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Tomáš Roubal

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Školitelé: **doc. Ing. Radek Cibulka, Ph.D.**
doc. RNDr. Petr Stehlík, Ph.D. (předchozí)

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Openness of mappings

Tomáš Roubal

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Supervisors: **doc. Ing. Radek Cibulka Ph.D.**
doc. RNDr. Petr Stehlík Ph.D. (former)

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Declaration

I declare that this thesis is my original work, unless clearly stated otherwise. This thesis is also based on my rigorous thesis [57], which was successfully defended in September 2018.

Pilsen 2021

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Abstrakt

V práci studujeme zobecněné verze metrické regularity, například nelineární a směrová regularita. Rovněž studujeme podobné zobecnění metrické subregularity a semiregularity a odvozujeme postačují podmínky pro tyto vlastnosti v případě jednoznačných zobrazení v konečné dimenzi.

Prvním cílem práce je definovat metrickou regularitu, metrickou subregularitu a metrickou semiregularity jednoznačných i mnohoznačných zobrazení. Formulujeme několik ekvivalentních vlastností a také uvedeme postačující i nutné podmínky pro jejich platnost. Dále se zabýváme stabilitou zmíněných vlastností vzhledem k jednoznačné i mnohoznačné perturbaci.

Druhým cílem je poskytnout postačující podmínky pro směrovou semiregularitu a semiregularitu s vazbou jednoznačných zobrazení v konečné dimenzi založených na aproximaci lineárním zobrazením a svazkem lineárních zobrazení. Zaměříme se na výpočet modulů (semi)regularity lineárních zobrazení.

Posledním cílem je zobecnit kritéria Ioffeho typu do kvazimetrických prostorů a tím získat kritéria pro nelineární a směrové verze uvedených vlastností.

Klíčová slova: kritéria regularity, metrická regularita, metrická subregularita, metrická semiregularita, nelineární regularita, směrová regularita, otevřenost zobrazení, kvazimetrický prostor, semiregularita s vazbou, modulus regularity lineárního zobrazení, kritéria Ioffeho typu, Ekelandův variační princip.

Abstract

In this thesis, we study criteria for generalized notions of metric regularity for single-valued and set-valued mappings, such as nonlinear and directional versions and the combination of both. We also study similar generalizations of metric subregularity and semiregularity and we focus on the criteria for constrained and directional semiregularity of single-valued mappings in finite dimensional spaces.

The first aim of this thesis is to discuss metric regularity, metric subregularity, and metric semiregularity of both single-valued and set-valued mappings. Several equivalent properties are formulated and the sufficient as well as the necessary conditions are presented. Further, we discuss the stability of these properties with respect to single-valued and set-valued perturbations.

The second aim is to provide sufficient conditions for directional and constrained semiregularity of single-valued mappings in finite dimensional spaces via an approximation by a linear mapping and by a bunch of linear mappings. We also focus on the computation of directional (semi)regularity modulus of linear mappings.

The last aim is to extend Ioffe-type criteria to quasi-metric spaces and thus to achieve criteria for nonlinear and directional versions of the mentioned properties.

Keywords: regularity criteria, metric regularity, metric subregularity, metric semiregularity, nonlinear regularity, directional regularity, openness of mapping, quasi-metric space, constrained semiregularity, modulus of regularity of linear mapping, Ioffe-type criteria, Ekeland variational principle.

Preface

In this thesis, we study criteria for certain various of regularity of single-valued and set-valued mappings. Some of them are well-known and studied in the literature and the other are new.

Metric regularity as well as corresponding equivalent properties called linear openness and Aubin property of the inverse, were entrenched in the literature, e.g. [3, 6, 7, 21, 23, 35, 37, 44, 48, 49], during several last decades. Although, Aubin property is also known under several different names. We call these three equivalent properties just regularity for short.

Metric regularity can be weakened in two ways by fixing one of the points involved in its definition. The first resulting property is called metric subregularity and is equivalent to pseudo-openness and calmness of the inverse, e.g. [1, 10, 23, 36]. We call these three equivalent properties only subregularity for short. The second one is metric semiregularity, e.g. [2, 14, 23, 26, 32, 33, 35, 42, 65], which is known under several under different names. Note that there are equivalent properties called linear openness at the reference point and recessiveness of the inverse. We call these three equivalent properties only semiregularity for short.

The first two chapters contain definitions of the properties and a (brief) historical survey of known results, respectively. The chapters are based on the author's rigorous thesis[1]:

[57] Roubal, T. *Regularity of Mappings.* University of West Bohemia, Pilsen, 2018

The third chapter deals with the criteria, in the spirit of [14, Theorem 3.4] and [54, Theorem 1], which guarantee constrained versions of the openness at the reference point in finite dimensional spaces. They are based on an approximation of a (nonlinear) single-valued mapping either by a linear mapping or by a bunch of a linear mappings. Moreover, we compute a constrained semiregularity modulus of linear mapping and establish its uniformity with respect to the elements of the bunch. This is based on:

[15] Cibulka, R., Fabian, M., and Roