ is also located above this line for $y \in (1/2,\ 1)$. Since the nullclines are continuous curves and $N_x$ intersects the line in point $A$ that is located between the points $C$ and $D$, where $N_y$ does intersect this line, the intersection of these curves has to be situated there. Thus, the stationary solution satisfies $x \in (0,\ 1/2)$, $y \in (1/2,\ 1)$ and $y > x/\gamma$.

The nullcline $N_y$ is located below the line $y = x/\gamma$ for $y \in (1,\ +\infty)$ and for $x \in (1/2,\ 1)$ the nullcline $N_x$ is also located there. In this case, the curve representing $N_y$ intersects the line in point $D$ which is located between points $A$ and $B$. Thus, a stationary solution has to be located there. Moreover, since we discuss the case $y < x\gamma$, stationary solution has to satisfies $x \in (\gamma,\ 1)$, $y \in (1,\ 1/\gamma)$ and $y < x/\gamma$.

□

REMARK 2.4. *The statement of Lemma 2.3 can be more specific if we assume* $\alpha,\ \beta \in (0,\ 4)$. *Under this assumption, there exists exactly three stationary solutions of* (2.7) *with* $\gamma \in (1/2,\ 1]$ *and exactly two stationary solutions for* $\gamma = 1/2$.. *Since this is not the case we are primarily interested here, we leave it here without a proof.*

Based on Remark 2.2 and Lemma 2.3 we can focus only on the case $\gamma \in (0,\ 1/2)$. As indicated, we are going to show that if a non-trivial stationary solution exists then it has to be located in one of a specific regions of the first quadrant.

LEMMA 2.5. *Let* $\mathbf{x} = [x^*, y^*]$ *be a non-trivial stationary solution of* (2.7) *with the capacity rate* $\gamma \in (0,\ 1/2)$, *then it is located in one of the following regions*

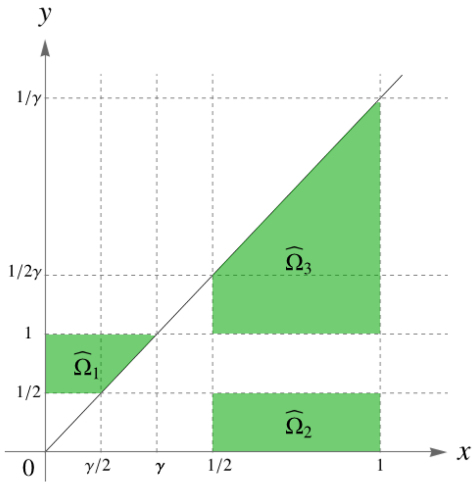



Figure 2.2: Regions of possible location of stationary solution of equation (2.7) given by Lemma 2.5.

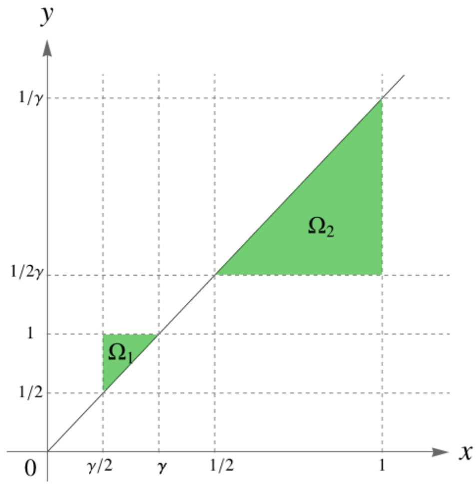



Figure 2.3: Reduced regions of possible location of stationary solution of equation (2.7) given by Lemma 2.6.

$$
\begin{aligned}
\widehat{\Omega}_1 &= \left\{(x, y) \in \mathbb{R}^2 : x \in (0, \gamma),\, y \in \left(\frac{1}{2}, 1\right),\, y > \frac{x}{\gamma}\right\}, \\
\widehat{\Omega}_2 &= \left\{(x, y) \in \mathbb{R}^2 : x \in \left(\frac{1}{2}, 1\right),\, y \in \left(0, \frac{1}{2}\right)\right\}, \\
\widehat{\Omega}_3 &= \left\{(x, y) \in \mathbb{R}^2 : x \in \left(\frac{1}{2}, 1\right),\, y \in \left(1, \frac{1}{\gamma}\right),\, y < \frac{x}{\gamma}\right\}.
\end{aligned}
\tag{2.9}
$$

*Proof.* The stationary solution of (2.7) satisfies

$$
\begin{cases}
y &= \frac{1}{\gamma}\left(x - \alpha f(x)\right), \\
x &= \gamma\left(y - \beta f(y)\right),
\end{cases}
\tag{2.10}
$$

where the function $f$ is defined as

$$
f(s) := s(1-s)\left(s - \frac{1}{2}\right),
\tag{2.11}
$$

i.e., the function (1.9) with $a = 1/2$. All the parameters $\alpha, \beta, \gamma$ are positive. Since $\gamma y - x = -\gamma(x/\gamma - y)$, both equations in (2.10) cannot be satisfied if $\operatorname{sgn}(f(x)) = \operatorname{sgn}(f(y))$. This leads to two possible cases.

In the first case we have

$$
\begin{aligned}
\operatorname{sgn}(f(x)) = -1 \quad &\Longrightarrow \quad x \in (0, 1/2) \cup (1, +\infty), \\
\operatorname{sgn}(f(y)) = 1 \quad &\Longrightarrow \quad y \in (1/2, 1).
\end{aligned}
$$

and

$$
\gamma y - x > 0.
\tag{2.12}
$$

Since $y \in (1/2, 1)$ and $\gamma \in (0, 1/2)$, the inequality (2.12) can not be satisfied for $x \in (1, +\infty)$. On the contrary, it could be satisfied for some $x \in (0, 1/2)$. Since $y \in (1/2, 1)$ we have

$$
\begin{aligned}
\gamma - x > \gamma y - x &> 0, \\
x &< \gamma.
\end{aligned}
$$

Together, this define the region $\widehat{\Omega}_1$ from (2.9).

In the second case we have

$$\begin{aligned} \operatorname{sgn}(f(x)) = 1 \quad &\Longrightarrow \quad x \in (1/2,\, 1), \\ \operatorname{sgn}(f(y)) = -1 \quad &\Longrightarrow \quad y \in (0,\, 1/2) \cup (1, +\infty). \end{aligned}$$

Obviously the opposite inequality than (2.12) has to be satisfied. That is

$$\gamma y - x < 0. \tag{2.13}$$

For $y \in (0,\, 1/2)$ the inequality (2.13) holds without any extra limitations. On the contrary, the case $y \in (1,\, +\infty)$ has to be reduced. Since $x \in (1/2,\, 1)$, we have

$$\begin{aligned} \gamma y - 1 < \gamma y - x &< 0, \\ y &< \frac{1}{\gamma}. \end{aligned}$$

Thus, we have derived the second and the third region in the statement of this lemma, i.e., $\widehat{\Omega}_2$, $\widehat{\Omega}_3$ in (2.9). $\square$

To visualize the possible location of a non-trivial stationary solutions of system (2.7) with $\gamma \in (0,\, 1/2)$ determined by Lemma 2.5 see Fig. 2.2.

We are going to determine conditions for each of the region given by (2.9) that will guarantee that any stationary solution can not belong in here. First of all, we determine conditions which eliminate $\widehat{\Omega}_2$ and greatly reduce $\widehat{\Omega}_1$ and $\widehat{\Omega}_3$.

LEMMA 2.6. *Let* $\mathbf{x} = [x,\, y]$ *be a non-trivial stationary solution of* (2.7) *with* $\gamma \in (0,\, 1/2)$ *and* $\alpha,\, \beta < 4$, *then it satisfies* $\mathbf{x} \in \Omega_1 \cup \Omega_2$, *where*

$$\begin{aligned} \Omega_1 &= \left\{ (x,\, y) \in \mathbb{R}^2 : x \in \left(\frac{\gamma}{2},\, \gamma\right),\, y \in \left(\frac{1}{2},\, 1\right),\, y > \frac{x}{\gamma} \right\}, \\ \Omega_2 &= \left\{ (x,\, y) \in \mathbb{R}^2 : x \in \left(\frac{1}{2},\, 1\right),\, y \in \left(\frac{1}{2\gamma},\, \frac{1}{\gamma}\right),\, y < \frac{x}{\gamma} \right\}. \end{aligned} \tag{2.14}$$

*Proof.* We define a function

$$g_\delta(s) := s - \delta f(s), \tag{2.15}$$

where the function $f$ is given by (2.11). The function $g_\delta$ corresponds to multiples of right hand sides of equations in (2.10). The function $g_\delta$ is a monotone function for $\delta \in (0,\, 4)$.

Since the function $f$ defined by (2.11) is concave on $(1/2,\, 1)$ it satisfies

$$f(s) = s(1-s)\left(s - \frac{1}{2}\right) \leq \frac{1}{4}\left(s - \frac{1}{2}\right), \quad s \in \left(\frac{1}{2},\, 1\right). \tag{2.16}$$

For $\alpha,\, \beta < 4$ are the functions given by each equation of the system (2.10) monotone. Thus, we can use the concavity to obtain

$$\begin{aligned} y &= \frac{x - \alpha f(x)}{\gamma} \geq \frac{x - \frac{\alpha}{4}\left(x - \frac{1}{2}\right)}{\gamma} = \frac{x\left(1 - \frac{\alpha}{4}\right) + \frac{\alpha}{8}}{\gamma}, \\ x &= \gamma\left(y - \beta f(y)\right) \geq \gamma\left(y - \frac{\beta}{4}\left(y - \frac{1}{2}\right)\right) = \gamma\left(y\left(1 - \frac{\beta}{4}\right) + \frac{\beta}{8}\right). \end{aligned}$$

Since $\alpha,\, \beta < 4$, the terms $(1 - \alpha/4)$ and $(1 - \beta/4)$ are positive and we can take infimum over interval $(1/2,\, 1)$ to continue in inequalities

$$\begin{aligned} y &\geq \frac{x\left(1 - \frac{\alpha}{4}\right) + \frac{\alpha}{8}}{\gamma} > \frac{\frac{1}{2}\left(1 - \frac{\alpha}{4}\right) + \frac{\alpha}{8}}{\gamma} = \frac{1}{2\gamma}, \\ x &\geq \gamma\left(y\left(1 - \frac{\beta}{4}\right) + \frac{\beta}{8}\right) > \gamma\left(\frac{1}{2}\left(1 - \frac{\beta}{4}\right) + \frac{\beta}{8}\right) = \frac{\gamma}{2}. \end{aligned}$$

Since we have used the concavity of the first function of (2.10) in $x = 1/2$ to obtain $y > 1/2\gamma$, we have eliminated $\widehat{\Omega}_2$ and reduced $\widehat{\Omega}_3$ from (2.9) to $\Omega_2$ from (2.14). The second equation of (2.10) is restricted in $y = 1/2$, thus it reduces $\widehat{\Omega}_1$ to $\Omega_1$. $\square$

The reduced regions (2.14), where the non-trivial stationary solution of (2.7) may be located while the conditions given by Lemma 2.6 are satisfied are shown at Fig. 2.3. Here, the green regions (possible location of a stationary solution) are just the two triangles. For each of these triangular regions we introduce conditions that eliminate the possibility that any stationary solution is located there.

LEMMA 2.7. *Assume that the conditions from Lemma 2.6 are satisfied. Let*

$$\alpha > \max\left\{\frac{2\sqrt{3}}{9\left(\gamma-1\right)\left(\gamma-2\right)}\beta,\ \frac{\sqrt{3}}{18\left(2\gamma-1\right)\left(\gamma-1\right)}\beta\right\}, \tag{2.17}$$

*then the system* (2.7) *has no stationary solution located in the region* $\Omega_1$ *given by* (2.14).

*Proof.* For the function $f$ defined by (2.11) we have

$$\max_{s\in(0,1)} f(s) = \frac{1}{12\sqrt{3}}. \tag{2.18}$$

We use the maximum (2.18) for the second equation of (2.10) to obtain

$$x = \gamma\left(y - \beta y(1-y)\left(y-\frac{1}{2}\right)\right) \geq \gamma\left(y - \frac{\beta}{12\sqrt{3}}\right).$$

Assume the border line

$$y = \frac{1}{\gamma}x + \frac{\beta}{12\sqrt{3}} \tag{2.19}$$

and let us find the value this function attains in $x_1 = \gamma/2$ and $x_2 = \gamma$, i.e., on border of the region $\Omega_1$ given by (2.14). The values are $y_1^1 = 1/2 + \beta/(12\sqrt{3})$ and $y_2^1 = 1 + \beta/(12\sqrt{3})$.

Now we can find the values the function given by the first equation in (2.10) attains for the same $x_1$ and $x_2$. Substituting $x_1$ and $x_2$ into

$$y = \frac{1}{\gamma}\left(x - \alpha x(1-x)\left(x-\frac{1}{2}\right)\right) \tag{2.20}$$

we get $y_1^2 = (4 + \alpha(\gamma-2)(\gamma-1))/8$ and $y_2^2 = 1 + \alpha(1/2 - 3\gamma/2 + \gamma^2)$.

If the assumption (2.17) holds, then $y_1^1 < y_1^2$ and $y_2^1 < y_2^2$. Assume by contradiction that there exists the stationary solution of (2.7) which is located in $\Omega_1$ given by (2.14). The stationary solution has to be located above the line given by points $[x_1, y_1^2]$ and $[x_2, y_2^2]$ and below the line given by (2.19). But under the assumption (2.17) this situation can not occur at the same time, a contradiction. $\square$

REMARK 2.8. *Two expressions in* (2.17) *are equal for* $\gamma = 2/7$. *In this case the parameter* $\alpha$ *has to be greater than* $49\beta/(90\sqrt{3})$. *For* $\gamma \in (0, 2/7)$, *the first expression is greater and so active. While* $\gamma \in (2/7, 1/2)$, *the second expression is active.*

LEMMA 2.9. *Assume that the conditions from Lemma 2.6 are satisfied. Let*

$$\alpha < \frac{9\beta\left(2\gamma-1\right)\left(\gamma-1\right)}{2\sqrt{3}\gamma^2}, \tag{2.21}$$

*then the system* (2.7) *has no stationary solution located in the region* $\Omega_2$ *given by* (2.14).

*Proof.* This time, we use the maximum (2.18) for the first equation of (2.10) to obtain

$$y = \frac{1}{\gamma}\left(x - \alpha x(1-x)\left(x - \frac{1}{2}\right)\right) \geq \frac{1}{\gamma}\left(x - \frac{\alpha}{12\sqrt{3}}\right).$$

Since we have $\alpha \in (0, 4)$, this partly reduces the region $\Omega_2$ given by (2.14). We want to prove that the function given by the second equation in (2.10) is located in the removed part of $\Omega_2$. The boundary line of this removed part has form

$$y = \frac{1}{\gamma}\left(x - \frac{\alpha}{12\sqrt{3}}\right) \tag{2.22}$$

and it intersects the line $y = 1/(2\gamma)$ (the bottom boundary of the region $\Omega_2$) in $x_1 = 1/2 + \alpha/12\sqrt{3}$ and its slope is $1/\gamma$.

The function given by the second equation of (2.10) intersects the line $y = 1/(2\gamma)$ for

$$x_2 = \frac{1}{\gamma}\left(\frac{1}{2\gamma} - \beta f\left(\frac{1}{2\gamma}\right)\right) = \frac{\beta - 3\beta\gamma + 2\gamma^2(\beta+2)}{8\gamma^2}$$

and by concavity (2.16) of the function $f$ given by (2.11) we obtain

$$x = \gamma\left(y - \beta y(1-y)\left(y - \frac{1}{2}\right)\right) \geq \gamma\left(y + \frac{\beta\left(2\gamma^2 - 6\gamma + 3\right)}{4\gamma^2}\left(y - \frac{1}{2\gamma}\right)\right), \quad y \geq \frac{1}{2\gamma}.$$

Thus the second border line has a form

$$y = \frac{4\gamma}{4\gamma^2 + \beta\left(2\gamma^2 - 6\gamma + 3\right)}x + \frac{\beta\left(2\gamma^2 - 6\gamma + 3\right)}{8\gamma^2}. \tag{2.23}$$

Clearly, for $\beta > 0$ and $\gamma \in (0, 1/2)$ is the slope of (2.22) greater then slope of (2.23). If the assumption (2.21) holds, then the $x_2$ is greater than $x_1$.

Now, assume by contradiction that a stationary solution of (2.7) is located in $\Omega_2$ given by (2.14). The stationary solution has to be located above the line (2.22) and under the line (2.23). But under the assumption of (2.21), both of these can not be satisfied in the same time, a contradiction. □

Finally we can prove the main result of this section, non-existence of non-trivial stationary solutions of (2.7).

THEOREM 2.10. *Assume that $\alpha$, $\beta \in (0, 4)$, $\gamma \in (0, 1/2)$ satisfy*

$$\max\left\{\frac{2\sqrt{3}}{9(\gamma-1)(\gamma-2)}, \frac{\sqrt{3}}{18(2\gamma-1)(\gamma-1)}\right\}\beta < \alpha < \frac{9(2\gamma-1)(\gamma-1)}{2\sqrt{3}\gamma^2}\beta, \tag{2.24}$$

*then the system* (2.7) *has the unique stationary solution* $(x^*, y^*) = (0, 0)$.

*Proof.* Statement of this theorem comes from the previous lemmas. Lemma 2.5 gives us possible location of a non-trivial stationary solutions of system (2.7) with $\gamma \in (0, 1/2)$. If $\alpha$, $\beta \in (0, 4)$ then by Lemma 2.6 the regions where stationary solution may belong are reduced to two triangular regions (2.14).

Under the assumption of Lemma 2.7, i.e., the first inequality of (2.24), a stationary solution of the system (2.7) can not be located in $\Omega_1$ given by (2.14). The second inequality of (2.24) is given by Lemma 2.9 and guarantees that stationary solution of (2.7) cannot be located neither in the second region $\Omega_2$ given by (2.14).

Thus, under assumptions of this theorem, non-trivial stationary solution of (2.7) cannot exist. The existence of the trivial stationary solution is obvious. □

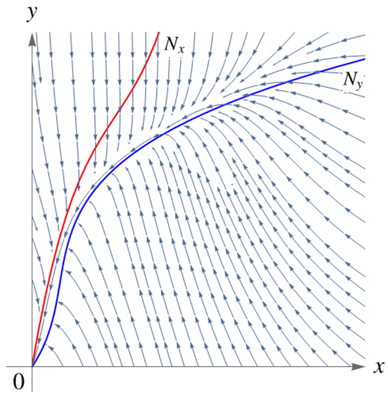



Figure 2.4: Nullclines and the vector field for the system (2.25).

REMARK 2.11. *Note that the assumption* (2.24) *of Theorem 2.10 gives for a fixed value of parameter $\gamma$ boundaries of parameter $\alpha$ as two linear functions of the parameter $\beta$. In fact, we could strengthen the assumption on the parameter $\gamma$ to guarantee that parameters $\alpha$ and $\beta$ satisfying assumption* (2.24) *exist. In this section, where we assume the system* (1.4) *with a fixed value of viability parameter $a = 1/2$, i.e., the system* (2.1)*, it is possible to do so.*

*By the comparison of the left and right sides in* (2.24) *we obtain that for the certainty of existence of parameters $\alpha$ and $\beta$ satisfying* (2.24) *the ratio of capacities has to satisfy*

$$0 < \gamma < \frac{1}{36}(27 + \sqrt{3} - \sqrt{6(14 + 9\sqrt{3})}) \approx 0.4279.$$

In the following example we illustrate the assumptions of Theorem 2.10 for the system (2.7) with fixed parameters $\alpha$, $\beta$ and $\gamma$.

EXAMPLE 2.12. *Consider the system* (2.7) *with parameters $\alpha = \beta = 2$ and $\gamma = 1/3$. That is*

$$\begin{cases} \dfrac{dx}{d\tau} &= \frac{y}{3} - x + 2x\,(1-x)\left(x - \frac{1}{2}\right), \\ \dfrac{dy}{d\tau} &= 3x - y + 2y\,(1-y)\left(y - \frac{1}{2}\right). \end{cases} \tag{2.25}$$

*Obviously, the parameters $\alpha$, $\beta$, $\gamma$ satisfy the assumptions of Lemma 2.6. Since $\gamma = 1/3 > 2/7$, we have to verify if $\alpha$ is greater than the second expression of* (2.17) *only, see Remark 2.8. The inequality has the form*

$$\alpha > \frac{\sqrt{3}}{18\,(2\gamma - 1)\,(\gamma - 1)}\beta.$$

*After substituting the values $\alpha = \beta = 2$ and $\gamma = 1/3$ we obtain*

$$2 > \frac{\sqrt{3}}{2}.$$

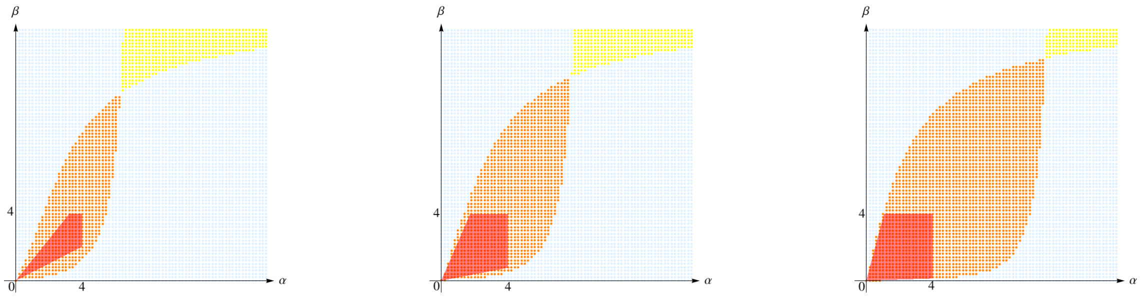
*Consequently the assumption of Lemma 2.9 is also satisfied since*

$$2 < \frac{9}{\sqrt{3}},$$

*Thus, based on Theorem 2.10, the system* (2.25) *has only one stationary solution - the trivial one.*

*To see that there does not exist a non-trivial stationary solution of* (2.25)*, we express a nullclines of this system*

$$N_x = \left\{(x, y) \in \mathbb{R}_0^{+2} : y = 3\left(x - 2x(1-x)\left(x - \frac{1}{2}\right)\right)\right\},$$
$$N_y = \left\{(x, y) \in \mathbb{R}_0^{+2} : x = \frac{y - 2y(1-y)\left(y - \frac{1}{2}\right)}{3}\right\} \tag{2.26}$$




(a) $\gamma = 2/5$. (b) $\gamma = 1/3$. (c) $\gamma = 1/5$.

Figure 2.5: Comparison of regions given by the conditions of Theorem 2.10 (red region) and the numerical results for fixed parameter $\gamma$. Orange points represent combinations of parameters $\alpha$ and $\beta$ for which the system (2.7) has unique stationary solution. Yellow and light blue colored points represent values of these parameters for which the equation has five and three stationary solutions, respectively.

*and plot them with the vector field given by* (2.25). *It is shown at Fig. 2.4. We can see that these nullclines intersect only in point* $(0, 0)$.

In the way to prove lemmas which form Theorem 2.10 we have made many limitations that reduce the possible combination of parameters $\alpha$, $\beta$, $\gamma$ under which the system (2.7) has the unique stationary solution. To see that this can be improved, we run a simple simulation, that for a fixed $\gamma$ runs through combinations of parameters $\alpha$ and $\beta$ and counts the number of stationary solutions of the system (2.7). In Fig. 2.5 we show the results of these computations for different $\gamma$ with red region representing results of Theorem 2.10.

The first observation based on Fig. 2.5, is that the assumptions of Theorem 2.10 can be greatly improved for $\alpha, \beta \geq 4$. Moreover, for $\gamma = 2/5$ we can see that our assumptions do not give the best estimate for which a non-trivial stationary solutions cannot exist. On the contrary, in the case $\gamma = 1/5$ we determined almost the same region as we found by the numeric algorithm. For even smaller value of $\gamma$ we have almost determined the exactly same region as found by numeric algorithm.

Finally, we rewrite Theorem 2.10 in the terms of the original problem (2.1).

COROLLARY 2.13. *Assume that* $D > \frac{\max\{\lambda_1, \lambda_2\}}{4}$, $k_2 < \frac{k_1}{2}$ *and*

$$\max\left\{\frac{2\sqrt{3}k_1^2}{9\,(k_2-k_1)\,(k_2-2k_1)}, \frac{\sqrt{3}k_1^2}{18\,(2k_2-k_1)\,(k_2-k_1)}\right\} < \frac{\lambda_1}{\lambda_2} < \frac{9\,(2k_2-k_1)\,(k_2-k_1)}{2\sqrt{3}k_2^2},$$

*then the system* (2.1) *has the unique stationary solution* $(x_1^*, x_2^*) = (0, 0)$.

### 2.1.2 Generalization

In the previous section, we derived the conditions for existence of the unique stationary solution of (2.1). That is the system (1.4) with fixed viability parameter $a = 1/2$. Here, we discuss the generalization of these conditions for (1.4), i.e., the system with viability parameter $a \in (0, 1)$.

In fact, we prove an analogy of the Theorem 2.14 (more precisely, its Corollary 2.13) the same way as we did in Section 2.1.1. Instead of proving all auxiliary lemmas, we prove the following theorem in a short way with pointing out differences in the proofs of Lemmas 2.3, 2.5, 2.6, 2.7 and 2.9 for the system (1.4).

THEOREM 2.14. *Assume that* $D > \max\{\lambda_1, \lambda_2\}\frac{a^2-a+1}{3}$, $k_2 < ak_1$,

$$\frac{\lambda_1}{\lambda_2} > 2\max\left\{\frac{k_1^2\left((a+1)\left(a-\frac{1}{2}\right)(a-2)+(1+a(a-1))^{3/2}\right)}{27a^2(k_2-k_1)(ak_2-k_1)},\right.$$
$$\left.\frac{k_1^2\left((a+1)\left(a-\frac{1}{2}\right)(a-2)+(1+a(a-1))^{3/2}\right)}{27(k_2-ak_1)(k_2-k_1)}\right\} \tag{2.27}$$

*and*

$$\frac{\lambda_1}{\lambda_2} < 2\frac{(ak_1-k_2)(k_2-k_1)\left((a+1)\left(a-\frac{1}{2}\right)(a-2)-(1+a(a-1))^{3/2}\right)}{(a-1)k_2^2}. \tag{2.28}$$

*Then the system* (1.4) *has the unique stationary solution* $(x_1^*, x_2^*) = (0, 0)$.

*Proof.* Similarly as in the previous section for the system (2.1), we use the substitution (2.6) for (1.4) to reduce some parameters. We obtain a system

$$\begin{cases} \dfrac{dx}{d\tau} &= \gamma y - x + \alpha f(x),\\ \dfrac{dy}{d\tau} &= \frac{1}{\gamma}x - y + \beta f(y), \end{cases} \tag{2.29}$$

where the parameters $0 < \alpha = \lambda_1/D$ and $0 < \beta = \lambda_2/D$ stand for ratios of the growth rate of each population and the diffusion, $0 < \gamma = k_2/k_1$ is the ratio of capacities, $a \in (0,1)$ is the viability parameter and the function $f$ is given by (1.9).

Remark 2.2 holds for this system too. Thus, we assume $\gamma \in (0, 1]$. Analogy of Lemma 2.3 for the system (2.31) would guarantee following statements

1. For $\gamma = 1$, the system (2.31) has at least three stationary solutions $\mathbf{x}_2 = (0, 0)$, $\mathbf{x}_2 = (a, a)$ and $\mathbf{x}_3 = (1, 1)$.

2. For $\gamma = a$, the system (2.31) has at least two stationary solutions $\mathbf{x}_1 = (0, 0)$ and $\mathbf{x}_2 = (a, 1)$.

3. For $\gamma \in (a, 1)$ the system (2.31) has the trivial stationary solution and at least two more.

This implies that if we are looking for conditions for which non-trivial stationary solutions do not exist, we have to assume $\gamma \in (0, a)$.

Analogously to the proof of Lemma 2.5, we can use signs of the functions $f(x)$ and $f(y)$, which are given by (1.9), thus the sign is given by (1.10), to determine possible regions where a non-trivial stationary solution of (2.31) can be located. These regions are

$$\begin{aligned} \widehat{\Omega}_1 &= \left\{(x, y) \in \mathbb{R}^2 : x \in (0, \gamma),\, y \in (a, 1),\, x < \gamma y\right\},\\ \widehat{\Omega}_2 &= \left\{(x, y) \in \mathbb{R}^2 : x \in (a, 1),\, y \in (0, a)\right\},\\ \widehat{\Omega}_3 &= \left\{(x, y) \in \mathbb{R}^2 : x \in (a, 1),\, y \in (1, 1/\gamma),\, x > \gamma y\right\}. \end{aligned} \tag{2.30}$$

Further, we wish to reduce these regions analogously as we did in Lemma 2.6.

In the proof of the Lemma 2.6, we used the concavity of the function $f$ given by (1.9) and the monotonicity of right hand sides of equations in (2.10) to reduce the regions (2.30). Stationary solutions of the system (2.31) satisfy

$$\begin{cases} y &= \frac{1}{\gamma}\left(x - \alpha f(x)\right),\\ x &= \gamma\left(y - \beta f(y)\right), \end{cases} \tag{2.31}$$

where the function $f$ is given by (1.9). We define a function

$$h_\delta(s) := s - \delta f(s), \tag{2.32}$$

which corresponds to the right hand sides of the equations in this system. The monotonicity of the function $h$ is important for the reduction of the regions where a stationary solution may be located. The function $h$ is monotone for $\delta \in \left(0, 3/(a^2 - a + 1)\right)$.

Assumption of $\alpha, \beta \in \left(0, 3/(a^2 - a + 1)\right)$ is sufficient to formulate an analogical lemma as Lemma 2.6, since the other condition we have to satisfy while applying the concavity is $\alpha, \beta \in (0, 1/(a(1-a)))$ which is implied by the previous assumption. Thus, we can reduce the regions (2.30). The reduced regions are given as follows

$$\begin{aligned} \Omega_1 &= \left\{(x, y) \in \mathbb{R}^2 : x \in (a\gamma, \gamma),\, y \in (a, 1),\, x < \gamma y\right\}, \\ \Omega_2 &= \left\{(x, y) \in \mathbb{R}^2 : x \in (a, 1),\, y \in (a/\gamma, 1/\gamma),\, x > \gamma y\right\}. \end{aligned} \tag{2.33}$$

Further, we want an analogy of assumptions from Lemmas 2.7 and 2.9 to eliminate the possibility that a stationary solution will be located $\Omega_1$ or $\Omega_2$ given by (2.33). Here comes a first complication with parameter $a$ in the system (2.31). In the proofs of Lemmas 2.7 and 2.9 we have used the maximum of function $f$ given by (1.9) with fixed $a = 1/2$, that is $1/(12\sqrt{3})$. For the general viability parameter $a \in (0, 1)$ the function (1.9) has a more complicated maximum value. More precisely,

$$\max_{x \in (a,1)} f(x) = \frac{1}{27}\left(1 - 2a + \sqrt{a^2 - a + 1}\right)\left(2 - a - \sqrt{a^2 - a + 1}\right)\left(1 + a + \sqrt{a^2 - a + 1}\right). \tag{2.34}$$

Following the proof of Lemma 2.7, we obtain values

$$y_1^1 = a + \beta \max_{x \in (a,1)} f(x), \quad y_2^1 = 1 + \beta \max_{x \in (a,1)} f(x),$$

where $\max f(x)$ is given by (2.34). These values correspond to the values of upper boundary linear function which bound the first function of (2.31) for $x_1 = a\gamma$ and $x_2 = \gamma$, i.e., the boundary of region $\Omega_1$ given by (2.33). The second equation of (2.31) intersect the lines $x_1 = a\gamma$ and $x_2 = \gamma$ for

$$y_1^2 = a + \alpha a^2(\gamma - a)(a\gamma - 1), \quad y_2^2 = 1 - \alpha\left(\gamma(\gamma + 1) + a(1 - \gamma)\right).$$

If

$$\alpha > 2\beta \frac{(a + 1)\left(a - \frac{1}{2}\right)(a - 2) + (1 + a(a - 1))^{3/2}}{27a^2(\gamma - 1)(a\gamma - 1)}, \tag{2.35}$$

then $y_1^1 < y_1^2$ and if

$$\alpha > 2\beta \frac{(a + 1)\left(a - \frac{1}{2}\right)(a - 2) + (1 + a(a - 1))^{3/2}}{27(\gamma - a)(\gamma - 1)}, \tag{2.36}$$

then $y_2^1 < y_2^2$.

Assumptions (2.35) and (2.36) together, eliminate that possibility that any stationary solution of (2.31) is located in $\Omega_1^2$ given by (2.33). See the proof of Lemma 2.7.

For elimination of the region $\Omega_2^2$ given by (2.33) also, we proceed the same way, as we did in the proof of Lemma 2.9. That is, we limit the function $f$, given by (1.9), in the first equation of (2.31) by the maximum (2.34). Further, we use concavity of the function $f$ given by (1.9) for the second equation of (2.31). This way, we obtain two boundary lines - linear functions.

Clearly, a slope of the first function, which is defined by the maximum of the function $f$, is $1/\gamma$. The slope of the second boundary line have more complicated form. But it can be shown that it satisfies

$$\frac{\gamma}{\gamma^2 + a\gamma(\gamma - 2) + a^2(3 - 2\gamma)} < \frac{1}{\gamma},$$

| $a$ | 1/8 | 1/4 | 3/8 | 1/2 | 5/8 | 3/4 | 7/8 |
|---|---|---|---|---|---|---|---|
| $\gamma_{up}$ | 0.0368 | 0.1613 | 0.2961 | 0.4279 | 0.5648 | 0.7060 | 0.8510 |

Table 2.1: Values of $\gamma_{up}$ obtained by numerical computation for some fixed values of the viability parameter $a$, which guarantee that for ratio of capacities $k_2/k_1 = \gamma \in (0, \gamma_{up})$ the assumptions (2.27) and (2.28) of Theorem 2.14 do not contradict each other (rounded to four decimal digits).

i.e, we have the same ordering of the slopes as in the proof of Lemma 2.9. Now we find for which value of $x$ the first boundary line intersect the line $y = a/\gamma$ which define a bottom boundary of $\Omega_2^2$ given by (2.33). We obtain

$$x_1 = a + b\left(a - a^2 - \gamma(1 + a\gamma)\right),$$

where $\max f(x)$ is given by (2.34). To obtain the second intersection, we have to just substitute the value $y = a/\gamma$ into the second equation of (2.31). Thus

$$x_2 = a + \alpha \max_{x\in(a,1)} f(x).$$

If

$$\alpha < 2\beta \frac{(a-\gamma)(\gamma-1)\left((a+1)\left(a-\frac{1}{2}\right)(a-2) - (1+a(a-1))^{3/2}\right)}{(a-1)\gamma^2}. \tag{2.37}$$

then $x_1 < x_2$. Together with the ordering of slopes, these lines have no intersection in $\Omega_2^2$ given by (2.33).

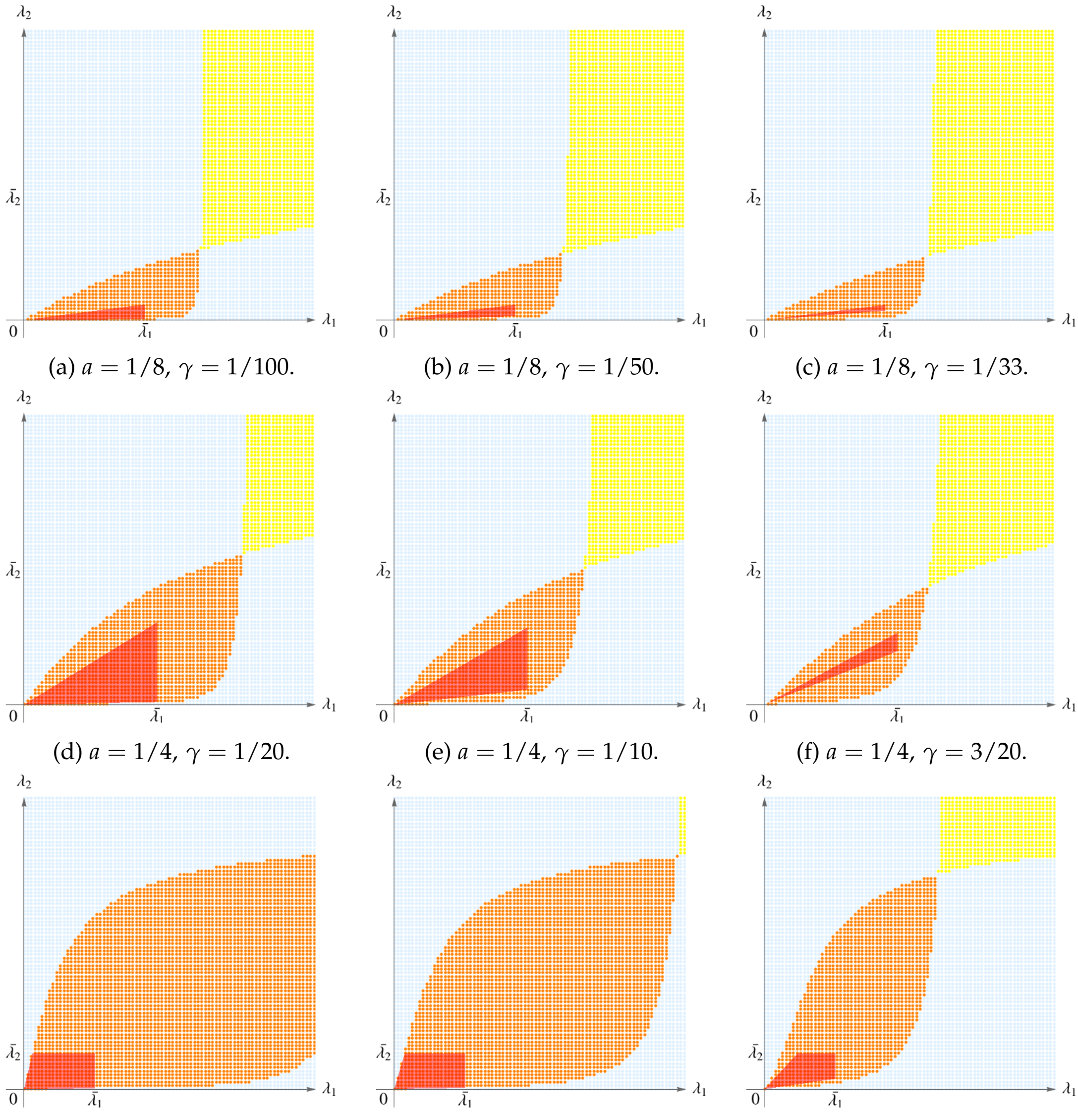
To summarize it, assumption $\gamma \in (0, a)$ is necessary, since otherwise a non-trivial stationary solutions of (2.31) always exist. $\alpha, \beta \in \left(0, 3/(a^2 - a + 1)\right)$ guarantee that if any non-trivial stationary solution of (2.31) does exist, it has to be located in one of the regions $\Omega_1$ and $\Omega_2$ given by (2.33). Assumptions (2.35) and (2.36) eliminate possibility that a non-trivial stationary solution is located in $\Omega_1^2$. Last, assumption (2.37) eliminate the region $\Omega_2^2$.

Rewriting all the assumptions to original parameters from the system (1.4), we obtain the statement of this theorem. □

Similarly as the assumption (2.24) in Theorem 2.10 gave boundaries for the parameter $\alpha$ as two linear functions of the parameter $\beta$, assumptions (2.27) and (2.28) of Theorem 2.14 give interval where the ratio of $\lambda_1$ and $\lambda_2$ has to belong. We would like to have similar result as stated in Remark 2.11. That is the answer on a question for which values of $a$ and $\gamma$ is this interval non-empty. Here, it is not so simple to answer this question. Instead, we bring some numerical results for few fixed values of the viability parameter $a$ which give us the maximal value $\gamma_{up}$ such that for all ratios of capacities $k_2/k_1 = \gamma \in (0, \gamma_{up})$ the interval for ratio of reactions parameters $\lambda_1/\lambda_2$ is non-empty, see Table 2.1.

For the fixed value of diffusion parameter $D = 1$ and various values of parameters $a$ and $\gamma$ satisfying statement of the previous paragraph, we run the same numerical simulation as in Section 2.1.1, which determine how many stationary solutions of the system (1.4) exist for a pair $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2$. Based on the number of stationary solution we assign a color to this point. The choice of parameters $D$, $a$ and $\gamma$ guarantee that the assumptions of Theorem 2.14 does not contradict each other. Thus the region of parameters $\lambda_1$ and $\lambda_2$ satisfying the assumptions (2.27) and (2.28) is non-empty and can be plot over the results of the numerical simulation.

Results of the numerical simulation and the region given by assumptions of Theorem (2.14) can be viewed in Fig. 2.6. There, orange, yellow and blue points represent pairs $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2$ such that the system (1.4) has one, three and five stationary solutions, respectively. The red region is given by assumptions of Theorem (2.14), i.e., the region for which we have analytically proved the existence of a unique stationary solution. By the comparison of the numerical and the analytical results we see that assumptions of Theorem 2.14 can be greatly improved. Especially, the assumption $D < \max\{\lambda_1, \lambda_2\}(a^2 - a + 1)/3$, which guarantee the monotonicity of the function $h$ given by (2.32) corresponding to the separated equations of (2.31) is too strong.



(a) $a = 1/8$, $\gamma = 1/100$. (b) $a = 1/8$, $\gamma = 1/50$. (c) $a = 1/8$, $\gamma = 1/33$.

(d) $a = 1/4$, $\gamma = 1/20$. (e) $a = 1/4$, $\gamma = 1/10$. (f) $a = 1/4$, $\gamma = 3/20$.

(g) $a = 5/8$, $\gamma = 1/4$. (h) $a = 5/8$, $\gamma = 1/3$. (i) $a = 5/8$, $\gamma = 1/2$.

Figure 2.6: Comparison of regions given by conditions of Theorem 2.14 (red region) and the numerical results for the fixed diffusion parameter $D = 1$ and different values of parameters $a$, $\gamma = k_2/k_1$ satisfying condition given by Tab. 2.1. $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are the greatest values of $\lambda_1$ and $\lambda_2$ satisfying the first assumption of this theorem ($D < \max\{\lambda_1, \lambda_2\}(a^2 - a + 1)/3$). Orange points represent combinations of parameters $\lambda_1$ and $\lambda_2$ for which the system (1.4) has unique stationary solution. Yellow and light blue colored points represent values of these parameters for which the equation has five and three stationary solutions, respectively.

## 2.2 4-cycle

In this section we present a simple example of a graph, for which the statement of Theorem 2.14 can be used. That means, we are deriving the same result for (1.3) as we did for the system (1.4).

The simplest graph for which the results of Section (2.1.2) can be used is a cycle with four vertices, i.e., $C_4$. We assume a special case of this graph, where the capacities and growth rates of non-adjacent vertices are the same. In this case, the system (1.3) has a form

$$\begin{aligned}
\frac{dx_1}{dt} &= D(x_4 - 2x_1 + x_2) + \lambda_1 x_1 \left(1 - \frac{x_1}{k_1}\right)\left(\frac{x_1}{k_1} - a\right), \\
\frac{dx_2}{dt} &= D(x_1 - 2x_2 + x_3) + \lambda_2 x_2 \left(1 - \frac{x_2}{k_2}\right)\left(\frac{x_2}{k_2} - a\right), \\
\frac{dx_3}{dt} &= D(x_2 - 2x_3 + x_4) + \lambda_1 x_3 \left(1 - \frac{x_3}{k_1}\right)\left(\frac{x_3}{k_1} - a\right), \\
\frac{dx_4}{dt} &= D(x_3 - 2x_4 + x_1) + \lambda_2 x_4 \left(1 - \frac{x_4}{k_2}\right)\left(\frac{x_4}{k_2} - a\right),
\end{aligned} \tag{2.38}$$

where $D > 0$ is the diffusion coefficient, $\lambda_i > 0$ and $k_i > 0$, $i = 1, 2$, stand for the population growth rates and the capacities of territories and $a \in (0, 1)$ is the viability parameter.

For $D$ close to zero, the system (2.38) has $3^4$ solutions. We are interested in the opposite case, i.e., existence of the unique stationary solution. For this purpose, we use one auxiliary lemma, which will guarantee, that the results derived for the system (1.4) can be used. We start with an analogical substitution as (2.6), which has been used for the systems (1.4) and (2.1). That is

$$\bar{x}_1 = \frac{x_1}{k_1}, \quad \bar{x}_2 = \frac{x_3}{k_1}, \quad y_1 = \frac{x_2}{k_2}, \quad y_2 = \frac{x_4}{k_2}, \quad \tau = 2Dt.$$

We omit the bars over $\bar{x}_1$ and $\bar{x}_2$. This leads to a system

$$\begin{aligned}
\frac{dx_1}{d\tau} &= \gamma\left(\frac{y_1 + y_2}{2}\right) - x_1 + \frac{\alpha}{2} f(x_1), \\
\frac{dy_1}{d\tau} &= \frac{1}{\gamma}\left(\frac{x_1 + x_2}{2}\right) - y_1 + \frac{\beta}{2} f(y_1), \\
\frac{dx_2}{d\tau} &= \gamma\left(\frac{y_1 + y_2}{2}\right) - x_2 + \frac{\alpha}{2} f(x_2), \\
\frac{dy_2}{d\tau} &= \frac{1}{\gamma}\left(\frac{x_1 + x_2}{2}\right) - y_2 + \frac{\beta}{2} f(y_2),
\end{aligned} \tag{2.39}$$

where $\gamma = k_2/k_1 > 0$ is ratio of the capacities, $\alpha = \lambda_1/D > 0$ and $\beta = \lambda_2/D > 0$ are ratios of the growth rates and the diffusion. The function $f(s)$ is defined as the bistable cubic (1.9).

Now we derive conditions under which the system of algebraic equations for the stationary solutions of the system (2.39) is equivalent with the system for a stationary solutions of the system (2.7).

LEMMA 2.15. *Let $a \in (0, 1)$, $\gamma \in (0, 1]$ and*

$$\alpha, \beta < \frac{6}{a^2 - a + 1},$$

*then all stationary solutions of the system* (2.39) *have a form $(x^*, y^*, x^*, y^*)$ and satisfy*

$$\begin{aligned}
0 &= \gamma y^* - x^* + \frac{\alpha}{2} f(x^*), \\
0 &= \frac{1}{\gamma} x^* - y^* + \frac{\beta}{2} f(y^*).
\end{aligned} \tag{2.40}$$

*Proof.* Stationary solutions of (2.39) have to satisfy the following system of algebraic equations

$$\begin{aligned}
0 &= \gamma\left(\frac{y_1 + y_2}{2}\right) - x_1 + \frac{\alpha}{2} f(x_1), \\
0 &= \frac{1}{\gamma}\left(\frac{x_1 + x_2}{2}\right) - y_1 + \frac{\beta}{2} f(y_1), \\
0 &= \gamma\left(\frac{y_1 + y_2}{2}\right) - x_2 + \frac{\alpha}{2} f(x_2), \\
0 &= \frac{1}{\gamma}\left(\frac{x_1 + x_2}{2}\right) - y_2 + \frac{\beta}{2} f(y_2),
\end{aligned} \tag{2.41}$$

where the terms $s - \delta/2 f(s)$ can be written as the function $h_\delta(s)$ given by (2.32).

Since the first and the third equations in (2.41), same as the second and fourth, contain the same term, it follows that

$$\begin{aligned} h_{\frac{\alpha}{2}}(x_1) &= h_{\frac{\alpha}{2}}(x_2), \\ h_{\frac{\beta}{2}}(y_1) &= h_{\frac{\beta}{2}}(y_2). \end{aligned} \tag{2.42}$$

For $\delta \in (0, 3/(a^2 - a + 1))$ the function $h_\delta(s)$ has a positive derivative for all $s \in \mathbb{R}$. This means that the function is injective (strictly monotone). Thus, for $\alpha, \beta \in (0, 6/(a^2 - a + 1))$ the equations (2.42) are satisfied if and only if $x_1 = x_2$ and $y_1 = y_2$.

It follows, that for $\alpha, \beta \in (0, 6/(a^2 - a + 1))$ the system (2.41) contain the same equations. Thus, it can be reduced to the system (2.40). □

The system (2.40) was analyzed in Section 2.1.1 and 2.1.2. The only differences here are the halves of reaction coefficients $\alpha$ and $\beta$ (ratios of growth rates and diffusion). Clearly, simple substitution $\bar{\alpha} = \alpha/2$ and $\bar{\beta} = \beta/2$ solves this problem. We obtain the following theorem.

COROLLARY 2.16. *Let* $D > \max\{\lambda_1, \lambda_2\}\frac{(a^2-a+1)}{6}$, $k_2 < ak_1$ *and assume that* (2.27) *and* (2.28) *hold. Then the system* (2.38) *has the unique stationary solution* $(x_1^*, x_2^*, x_3^*, x_4^*) = (0, 0, 0, 0)$.

## 2.3 Complete bipartite graph

The same approach as in Section 2.2 can be applied for a complete bipartite graphs to obtain similar results. A complete bipartite graph is a graph with two disjoint sets of vertices, where every vertex from the first set is connected to all vertices from the other set but to non from the same set. Complete bipartite graph with sets of size $m$ and $n$ is denoted by $K_{m,n}$. Note that $C_4$ is a special case of complete bipartite graph, i.e. $C_4 = K_{2,2}$.

Assume a complete bipartite graph $K_{m,n}$, where the capacities and the growth rates of territories represented by a vertex from the same set are equal. Then the system (1.3) has a form

$$\begin{aligned} \frac{dx_i}{dt} &= D\left(\left(\sum_{j=m+1}^{m+n} x_j\right) - nx_i\right) + \lambda_1 x_i\left(1 - \frac{x_i}{k_1}\right)\left(\frac{x_i}{k_1} - a\right), \quad i = 1, ..., m, \\ \frac{dx_i}{dt} &= D\left(\left(\sum_{j=1}^{m} x_j\right) - mx_i\right) + \lambda_2 x_i\left(1 - \frac{x_i}{k_2}\right)\left(\frac{x_i}{k_2} - a\right), \quad i = m+1, ..., m+n. \end{aligned} \tag{2.43}$$

After substituting

$$\bar{x}_i = x_i/k_1,\ i = 1, ..., m, \quad y_i = x_{i+m}/k_2,\ i = 1, ..., n, \quad \tau = dt,$$

and omitting the bar over $x_i$ we obtain system

$$\begin{aligned} \frac{dx_i}{d\tau} &= \gamma \sum_{j=1}^{n} y_j - nx_i + \alpha f(x_i), \quad i = 1, ..., m \\ \frac{dy_i}{d\tau} &= \frac{1}{\gamma} \sum_{j=1}^{m} x_j - my_i + \beta f(y_i), \quad i = 1, ..., n, \end{aligned} \tag{2.44}$$

where the function $f$ is given by (1.9) and $\alpha = \lambda_1/d$, $\beta = \lambda_2/d$, $\gamma = k_2/k_1$ (all positive) stand for ratio of the growth rates with the diffusion and ratio of the capacities.

Stationary solutions of the system (2.44) have to satisfy

$$\begin{aligned} 0 &= \gamma \frac{\sum_{j=1}^{n} y_j}{n} - h_{\frac{\alpha}{n}}(x_i), \quad i = 1, ..., m \\ 0 &= \frac{1}{\gamma} \frac{\sum_{j=1}^{m} x_j}{m} - h_{\frac{\beta}{m}}(y_i), \quad i = 1, ..., n, \end{aligned}$$

where the function $h_\delta$ is given by (2.32).

Again, we have $m$ and $n$ equations with the same term. Thus, we obtain a chain of equalities

$$\begin{aligned} h_{\alpha/n}(x_1) = h_{\alpha/n}(x_2) = ... = h_{\alpha/n}(x_m), \\ h_{\beta/m}(y_{m+1}) = h_{\beta/m}(y_{m+2}) = ... = h_{\beta/m}(y_{m+n}). \end{aligned} \tag{2.45}$$

Since $h_\delta(s)$ is injective function for $\delta < 3/(a^2 - a + 1)$, the equations (2.45) with $\alpha < 3n/(a^2 - a + 1)$ and $\beta < 3m/(a^2 - a + 1)$ are satisfied if and only if

$$x_1 = x_2 = ... = x_m, \quad y_1 = y_2 = ... = y_n.$$

Thus, stationary solutions of the system (2.44) has form

$$(\underbrace{x^*, x^*, ..., x^*}_{m}, \underbrace{y^*, y^*, ..., y^*}_{n})$$

and satisfy

$$\begin{aligned} 0 &= \gamma y^* - x^* + \frac{\alpha}{n} f(x^*), \\ 0 &= \frac{1}{\gamma} x^* - y^* + \frac{\beta}{m} f(y^*). \end{aligned}$$

Again, we can use Theorem 2.14 to obtain the following result.

COROLLARY 2.17. *Let* $D > \max\left\{\frac{\lambda_1}{n}, \frac{\lambda_2}{m}\right\} \frac{a^2-a+1}{3}$, $k_2 < ak_1$ *and assume that inequalities* (2.27) *and* (2.28) *hold. Then the system* (2.43) *has the unique stationary solution*

$$(x_1^*, x_2^*, ..., x_{m+n}^*) = (0, 0, ..., 0).$$

# RDE in heterogeneous lattices 3

It has been shown, that the lattice Nagumo equation (1.5) can exhibit rich behavior. Unlike for continuous Nagumo equation (1.1), a traveling waves in the semi-discrete Nagumo equation can be found only for sufficiently strong diffusion, see [15]. Common traveling wave is a solution which connects two stationary solutions. Based on the homogeneity of the system (1.5), at least three bounded stationary solutions exist for any parameter values. In this chapter, we show that this is not the case for heterogeneous lattice equation (1.2).

The main goal of this chapter is to show that the system (1.2) may have trivial solution as a unique non-negative bounded stationary solution. This is done in Section 3.1 by applying the results from Chapter 2 for the two-dimensional system (2.1) on difference equations corresponding to the stationary solutions of a simplified system (1.2) with fixed viability parameter. Further, in Section 3.3 we show that other non-negative stationary solutions of this system exist, but they are unbounded. Finally, in Section 3.2 we discuss the generalization of these results for the system (1.2) with a general viability parameter.

## 3.1 Non-existence of bounded stationary solutions

Similarly as in Chapter 2, we start with a simpler form of (1.2), where the viability parameter is fixed $a = 1/2$. That is

$$\begin{cases} \frac{dx_i}{dt} &= D\left(x_{i-1} - 2x_i + x_{i+1}\right) + \lambda_1 x_i \left(1 - \frac{x_i}{k_1}\right)\left(\frac{x_i}{k_1} - \frac{1}{2}\right), \quad i = 2k,\, k \in \mathbb{Z}, \\ \frac{dx_i}{dt} &= D\left(x_{i-1} - 2x_i + x_{i+1}\right) + \lambda_2 x_i \left(1 - \frac{x_i}{k_2}\right)\left(\frac{x_i}{k_2} - \frac{1}{2}\right), \quad i = 2k-1,\, k \in \mathbb{Z}, \end{cases} \tag{3.1}$$

where $D > 0$ is the diffusion coefficient, $\lambda_1, \lambda_2 > 0$ are the coefficients of the growth rates and $k_1, k_2 > 0$ are the capacities of territories.

Following the ideas of Section 2.3, we show that adjusted conditions from Theorem 2.10 guarantee that the system (3.1) has no other bounded stationary solution besides the trivial one.

We start with a similar substitution for (3.1) as in Section 2.3

$$\bar{x}_{\frac{i}{2}} = \frac{x_i}{k_1},\, i = 2k, k \in \mathbb{Z}, \quad y_{\frac{i-1}{2}} = \frac{x_i}{k_2},\, i = 2k-1,\, k \in \mathbb{Z}, \quad \tau = dt. \tag{3.2}$$

For simplicity, we omit the bar in $\bar{x}_i$ in the following. This leads to a system

$$\begin{cases} \frac{dx_i}{d\tau} &= \gamma(y_{i-1} + y_i) - 2x_i + \alpha x_i\left(1 - x_i\right)\left(x_i - \frac{1}{2}\right), \quad i \in \mathbb{Z}, \\ \frac{dy_i}{d\tau} &= \frac{1}{\gamma}(x_i + x_{i+1}) - 2y_i + \beta y_i\left(1 - y_i\right)\left(y_i - \frac{1}{2}\right), \quad i \in \mathbb{Z}, \end{cases} \tag{3.3}$$

where $\gamma = k_2/k_1 > 0$ stands for the ratio of capacities and $\alpha = \lambda_1/D$, $\beta = \lambda_2/D > 0$ are ratios of the growth rates and the diffusion.

REMARK 3.1. *For the system* (3.3)*, we assume, without loss of generality,* $\gamma \in (0, 1]$. *Otherwise, we use analogical substitution as in Remark 2.2.*

Stationary solutions of this system satisfy an infinite system of algebraic equations

$$\begin{cases} 0 &= \gamma(y_{i-1} + y_i) - 2x_i + \alpha x_i\,(1 - x_i)\left(x_i - \frac{1}{2}\right), \quad i \in \mathbb{Z}, \\ 0 &= \frac{1}{\gamma}(x_i + x_{i+1}) - 2y_i + \beta y_i\,(1 - y_i)\left(y_i - \frac{1}{2}\right), \quad i \in \mathbb{Z}. \end{cases} \tag{3.4}$$

The system (3.4) can be viewed as an iterative scheme

$$\begin{cases} x_{i+1} &= 2\gamma g_{\frac{\beta}{2}}(y_i) - x_i, \quad i \in \mathbb{Z}, \\ y_{i+1} &= \frac{2}{\gamma} g_{\frac{\alpha}{2}}(x_{i+1}) - y_i, \quad i \in \mathbb{Z}, \end{cases}$$

For our needs we rewrite it in a bit different form, which we refer to as a mirror scheme. More precisely, the following relations are called the forward mirror scheme

$$\begin{cases} x_{i+1} - \gamma g_{\frac{\beta}{2}}(y_i) &= \gamma g_{\frac{\beta}{2}}(y_i) - x_i, \quad i \in \mathbb{Z}, \\ y_{i+1} - \frac{1}{\gamma} g_{\frac{\alpha}{2}}(x_{i+1}) &= \frac{1}{\gamma} g_{\frac{\alpha}{2}}(x_{i+1}) - y_i, \quad i \in \mathbb{Z}, \end{cases} \tag{3.5}$$

where the function $g_\delta(s)$ is defined by (2.15).

Analogously we can rewrite the system (3.4) to a backward mirror scheme

$$\begin{cases} y_{i-1} - \frac{1}{\gamma} g_{\frac{\alpha}{2}}(x_i) &= \frac{1}{\gamma} g_{\frac{\alpha}{2}}(x_i) - y_i, \quad i \in \mathbb{Z}, \\ x_{i-1} - \gamma g_{\frac{\beta}{2}}(y_{i-1}) &= \gamma g_{\frac{\beta}{2}}(y_{i-1}) - x_i, \quad i \in \mathbb{Z}, \end{cases} \tag{3.6}$$

where the function $g_\delta(s)$ is defined by (2.15).

For $\alpha$, $\beta < 8$, $\gamma \in (0, 1/2)$ and

$$\max\left\{\frac{2\sqrt{3}}{9\,(\gamma-1)\,(\gamma-2)}\beta,\ \frac{\sqrt{3}}{18\,(2\gamma-1)\,(\gamma-1)}\beta\right\} < \alpha < \frac{9\,(2\gamma-1)\,(\gamma-1)}{2\sqrt{3}\gamma^2}\beta, \tag{3.7}$$

a system

$$\begin{cases} y &= \frac{1}{\gamma} g_{\frac{\alpha}{2}}(x), \\ x &= \gamma g_{\frac{\beta}{2}}(y), \end{cases} \tag{3.8}$$

has unique stationary [0, 0] (follows directly from Theorem 2.10). This entitles us to split the first quadrant into three regions.

DEFINITION 3.2. *Let conditions* $\alpha$, $\beta < 8$, $\gamma \in (0, 1/2)$ *and* (3.7) *hold. We define curves given by the equations from* (3.8)

$$\begin{aligned} \phi_1 &: \left(x;\ \frac{1}{\gamma} g_{\frac{\alpha}{2}}(x)\right),\ x \in \mathbb{R}, \\ \phi_2 &: \left(\gamma g_{\frac{\beta}{2}}(y);\ y\right),\ y \in \mathbb{R}. \end{aligned} \tag{3.9}$$

*which divide the first quadrant into three regions*

$$\begin{aligned} A &= \left\{(x, y) \in \mathbb{R}^2 : x > 0,\ y > \frac{1}{\gamma} g_{\frac{\alpha}{2}}(x)\right\}, \\ B &= \left\{(x, y) \in \mathbb{R}^2 : x < \gamma g_{\frac{\beta}{2}}(y),\ y < \frac{1}{\gamma} g_{\frac{\alpha}{2}}(x)\right\}, \\ C &= \left\{(x, y) \in \mathbb{R}^2 : x > \gamma g_{\frac{\beta}{2}}(y),\ y > 0\right\}. \end{aligned} \tag{3.10}$$

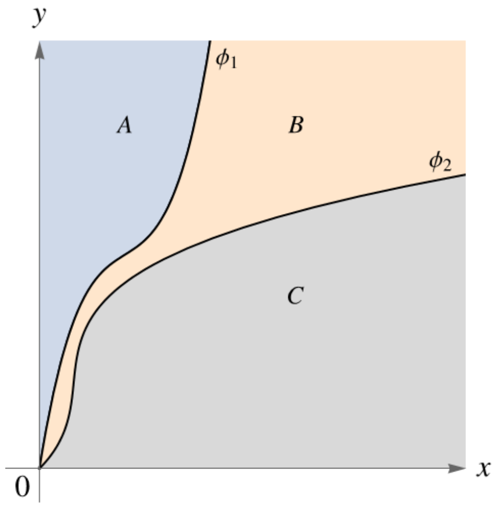



Figure 3.1: An example of curves $\phi_1$, $\phi_2$ given by (3.9) with parameters $\alpha = \beta = 6$ and $\gamma = 0.4$ and regions $A, B, C$ defined by (3.10) and bounded by these curves.

Example of curves (3.9) and regions (3.10) from Definition 3.2 can be seen in Figure 3.1.

REMARK 3.3. *The system of equations* (3.8) *is closely related to the forward and backward mirror schemes* (3.5) *and* (3.6). *In particular, it is formed by the functions which occur in these schemes. Clearly, any solution of* (3.8) *can be periodically extended to satisfy* (3.4).

The following auxiliary result justifies the fact that we have named the systems (3.5) and (3.6) forward and backward mirroring schemes.

LEMMA 3.4. *Assume that $\alpha$, $\beta < 8$ and $[x_i, y_i]$ is a point in the first quadrant, i.e., $x_i$, $y_i > 0$, then*

1. *the odd steps of the forward mirror scheme given by* (3.5) *and even steps of the backward mirror scheme* (3.6) *map $[x_i, y_i]$ onto the unique point located in the opposite side of the curve $\phi_2$ given by* (3.9)*,*
2. *the even steps of the forward mirror scheme given by* (3.5) *and odd steps of the backward mirror scheme* (3.6) *map $[x_i, y_i]$ onto the unique point located in the opposite side of the curve $\phi_1$ given by* (3.9).

*Moreover, the point $[x_i, y_i]$ with its image given by these schemes has the same distance from the mirroring curve $\phi_1$ or $\phi_2$ in the corresponding direction, i.e., vertical or horizontal.*

*Proof.* The function $g_\delta(s)$ given by (2.15) is injective if $\delta < 4$, see the proof of Lemma 2.15. Thus, assuming $\alpha$, $\beta < 8$ guarantee that the images of $[x_i, y_i]$ given by the forward and backward mirror schemes (3.5) and (3.6) are unique.

Denote $\bar{x}_i$ a value of curve $\phi_2$ defined by (3.9) in $y_i$, i.e.,

$$\bar{x}_i = \gamma g_{\frac{\beta}{2}}(y_i).$$

Then the first step of forward mirror scheme (3.5) has a form

$$x_{i+1} - \bar{x}_i = \bar{x}_i - x_i.$$

This can be interpreted as follows: the value $x_{i+1}$ is in the same distance[1] but in opposite direction as the value $x_i$ from the fixed value $\bar{x}_i$ which is given by the curve $\phi_2$. A similar relation can be

[1] The equality holds in the absolute value too. Since $x_{i+1}$, $x_i$, $\bar{x}_i \in \mathbb{R}$, the absolute value gives us an equality of the metrics on the reals numbers.

derived analogically from the second equation of (3.5) for values $y_{i+1}$, $y_i$ with a value $\bar{y}_{i+1}$ of the curve $\phi_1$.

The statement for the other steps of the forward and backward mirror scheme (3.5) and (3.6) follows from the fact, that these steps has the same structure, just different indexes. □

The main result of this section states sufficient conditions for the nonexistence of nontrivial bounded nonnegative stationary solutions.

THEOREM 3.5. *Assume that $\alpha$, $\beta < 8$, $\gamma \in (0, 1/2)$ and* (3.7) *hold, then the system* (3.3) *has the unique non-negative stationary solution in the space $\ell^\infty(\mathbb{Z})$ - the trivial one.*

*Proof.* Stationary solutions of (3.3) have to satisfy the system of algebraic equation (3.4). We have already rewritten this system into the forward mirroring scheme (3.5) and the backward mirroring scheme (3.6).

If assumptions of this theorem hold, then the curves (3.9) divide the first quadrant into three disjoint sets (3.10), see Definition 3.2.

We want to show, that for any starting pair $[x_i, y_i] \neq [0, 0]$ located in the first quadrant the mirroring schemes lead to an unbounded sequence.

We start with a pair $[x_i, y_i] \in A \cup B \cup \phi_1$ given by (3.9) and (3.10). By the first statement of Lemma 3.4, the first step of the forward mirror scheme (3.5) maps this point into the region $C$. By the second statement of this lemma, the second step of the forward mirror scheme maps the pair $[x_{i+1}, y_i]$ (located in $C$) into the region $A$. Moreover, for the starting pair $[x_i, y_i] \in A \cup B \cup \phi_1$ we have

$$\bar{x}_i - x_i > 0 \quad \Longrightarrow \quad x_{i+1} > x_i \tag{3.11}$$

and for $[x_{i+1}, y_i] \in C$

$$\bar{y}_{i+1} - y_i > 0 \quad \Longrightarrow \quad y_{i+1} > y_i. \tag{3.12}$$

Since the pair $[x_i, y_i]$ from region $A \cup B \cup \phi_1$ given by (3.9) and (3.10) is mapped onto the pair $[x_{i+1}, y_{i+1}]$ located in region $A \subset A \cup B \cup \phi_1$, the forward mirror scheme (3.5) can be applied again in the same way. Specially, from (3.11) and (3.12) we have

$$x_{i+k} > x_{i+k-1} > \ldots > x_{i+1} > x_i, \quad y_{i+k} > y_{i+k-1} > \ldots > y_{i+1} > y_i, \quad k \in \mathbb{N}. \tag{3.13}$$

Moreover, the distance of $x_{j+1}$ and $x_j$ (or $y_{j+1}$ and $y_j$) is for $j \geq i$ bounded from below by the distance of curves $\phi_1$ and $\phi_2$ in $x$ (or $y$) direction. If $\alpha$, $\beta < 8$, $\gamma < 1/2$ and the inequalities (3.7) hold, then there is only one solution of (3.9) - the trivial one, i.e., the distances of $x_{j+1}$ and $x_j$ (or $y_{j+1}$ and $y_j$) cannot go to zero for increasing non-negative sequence. Thus, the sequence $\{[x_n, y_n]\}_{n=i}^{+\infty}$ given by the forward mirror scheme (3.5) for a starting pair $[x_i, y_i] \in A \cup B \cup \phi_1$ is unbounded and satisfies

$$\lim_{n \to \infty} [x_n, y_n] = [+\infty, +\infty].$$

Part of this sequence with starting pair $[x_i, y_i] \in A$ can be seen in Figure 3.2.

Now assume that the starting pair $[x_i, y_i]$ is located in region $C \cup \phi_2$ given by (3.9) and (3.10). Since by the Lemma 3.4, the first step of the forward mirror scheme (3.5) maps this point into the opposite side of the curve $\phi_2$, we must be more careful. On contrary of (3.11), this time we obtain $x_{i+1} \leq x_i$. The conditions under which the sequence stops being decreasing and leads to previous case may be found. The fact that $x_{i+1} \leq x_i$ also directly implies that the sequence may leave the first quadrant during the decreasing phase, i.e., it may happen that there exists index $j$ such that $x_j < 0$ or $y_j < 0$ (see Figure 3.3).

Instead of complicated analysis of the forward mirror scheme (3.5) with a starting pair $[x_i, y_i]$ located in the region $C \cup \phi_2$ given by (3.10) and (3.9), we use the second mirror scheme we derived from (3.4), the backward mirror scheme (3.6).

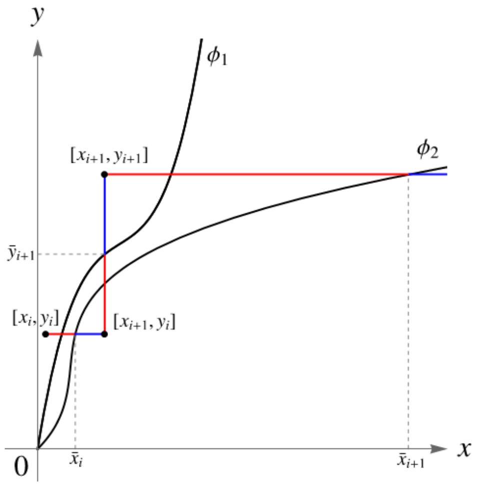



Figure 3.2: The forward mirror scheme (3.5) for a stationary solutions of (3.3) with starting pair $[x_i, y_i]$ located in region $A$ given by (3.10). The red and blue segments have the same length in every step of this scheme.

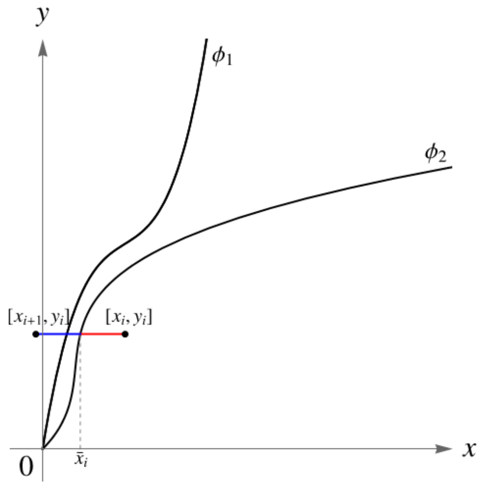



Figure 3.3: The forward mirror scheme (3.5) for a stationary solutions of (3.3) with starting pair $[x_i, y_i]$ located in region $C$ given by (3.10). The red and blue segments have the same length in every step of this scheme.

By the second statement of Lemma 3.4, for any $[x_i,\, y_i] \in B \cup C \cup \phi_2$ given by (3.9) and (3.10), the first step of backward mirror scheme (3.6) maps this pair into region $A$. By the second statement of this lemma, the image of $[x_i,\, y_i]$ is then mapped into region $C$. Moreover, for the starting pair $[x_i,\, y_i] \in C \cup \phi_2$ we have

$$\bar{y}_i - y_i > 0 \quad \Longrightarrow \quad y_{i-1} > y_i \tag{3.14}$$

and for $[x_i,\, y_{i-1}] \in C$

$$\bar{x}_{i-1} - x_i > 0 \quad \Longrightarrow \quad x_{i-1} > x_i. \tag{3.15}$$

Again, $[x_{i-1},\, y_{i-1}]$ is located in the subset of the starting region, i.e., $C \subset C \cup \phi_2$. Thus (3.14) and (3.15) hold for following steps too. That is

$$y_{i-j} > y_{i-j+1} > \ldots > y_{i-1} > y_i, \quad x_{i-j} > x_{i-j+1} > \ldots > x_{i-1} > x_i, \quad j \in \mathbb{N}.$$

The property of the distance of the two following pairs remains the same as for the forward mirror scheme (3.5). Thus, considering the backward mirror scheme with a starting pair $[x_i,\, y_i]$ from region $C \cup \phi_2$, given by (3.10), we can show that the sequence $\{[x_n, y_n]\}_{-\infty}^{n=i}$ satisfy

$$\lim_{n\to-\infty} [x_n, y_n] = [+\infty,\, +\infty].$$

For a completeness, we should briefly discuss the special cases of the starting pair $[x_i,\, y_i]$ which we passed silently. The first case we left out is the starting pair is $[x_i,\, y_i] = [0,\, 0]$. Since $g_{\alpha/2}(0) = g_{\beta/2}(0) = 0$, this starting pair leads to the trivial stationary solution of (3.3). Now, if only one element of $[x_i,\, y_i]$ is equal to zero, then the sequence clearly must have a negative part. For example, if $[x_i,\, 0]$, $x_i > 0$, then by the forward mirror scheme (3.5) we get $x_{i+1} = -x_i < 0$. This completes the proof. $\square$

Using the backward substitution for (3.2), we can state the following corollary for the original system (3.1).

COROLLARY 3.6. *Assume that* $D > \frac{\max\{\lambda_1, \lambda_2\}}{8}$, $k_2 < \frac{k_1}{2}$ *and*

$$\max\left\{ \frac{2\sqrt{3}}{9\left(\frac{k_2}{k_1} - 1\right)\left(\frac{k_2}{k_1} - 2\right)}, \frac{\sqrt{3}}{18\left(2\frac{k_2}{k_1} - 1\right)\left(\frac{k_2}{k_1} - 1\right)} \right\} < \frac{\lambda_1}{\lambda_2} < \frac{9\left(2 - \frac{k_1}{k_2}\right)\left(1 - \frac{k_1}{k_2}\right)}{2\sqrt{3}}, \tag{3.16}$$

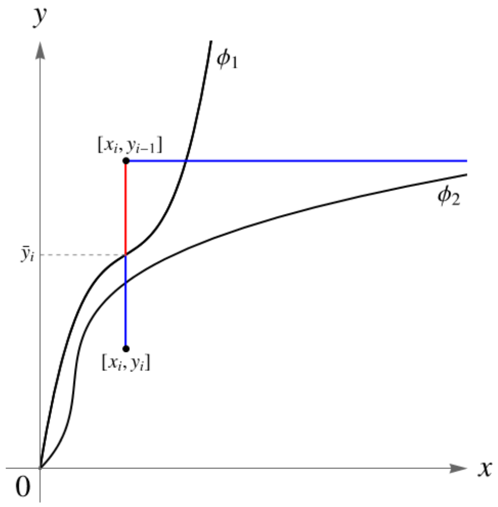



Figure 3.4: The backward mirror scheme (3.5) for a stationary solutions of (3.3) with starting pair $[x_i, y_i]$ located in region $A$ given by (3.10). The red and blue segments have the same length in every step of this scheme.

*then the system* (3.1) *has the unique non-negative stationary solution in the space* $\ell^\infty(\mathbb{Z})$ *- the trivial one.*

Thus, the heterogeneity of capacities of territories together with the large diffusion in the system (3.1) may lead to the full extinction of populations on all territories.

## 3.2 Existence of unbounded stationary solution

In the previous section, we showed that there are conditions which guarantee the unique bounded stationary solution of system (3.1), i.e., unique non-negative stationary solution in space $\ell^\infty(\mathbb{Z})$. This was done by the analysis of the so called forward and backward mirror schemes (3.5) and (3.6). Under assumptions of Theorem 3.5, sequence given by one of these mirror schemes for a starting pair is always unbounded. But we have no exact information what is going on, with the opposite part of this sequence. For example, the sequence that is unbounded by the backward mirror scheme, the application of the forward mirror scheme may lead to a sequence with negative values. See Figures 3.3 and 3.4, which show the forward and backward mirror scheme for the same starting pair.

We may ask, if there exists any starting pair for which neither the forward nor the backward mirror schemes lead to sequence with a negative part. For this purpose we leave the space $\ell^\infty(\mathbb{Z})$ and work in the sequence spaces $s(\mathbb{Z})$ and $s_+(\mathbb{Z})$ defined by

$$s(\mathbb{Z}) := \{(x_n)_{n\in\mathbb{Z}} : x_n \in \mathbb{R}\}\,,$$
$$s_+(\mathbb{Z}) := \{(x_n)_{n\in\mathbb{Z}} : x_n > 0\}\,.$$

We are able to apply the ideas of the proof of Theorem 2.10 to show the existence of an unbounded stationary solution in $s_+(\mathbb{Z})$.

COROLLARY 3.7. *Let the assumptions of Theorem 3.5 hold, then for any pair* $[x^*, y^*] \in \bar{B} \setminus [0, 0]$, *where B is given by* (3.10), *there exists unique non-negative unbounded stationary solution of* (3.3)

$$(..., x^*_{-1}, y^*_{-1}, x^*_0, y^*_0, x^*_1, y^*_1, ...) \tag{3.17}$$

*such that* $x^*_j = x^*$ *and* $y^*_j = y^*$ *for some index* $j \in \mathbb{Z}$.

*Moreover,*

1. *Odd and even entries of* (3.17) *are monotonic sequences, i.e.,*

$$
\begin{aligned}
x_j^* &\le x_{j+1}^* < x_{j+2}^* < \dots, \\
y_j^* &< y_{j+1}^* < y_{j+2}^* < \dots,
\end{aligned}
\tag{3.18}
$$

*where* $x_j = x_{j+1}$ *if and only if* $[x^*, y^*] \in \phi_2$ *given by* (3.9) *and*

$$
\begin{aligned}
x_j^* &< x_{j-1}^* < x_{j-2}^* < \dots, \\
y_j^* &\le y_{j-1}^* < y_{j-2}^* < \dots,
\end{aligned}
\tag{3.19}
$$

*where* $y_j = y_{j-1}$ *if and only if* $[x^*, y^*] \in \phi_1$ *given by* (3.9).

2. *Both sequences in* (3.18) *and* (3.19) *diverge to infinity, i.e.,*

$$\lim_{n\to\pm\infty} \left[x_{j+n}^*, y_{j+n}^*\right] = [+\infty, +\infty].$$

*Proof.* Assume that the assumptions of Theorem 3.5 hold. For any starting pair $[x_i, y_i] \in B$ given by (3.10) both, the forward and backward mirror schemes (3.5) and (3.6), lead to strictly increasing sequences. See the proof of Theorem 3.5.

Assume a starting pair $[x_i, y_i] \in \phi_1$ given by (3.9), i.e., located on one of the boundary curves of region $B$ given by (3.10). The forward iterative scheme leads to a strictly increasing sequence (shown in the proof of Theorem 3.5). Since

$$y_i = \frac{1}{\gamma} g_{\frac{\alpha}{2}}(x_i),$$

the first step of the backward mirror scheme (3.6) have a form

$$y_{i-1} - y_i = y_i - y_i,$$

i.e., $y_{i-1} = y_i$. The second step of this mirror scheme maps pair $[x_i, y_{i-1}] = [x_i, y_i]$ on the opposite side of the curve $\phi_2$ given by (3.9). That is $[x_{i-1}, y_{i-1}] \in A$ given by (3.10). We already know from the proof of Theorem 3.5, that the backward mirror scheme leads to strictly increasing sequence of pairs which diverges to $[+\infty, +\infty]$.

Analogously we can show that a starting pair $[x_i, y_i] \in \phi_2$ given by (3.9), which leads by backward mirror scheme (3.6) to an unbounded sequence, skips one step of forward mirror scheme ($x_{i+1} = x_i$), but right after it leads to strictly increasing sequence of pairs.

Note that the form of the forward and backward mirror scheme (3.5) and (3.6) stays the same independently on the index $i$. Thus we set $[x_i, y_i] = [x^*, y^*]$ and let the unbounded stationary solution of (3.3) attain these values at any arbitrary index $j$. ☐

We characterize the unbounded stationary solutions more precisely in the following result.

COROLLARY 3.8. *For the system* (3.3) *with the assumptions of Theorem 3.5, we have infinitely many unbounded non-negative stationary solution. More precisely:*

1. *There are countable many stationary solutions* (3.17) *which attain values of pair* $[x^*, y^*] \in \bar{B} \setminus [0, 0]$ *for some index j, i.e.,* $x_j^* = x^*$ *and* $y_j^* = y^*$.

2. *There are uncountable many pairs* $[x^*, y^*] \in \bar{B} \setminus [0, 0]$ *which lead to unbounded non-negative stationary solutions.*

3. *For any* $\epsilon > 0$, *there exist uncountable many non-negative unbounded stationary solutions of* (3.3), *such that there exists index* $j \in \mathbb{Z}$ *for which*

$$\left\| \left[x_j^*, y_j^*\right] \right\| \le \epsilon. \tag{3.20}$$

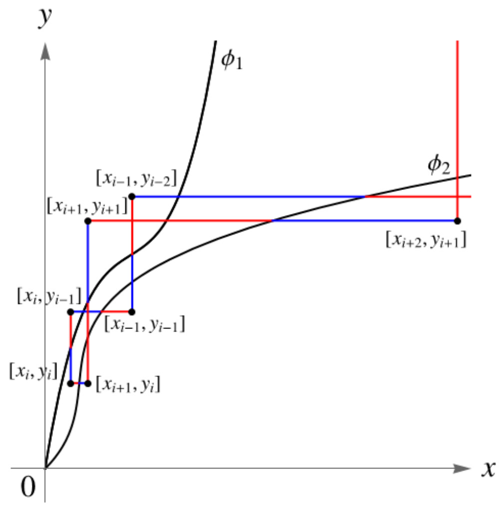



Figure 3.5: Non-negative unbounded sequence given by forward and backward mirror scheme (3.5) and (3.6) for a starting pair $[x_i, y_i]$ located in the region $B$ given by (3.10).

*Proof.* The first statement is just a rewritten version of the last paragraph of proof of Corollary 3.7.

Uncountability of the set of unbounded non-negative stationary solutions of (3.3) comes from fact that a pair $[x^*, y^*]$ can be located anywhere in the region $\bar{B} \setminus [0, 0]$ given by (3.10).

The last statement just emphasizes the fact that region $B$ given by (3.10) has always non-empty intersection with ball of arbitrary radius located in the origin. This can be supported by fact, that under assumptions of Theorem 3.5 the boundary curves (3.9) of the region $B$ can be interpreted as a functions of $x$ and satisfy

$$\phi_1'(0) = \frac{4+\alpha}{4\gamma} > \frac{4}{(4+\beta)\gamma} = \phi_2'(0).$$

Note that these curves are defined as single equations from (3.8). Derivative of a curve $\phi_2$, i.e., the second equation of (3.8) comes from the derivative of the inverse function [18, Theorem 5.9] □

Note that the proof of Corollary 3.7 gives the construction of non-negative unbounded stationary solutions of (3.3). If the system satisfies assumptions of Theorem 3.5, for any pair $[x_i, y_i] \in \bar{B} \setminus [0, 0]$, where $B$ is given by (3.10), the non-negative unbounded stationary solution of the system (3.3) can be constructed by the mirror schemes (3.5) and (3.6). We illustrate this by an example.

EXAMPLE 3.9. *Assume the system* (3.3) *with* $\alpha = \beta = 6$ *and* $\gamma = 2/5$. *Since* $\alpha, \beta < 8$, $\gamma \in (0, 1/2)$ *and*

$$\max\left\{\frac{25}{36\sqrt{3}}, \frac{25}{18\sqrt{3}}\right\} < 1 < \frac{9\sqrt{3}}{8},$$

*the assumptions of Theorem 3.5 are satisfied. This system has been used for all illustrations in this chapter, i.e., the regions* (3.10) *for this system can be seen on Figure 3.1.*

*We choose an arbitrary starting pair* $[x_i, y_i]$ *located in region B given by* (3.10), *and apply the forward and backward mirror schemes* (3.5) *and* (3.6) *to obtain the sequences corresponding to stationary solution of* (3.3). *For a pair* $[x_i, y_i] = [0.15, 0.5]$ *The sequence given by forward mirror scheme is*

$$\left\{[0.15, 0.5], [0.25, 1.45313], [2.4187, 109.398], \left[3.09935 \times 10^6, 4.46584 \times 10^{20}\right], \ldots\right\}$$

*and by the backward mirror scheme is*

$$\left\{\ldots, \left[7.62306 \times 10^7, 317.194\right], [3.25262, 1.59426], [0.510894, 0.919375], [0.15, 0.5]\right\}.$$

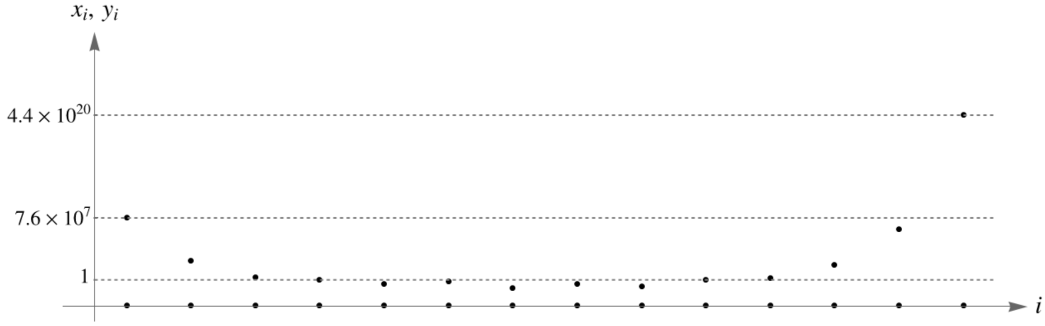



Figure 3.6: Part of the non-negative unbounded stationary solution (3.21) of the system (3.3) with $\alpha = \beta = 6$ and $\gamma = 2/5$. This stationary solution is constructed in Example 3.9 to demonstrate the construction of non-negative stationary solutions of the system (3.3), which satisfies the assumptions of Theorem 3.5, given by Corollary 3.7. The logarithmic scaling function for $y$ axis is chosen here.

*Putting these together, we obtain the non-negative unbounded stationary solution of* (3.3) *in a form*

$$\Big(\ldots 7.62306 \times 10^7,\ 317.194,\ 3.25262,\ 1.59426,\ 0.510894,\ 0.919375,\ 0.15,\ 0.5,$$
$$0.25,\ 1.45313,\ 2.4187,\ 109.398,\ 3.09935 \times 10^6,\ 4.46584 \times 10^{20},\ \ldots\Big). \tag{3.21}$$

To summarize previous parts of this chapter, Theorem 3.5 gives us conditions for the unique bounded non-negative stationary solution of the system (3.3) (or (3.1)). Corollaries 3.7 and 3.8 discuss the special subset of unbounded stationary solutions of (3.3) - the non-negative unbounded stationary solutions which have unbounded limits as $n \to \pm\infty$.

Obviously, these need not be all stationary solutions in $s_+(\mathbb{Z})$. There exist unbounded non-negative stationary solutions which blow up to $+\infty$ in one direction and tend to zero in the other one. Note that under assumptions of the Theorem 3.5, the only possible finite limit of stationary solution of (3.3) is zero.

COROLLARY 3.10. *Let the assumptions of Theorem 3.5 hold. There exist unbounded non-negative stationary solutions* (3.17) *of* (3.3) *which satisfy*

$$\lim_{n\to+\infty}[x_n, y_n] = [+\infty, +\infty], \qquad \lim_{n\to-\infty}[x_n, y_n] = [0, 0],$$

*or*

$$\lim_{n\to-\infty}[x_n, y_n] = [0, 0], \qquad \lim_{n\to-\infty}[x_n, y_n] = [+\infty, +\infty].$$

*Proof.* Let the assumptions Theorem 3.5 hold. From Corollary 3.7 we know that for any $[x_i, y_i] \in \bar{B} \setminus [0, 0]$, where $B$ is given by (3.10), both, the forward and backward mirror schemes (3.5) and (3.6) lead to strictly increasing sequences with limit $+\infty$. Moreover, from the proof of Theorem 3.5, we have the existence of an unbounded strictly increasing sequence by forward mirror scheme for any $[x_i, y_i] \in A$, given by (3.10). Similarly for $[x_i, y_i] \in C$, given by (3.10) we have the existence of an unbounded strictly increasing sequence by backward mirror scheme.

Clearly, we have to analyze sequences given by the forward mirror scheme (3.5) for the starting pair $[x_i, y_i] \in C$ and sequence given by the backward mirror scheme (3.6) for the starting pair $[x_i, y_i] \in A$, where the regions $A$ and $C$ are given by (3.10).

The proof is split into two parts corresponding to two properties of the stationary solution from the statement of this corollary - non-negativity and one sided finite limit. We construct subsets of $\mathbb{R}^2$, from which the pair and following sequences given by the forward and backward mirror schemes (3.5) and (3.6) are non-negative. Then the other subsets are made, such that any pair located in here, the sequences given by mirror schemes do not have a pair located in the region which guarantee unbounded both-sided limits. By properties of these sets we derive the statement of this corollary.

We start with a non-negativity of the stationary solution (3.17). The first step of the forward mirror scheme (3.5) maps a starting pair $[x_i, y_i]$ onto $[x_{i+1}, y_i]$ located in the opposite side of the curve $\phi_2$ given by (3.9) preserving the distance from the curve $\phi_2$. Analogously, the first step of the backward mirror scheme maps a pair $[x_i, y_i]$ onto $[x_i, y_{i-1}]$ located in the opposite side of the curve $\phi_1$ given by (3.9) keeping the distance. See Lemma 3.4.

It follows that any $[x_i, y_i] \in C$ given by (3.10) has to be closer to the curve $\phi_2$ in $x$ direction, than the distance of the corresponding point on the curve from the boundary of the first quadrant (the line $x = 0$) to $x_{i+1}$ given by forward mirror scheme (3.5) be greater then zero. Analogous statement follows for $y_{i-1}$ given by backward mirror scheme for a pair $[x_i, y_i] \in A$.

Rearranging the forward mirror scheme (3.5) and requiring $x_{i+1} > 0$ we obtain

$$0 < x_{i+1} < \frac{2}{\gamma} g_{\frac{\beta}{2}}(y_i) - x_i \implies x_i < \frac{2}{\gamma} g_{\frac{\beta}{2}}(y_i)$$

and analogically for the backward mirror scheme (3.6) with $y_{i-1} > 0$

$$0 < y_{i-1} < 2\gamma g_{\frac{\alpha}{2}}(x_i) - y_i \implies y_i < 2\gamma g_{\frac{\alpha}{2}}(x_i).$$

Thus, we define

$$\begin{aligned} A^+ &= \left\{ (x, y) \in \mathbb{R}^2 : x > 0,\ \frac{2}{\gamma} g_{\frac{\alpha}{2}}(x) > y > \frac{1}{\gamma} g_{\frac{\alpha}{2}}(x) \right\}, \\ C^+ &= \left\{ (x, y) \in \mathbb{R}^2 : 2\gamma g_{\frac{\beta}{2}}(y) > x > \gamma g_{\frac{\beta}{2}}(y),\ y > 0 \right\}. \end{aligned} \tag{3.22}$$

For any $[x_i, y_i]$ located in regions (3.22), the values $x_{i+1}$ and $y_{i-1}$ given by the first step of the forward mirror scheme (3.5) and the first step of the backward mirror scheme (3.6) for this pair are positive[2]. The regions $A^+$ and $C^+$ given by (3.22) can be seen at Figure 3.7.

Based on the statement of Lemma 3.4, the second step of the forward mirror scheme (3.5) works the same as the first step of the (3.6) and vice versa. Thus, we assume $[x_i, y_i]$ located in the reduced regions $A^+$ and $C^+$ (3.22). We know, that the first step of these schemes maps on a pair with positive values, but it may end up in the regions $A \setminus A^+$ or $C \setminus C^+$. By the construction of the regions (3.22), we know that such pairs will be mapped in the next step outside the interior of the first quadrant. Thus, $[x_i, y_i] \in C^+$ is not sufficient to $x_{i+n}, y_{i+n} > 0$ for any $n \in \mathbb{Z}$.

Obviously we have to reduce the regions (3.22) further. Assume $\alpha, \beta < 8$, then we can define the inverse functions

$$\begin{aligned} y = \frac{2}{\gamma} g_{\frac{\alpha}{2}}(x) &\implies x = g_{\frac{\alpha}{2}}^{-1}\left(\frac{\gamma y}{2}\right), \\ x = 2\gamma g_{\frac{\beta}{2}}(y) &\implies y = g_{\frac{\beta}{2}}^{-1}\left(\frac{x}{2\gamma}\right). \end{aligned}$$

since the function $g_\delta(s)$ given by (2.15) is bijective for $\delta < 4$, $s \in \mathbb{R}$, see the proof of Lemma 2.15.

Let us consider the second steps of the forward mirror scheme (3.5) and the backward mirror scheme (3.6). We then have

$$\begin{aligned} g_{\frac{\beta}{2}}^{-1}\left(\frac{x_{i+1}}{2\gamma}\right) &< y_{i+1} = \frac{2}{\gamma} g_{\frac{\alpha}{2}}(x_{i+1}) - y_i, \\ g_{\frac{\alpha}{2}}^{-1}\left(\frac{\gamma y_{i-1}}{2}\right) &< x_{i-1} = 2\gamma g_{\frac{\beta}{2}}(y_{i-1}) - x_i, \end{aligned}$$

[2]We could define regions (3.22), such that these values would be non-negative, but from the last paragraph of the proof of Theorem 3.5 we know, that any pair located on the axis (except the origin) is mapped onto a pair with a negative value.

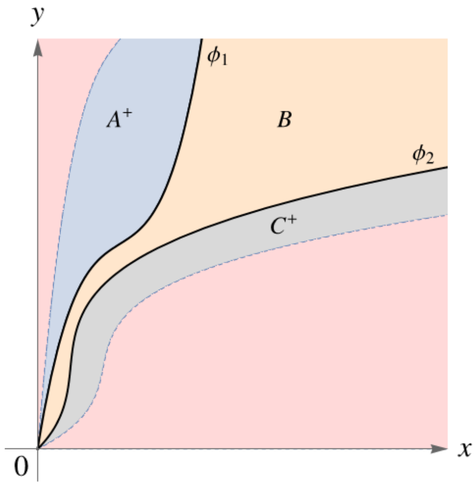



Figure 3.7: Reduced regions (3.22) which guarantee that for any pair $[x_i, y_i]$ from these regions the first step of forward and backward mirror scheme (3.5) and (3.6) returns a positive value. The red regions represent the reduction from original regions (3.10).

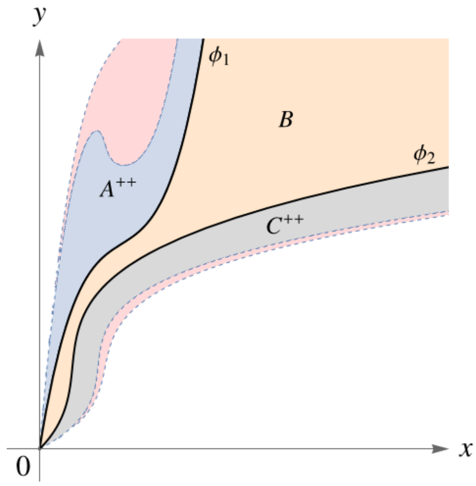



Figure 3.8: Reduced regions (3.24) which guarantee that for any pair $[x_i, y_i]$ from these regions the first two steps of forward and backward mirror scheme (3.5) and (3.6) returns a positive value. The red regions represent the reduction from regions (3.22).

i.e., the schemes do not map pairs into regions $A \setminus A^+$ and $C \setminus C^+$. Thus

$$\begin{aligned} y_i &< \frac{2}{\gamma} g_{\frac{\alpha}{2}}(x_{i+1}) - g_{\frac{\beta}{2}}^{-1}\left(\frac{x_{i+1}}{2\gamma}\right), \\ x_i &< 2\gamma g_{\frac{\beta}{2}}(y_{i-1}) - g_{\frac{\alpha}{2}}^{-1}\left(\frac{\gamma y_{i-1}}{2}\right). \end{aligned} \tag{3.23}$$

We define regions

$$\begin{aligned} A^{++} &= \left\{ (x, y) \in \mathbb{R}^2 : x > 0,\ \frac{2}{\gamma} g_{\frac{\alpha}{2}}(x) - g_{\frac{\beta}{2}}^{-1}\left(\frac{x}{2\gamma}\right) > y > \frac{1}{\gamma} g_{\frac{\alpha}{2}}(x) \right\}, \\ C^{++} &= \left\{ (x, y) \in \mathbb{R}^2 : 2\gamma g_{\frac{\beta}{2}}(y) - g_{\frac{\alpha}{2}}^{-1}\left(\frac{\gamma y}{2}\right) > x > \gamma g_{\frac{\beta}{2}}(y),\ y > 0 \right\}. \end{aligned} \tag{3.24}$$

Any $[x_i, y_i]$ located in regions $A^{++}$ and $C^{++}$ satisfy that the first two steps of the forward mirror scheme (3.5) and the backward mirror scheme (3.6) map onto pairs $[x_{i+1}, y_{i+1}]$ and $[x_{i-1}, y_{i-1}]$ with positive values. These regions can be seen at Figure 3.8.

Clearly we reduce the regions $A^{++}$ and $C^{++}$ for next steps of mirror schemes, i.e., $A^{3+} := A^{+++}$, $C^{3+} := C^{+++}$, .... The analytical expression of these regions may be very complicated. Instead of this, we apply simple observation. We know

1. There exists $[x_i, y_i] \in A \cup C$ such that the sequences given by the forward and backward mirror schemes are non-negative, see Corollary 3.7 (it is sufficient for any other pair of these sequences to be located in region $B$ given by (3.10), then the rest of the sequence is located in $A \cup C$), i.e., these sets are non-empty.

2. For any $n \in \mathbb{N}$ regions $A^{n+}$ and $C^{n+}$ are open sets by definitions of $A^+$ and $C^+$ in (3.22).

3. For any $n \in \mathbb{N}$ we have embedding $A^{(n+1)+} \subset A^{n+}$ and $C^{(n+1)+} \subset C^{n+}$.

Thus the sequences $A^{n+}$ and $B^{n+}$ converge to non-empty limits sets. We denote these limit sets as $A^{\infty+}$ and $B^{\infty+}$, which can be defined as

$$A^{\infty+} = \bigcap_{n\in\mathbb{N}} A^{n+}, \quad C^{\infty+} = \bigcap_{n\in\mathbb{N}} C^{n+}$$

and by the properties of $A^{n+}$ and $C^{n+}$ are half-closed sets (boundary curves which separate these sets from the region $B$ given by (3.10), that is the curves $\phi_1$ and $\phi_2$ given by (3.9), do not belong into these sets).

Together, the set of pairs $[x_i, y_i]$ which lead to a non-negative stationary solution by both mirror schemes is the set $A^{\infty+} \cup \phi_1 \cup B \cup \phi_2 \cup C^{\infty+}$, where $B$, $\phi_1$ and $\phi_2$ are given by (3.9) and (3.10). Moreover, the union of this set with the origin $[0, 0]$ or in $\mathbb{R}^2_+$ is a closed set.

Let us move to the second part of the proof. We want to determine cases for which the sequence of stationary solution of (3.3) given by the forward and backward mirror schemes (3.5) and (3.6) has both limits unbounded and tending to $+\infty$. From the proof of Corollary 3.7 we know that a sequence (stationary solution (3.17)) has both limits unbounded if and only if there exist index $j \in \mathbb{Z}$ such that $[x_j, y_j] \in \bar{B} \setminus [0, 0]$. Thus, we determine fist starting pairs $[x_i, y_i]$ which map onto points in $\bar{B} \setminus [0, 0]$. Clearly, the pairs $[y_i, x_i] \in \bar{B} \setminus [0, 0]$ are determined immediately.

We start with inverse functions

$$
\begin{aligned}
y &= \frac{1}{\gamma} g_{\frac{\alpha}{2}}(x) \implies x = g^{-1}_{\frac{\alpha}{2}}(\gamma y), \\
x &= \gamma g_{\frac{\beta}{2}}(y) \implies y = g^{-1}_{\frac{\beta}{2}}\left(\frac{x}{\gamma}\right).
\end{aligned}
$$

which define the boundaries of the region $B$ given by (3.10). Note that inverse functions exist since $\alpha$, $\beta < 8$ and the function $g_\delta(s)$ is s bijection for $\delta < 4$. Assume a starting pair $[x_i, y_i] \in C$, where $C$ is given by (3.10), then the forward mirror scheme (3.5) maps this pair into $\bar{B} \setminus [0, 0]$ if the distance of this pair and curve $\phi_2$ given by (3.9) is less or equal then the distance of $\phi_2$ and $\phi_1$ (distances here are meant in the $x$-direction). Analogously the backward mirror scheme (3.6) maps a pair $[x_i, y_i] \in A$, where $A$ is given by (3.10), into region $\bar{B} \setminus [0, 0]$ if the distance in $y$-direction of this pair and the curve $\phi_1$ is less or equal than the distance of $\phi_1$ and $\phi_2$. Thus, for this not to occur the first step of these schemes must satisfy

$$
\begin{aligned}
g^{-1}_{\frac{\alpha}{2}}(\gamma y_i) &> x_{i+1} = 2\gamma g_{\frac{\beta}{2}}(y_i) - x_i, \\
g^{-1}_{\frac{\beta}{2}}\left(\frac{x_i}{\gamma}\right) &> y_{i-1} = \frac{2}{\gamma} g_{\frac{\alpha}{2}}(x_i) - y_i,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
x_i &> 2\gamma g_{\frac{\beta}{2}}(y_i) - g^{-1}_{\frac{\alpha}{2}}(\gamma y_i), \\
y_i &> \frac{2}{\gamma} g_{\frac{\alpha}{2}}(x_i) - g^{-1}_{\frac{\beta}{2}}\left(\frac{x_i}{\gamma}\right).
\end{aligned}
$$

We define regions

$$
\begin{aligned}
A^- &= \left\{ (x, y) \in \mathbb{R}^2 : x > 0,\, y > \frac{2}{\gamma} g_{\frac{\alpha}{2}}(x) - g^{-1}_{\frac{\beta}{2}}\left(\frac{x}{\gamma}\right) \right\}, \\
C^- &= \left\{ (x, y) \in \mathbb{R}^2 : x > 2\gamma g_{\frac{\beta}{2}}(y) - g^{-1}_{\frac{\alpha}{2}}(\gamma y),\, y > 0 \right\}.
\end{aligned}
\tag{3.25}
$$

Any pair $[x_i, y_i] \in C^-$ is not mapped by the first step of the forward mirror scheme (3.5), same as a pair $[x_i, y_i] \in A^-$ by the first step of the backward mirror scheme (3.6), into region $\bar{B} \setminus [0, 0]$, where $B$ is given by (3.10). The regions (3.25) can be seen at Figure 3.9.

Again, the fact that a pair $[x_i, y_i]$ is not mapped into $\bar{B} \setminus [0, 0]$ in the first step of the forward or backward mirror scheme (3.5) or (3.6) does not imply that it wont be mapped there in following steps of these schemes, e.g., the forward mirror scheme may map a pair $[x_i, y_i] \in C^-$ onto $[x_{i+1}, y_i] \in A \setminus A^-$. Since the second step of the forward mirror scheme works the same as the first step of backward mirror scheme (Lemma 3.4), then a pair $[x_{i+1}, y_i]$ is mapped by this scheme onto $[x_{i+1}, y_{i+1}] \in \bar{B} \setminus [0, 0]$.

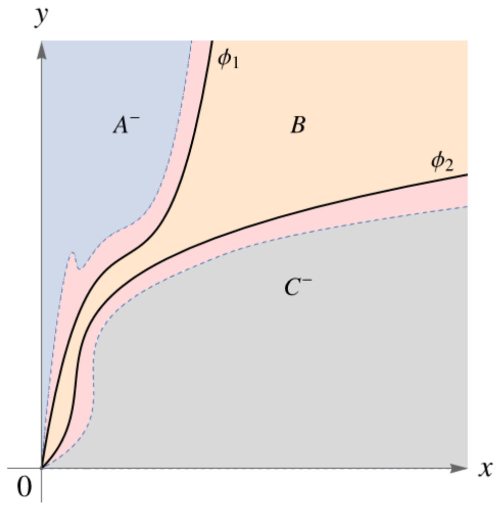



Figure 3.9: Reduced regions (3.25) which guarantee that for any pair $[x_i, y_i]$ from these regions a pair $[x_{i+1}, y_i]$ given by the first step of the forward mirror scheme (3.5) nor $[x_i, y_{i-1}]$ given by the backward mirror scheme (3.6) belongs to $\bar{B} \setminus [0, 0]$. The red regions represent the reduction from original regions (3.10).

Clearly, we have to reduce the regions (3.25) further, i.e., we define regions $A^{--}$ and $C^{--}$ which guarantee that a pair $[x_i, y_i]$ located in these regions wont be mapped into $\bar{B} \setminus [0, 0]$ in two steps of the forward nor the backward mirror schemes (3.5) and (3.6). More reduced regions cannot be easily expressed analytically but the same observation as for non-negative sequences holds. Since $(\bar{B} \setminus [0, 0]) \cup (A \setminus A^-) \cup (C \setminus C^-)$ is a closed set, the regions $A^{--}$ and $C^{--}$ are open in $\mathbb{R}^2_+$.

For the sequences of sets $A^{n-}$ and $B^{n-}$, i.e., sets which guarantee that any pair $[x_i, y_i]$ from this set is not mapped in $n$ steps of the forward nor the backward mirror scheme (3.5) and (3.6) into the region $\bar{B} \setminus [0, 0]$, we know

1. These sets are non-empty, e.g., since any pair $[x_i, y_i] \in A \setminus A^+$, where $A$ is given by (3.10) and $A^+$ by (3.22), is by the first step of the backward mirror scheme mapped outside the interior of the first quadrant it cannot belong to $A^{n-}$ for any $n \in \mathbb{N}$.

2. For any $n \in \mathbb{N}$ these regions are open in $\mathbb{R}^2_+$[3].

3. For any $n \in \mathbb{N}$ we have the embedding $A^{(n+1)-} \subset A^{n-}$ and $C^{(n+1)-} \subset C^{n-}$.

Thus the sequences of sets $A^{n-}$ and $B^{n-}$ converge to limit sets, we denote them $A^{\infty-}$ and $C^{\infty-}$. Again, this limit sets can be written as

$$A^{\infty-} = \bigcap_{n=1}^{+\infty} A^{n-}, \quad C^{\infty-} = \bigcap_{n=1}^{+\infty} C^{n-},$$

i.e., as a intersection of embedded open sets. Thus, the limit sets are closed in $\mathbb{R}^2_+$.

To complete the proof, we define the set $B^+ = \mathbb{R}^2_+ \setminus (A^{\infty-} \cup C^{\infty}-)$. Since $A^{\infty-}$ and $C^{\infty-}$ are closed, the set $B^+$ is open. Moreover, from the construction of the removed sets, it contains all the pair $[x_i, y_i]$ which are mapped into region $B$ by some step of the forward and the backward mirror schemes (3.5) and (3.6). Thus it is a subset of $A^{\infty+} \cup \phi_1 \cup B \cup \phi_2 \cup C^{\infty+}$ which is closed. It follows that there is a non-empty set $(A^{\infty+} \cup \phi_1 \cup B \cup \phi_2 \cup C^{\infty+}) \setminus B^+$, such that any pair $[x_i, y_i]$ from this set leads to non-negative stationary solution with only one unbounded limit. □

[3] we have no interest in the boundary of the first quadrant, i.e., lines $x = 0$ and $y = 0$.

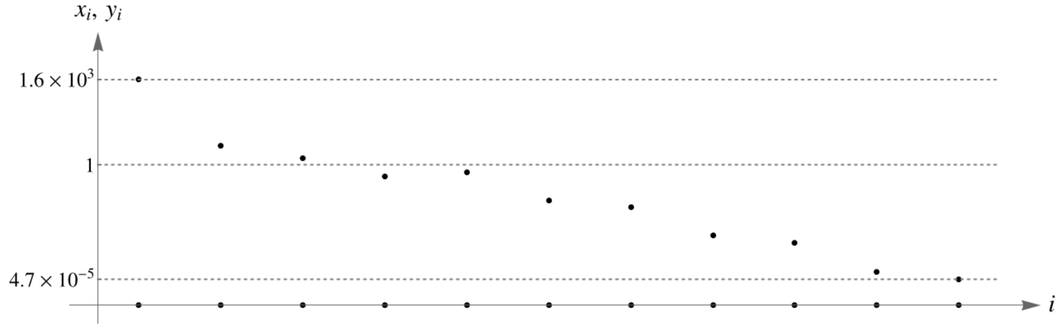



Figure 3.10: Part of the non-negative unbounded stationary solution of the system (3.3) with $\alpha = \beta = 6$ and $\gamma = 2/5$. This stationary solution is constructed in Example 3.11 for a starting pair found by numerical experiment to demonstrate the statement of Corollary 3.10. The logarithmic scaling function for $y$ axis is chosen here.

Similarly as in Example 3.9, where we have presented the construction of non-negative stationary solution of the system (3.1) with both sided unbounded limits, given by Corollary 3.7, we would like to illustrate construction of stationary solution given by Corollary 3.10. This time, we do not have explicitly given the region from which a starting pair leads to such solution by forward and backward mirror schemes (3.5) and (3.6). Such a starting pair has to be found numerically.

EXAMPLE 3.11. *Assume the system* (3.3) *with* $\alpha = \beta = 6$ *and* $\gamma = 2/5$, *that is the same system as in Example 3.9. This system satisfies the assumptions of Theorem 3.5, thus we know by Corollary 3.10 that a non-negative unbounded stationary solution with one sided finite limit exists.*

*Corollary 3.10 does not give explicitly the region from which a starting pair leads to such a solution by forward and backward mirror schemes* (3.5) *and* (3.6). *But from the proof of this corollary, we can construct the regions* $A^{n+}$ *and* $C^{n+}$ *which guarantee that a solution for starting pair from here is non-negative for* $n$ *steps of mirror schemes and the regions* $A^{n-}$ *and* $C^{n-}$ *which guarantee that one of a sequences given by these schemes is decreasing for* $n$ *steps.*

*By a numerical experiment, we have found a starting pair* $[x_i, y_i] = [0.3543688466971, 0.5]$ *which is located in* $C^{4+}$ *and* $C^{4-}$. *For this starting pair the forward mirror scheme leads to sequence*

$$\Big\{[x_i, y_i], \Big[4.5631 \times 10^{-2}, 2.4965 \times 10^{-2}\Big], \Big[2.0926 \times 10^{-3}, 1.0941 \times 10^{-3}\Big], \\ \Big[9.1359 \times 10^{-5}, 4.7674 \times 10^{-5}\Big], \Big[3.9801 \times 10^{-6}, 2.0858 \times 10^{-6}\Big], ...\Big\}. \quad (3.26)$$

*The sequence given by backward mirror scheme for this starting pair is not interesting in here, since it leads to increasing unbounded sequence similarly as in Example 3.9.*

*Part of stationary solution of* (3.1) *made by forward and backward mirror schemes* (3.5) *and* (3.5) *for a starting pair* $[x_i, y_i] = [0.3543688466971, 0.5]$ *can be found in Fig. 3.10. Note that even that the sequence* (3.26) *is decreasing here, it is only located in* $C^{4-}$ *and* $C^{4+}$, *thus we have no information if it leaves the first quadrant, converges to zero or starts growing for following steps.*

From the construction of regions $A^{n+}$, $A^{n-}$, $C^{n+}$ and $C^{n-}$ in the proof of Corollary 3.10 and by the $x_i$ value of starting pair in Example 3.11, we see that the region from which a starting pairs for forward and backward mirror schemes (3.5) and (3.6) lead to a stationary solution from the statement of Corollary 3.10 is not a very large. Even for only $n = 4$ we had to find a starting pair with $x_i$ value with 13 decimal digits. This leads to question how large is this region. One of the possibilities is that it has a zero-measure, i.e., it is made by two curves in $\mathbb{R}^2$, one in the region $A$ and the second in the region $B$, where $A$ and $B$ are given by (3.10). We leave this as an open question, see Conjecture 2 in Chapter 4.

## 3.3 Generalization

In this section, we briefly discuss the system (3.1) with general viability parameter, i.e., the system (1.2). We describe how an analogy of Corollary 3.6 can be derived for this system. This can be done by a simple fact, that in the proof of Theorem 3.5 we used only sufficient conditions for the existence of unique stationary solution of the two dimensional model corresponding to the system (3.3), i.e., the system (2.7).

In Section 2.1.2, we derived the conditions for the existence of a unique stationary solution of the system (1.4). This system is the two dimensional case of (1.2). Same as in Section 3.1, we can use the substitution (3.2) to simplify the system.

An infinite system of algebraic equation for a stationary solution of (1.2) after substitution (3.2) can be rewritten into an analogy of the forward mirror scheme (3.5) and the backward mirror scheme (3.6). Again, these schemes map a point $[x_i, y_i] \in \mathbb{R}^2$ onto opposite side of curves given by a separated equation of two dimensional system, i.e., the system (1.4). If assumptions of Corollary 2.13 hold, then these curves have a unique intersection. Moreover, these curves preserve mutual position as the curves (3.9).

Following steps from the proof of Theorem 3.5, we can show that any starting pair from the first region, leads by the analogy of the forward mirror scheme (3.5) or the backward mirror scheme (3.6) to strictly increasing sequence. Since under assumptions of Corollary 2.13, the two dimensional case has only one solution (the origin), these sequences cannot be bounded. Thus we can state the following corollary of Theorem 3.5.

COROLLARY 3.12. *Assume that* $D > \max\{\lambda_1, \lambda_2\}\frac{a^2-a+1}{6}$, $k_2 < ak_1$ *and inequalities* (2.27) *and* (2.28) *hold, then the system* (1.2) *has the unique non-negative bounded stationary solution in* $\ell^\infty(\mathbb{Z})$ *- the trivial one.*

Similarly, we can show the existence of infinite (uncountable) unbounded non-negative stationary solution of (1.2), i.e., analogies of Corollaries 3.7, 3.8 and 3.10.

# Conclusion 4

In this work, we studied modified version of the lattice Nagumo equation (1.5) and semi-discrete Nagumo equation on graph (1.6). Our main focus was on a unique stationary solution of these systems in the case of heterogeneous capacities of discrete patches.

In Chapter 2, we analyzed semi-discrete Nagumo equation on bipartite graphs, where the territories represented by vertices of this graph have different capacities. While the homogeneous semi-discrete Nagumo equation on a graph has at least three stationary solutions for any strength of the diffusion, our analysis of the heterogeneous semi-discrete Nagumo equation (1.3) shows that for this system it is not the case. In Sections 2.1.1 and 2.1.2, we study the simplest example of such graph with only two vertices for which the system (1.3) becomes (1.4). Even that this system is the simplest one, the results obtained here are essential, since they are widely used for systems on different discrete sturctures.

To derive the conditions for a unique stationary solution of (1.4), we made some restrictions, one of which is boundedness from below of the diffusion parameter $D$ by a function of reaction and viability parameters ($\lambda_1$, $\lambda_2$ and $a$), see the first assumption of 2.14. From numerical results at the end of the Section 2.1.1, where the viability parameter $a$ is fixed, and 2.1.2 we can see that this assumption is too strong. On the other hand, this allowed us to formulate the further assumptions in a relatively easy form. How to bypass the assumption on $D$, which guarantee monotonicity of auxiliary functions $g$ and $h$ given by (2.15) and (2.32) (see proofs of Lemma 2.6 and Theorem 2.14), is left as an open question.

QUESTION 1. Statement of Theorems 2.10 and 2.14 can be greatly improved by weakening the assumption on the monotonicity of functions $g$ and $h$, given by (2.15) and (2.32), which corresponds to functions given by equations in (2.10) and (2.31), i.e., the systems for stationary solutions of corresponding system.

In Sections 2.2 and 2.3, we use the results of Section 2.1.2 to prove that the modified version of semi-discrete Nagumo equation on complete bipartite graph (1.3), may have a unique stationary solutions too. Naturally there arise another question about deriving conditions for a unique stationary solution of more general graphs.

QUESTION 2. Is it possible to derive an analogous result as in Theorem 2.14 or its Corollaries 2.16 and 2.17 for more generals graphs, e.g., a general bipartite graph where territories represented by a vertices from the same set does not have exactly the same capacities but are bounded by maximum/minimum satisfying modified assumptions of Theorem 2.14?

The heterogeneous lattice Nagumo equation (1.2) was analyzed in Chapter 3. Here, we rewrote the system for stationary solutions into difference schemes called forward and backward mirror schemes. These schemes create for any point of $\mathbb{R}^2$ an infinite sequence corresponding to stationary solution. It is shown that under assumptions of Corollary 3.12, for any point in $\mathbb{R}^2_+$, one of the sequences given by mirror schemes is always unbounded. It follows that the origin is the only non-negative unbounded stationary solution.

In a finite dimensional case of heterogeneous semi-discrete Nagumo equation, i.e., the system (1.3), we can show by Hartman-Grobman theorem [3, Theorem 2.2.3] that the origin is locally asymptotically stable stationary solution. In an infinite dimensional case, this is more complicated. Since in the both cases, we have conditions for uniqueness of a stationary solution, the question of the global stability arise.

CONJECTURE 1. Let the assumptions of Corollary 3.12 hold, then for any initial condition

$$\mathbf{x}(0) = \mathbf{x}_0 \in \ell^\infty(\mathbb{Z}),\ x_{i,0} \geq 0,\ i \in \mathbb{Z}$$

the solution $\mathbf{x}(t)$ of (1.2) tends to the origin, i.e.,

$$\mathbf{x}(t) \to o \in \ell^\infty(\mathbb{Z}),\ t \to +\infty.$$

On the other hand, in Section 3.2, we showed that the system (1.2) has uncountable many unbounded stationary solutions. First, it is proved that there are uncountable many starting points in $\mathbb{R}^2_+$, for which both forward and backward mirror schemes lead to unbounded sequence. Further, we show that there are some point, for which a sequence made by one of the mirror scheme is unbounded but the other part of the sequence given by the second mirror scheme converges to origin. We showed that such a region of starting points does exist, but we cannot express it explicitly.

Corollary 3.10 state that there exist a starting pairs in $\mathbb{R}^2_+$ for which the sequences given by the forward and backward mirror schemes, given by (3.5) and (3.6), lead to one sided unbounded sequences, where the other sides converge to origin. The region of a such starting pairs is made by iterative reduction of the first quadrant based on requirements of following sequences. By the nature of construction and numerical experiments we believe that the final region has zero-measure.

CONJECTURE 2. Set of a starting pairs which lead by the forward and backward mirror schemes, given by (3.5) and (3.6), to the non-negative unbounded stationary solution of (3.1) satisfying the statement of Corollary 3.10 has zero-measure. It follows, that any stationary solution from this corollary has one of two possible profiles, i.e, if we define curves

$$\bar{\phi}_1(t),\ \bar{\phi}_2(t) \in \mathbb{R}^2_+,\ t \in \mathbb{R}$$

then such a stationary solution $(\ldots, x_{i-1}, y_{i-1}, x_i, y_i, x_{i+1}, y_{i+1}, \ldots)$ satisfy

$$\begin{aligned}&[x_i, y_i] \in \bar{\phi}_1,\ i \in \mathbb{Z},\ \text{or}\\ &[x_i, y_i] \in \bar{\phi}_2,\ i \in \mathbb{Z}.\end{aligned}$$

Since a traveling wave is a solution which connects two stationary solutions of a system and we have derived the conditions for unique bounded stationary solution of modified lattice Nagumo equation, it follows that under these assumptions any traveling wave cannot exists there. On the other hand, it can be easily shown that for similar capacities of territories at least three stationary solution exist. Thus, finding traveling waves in this system may be a subject of future study.

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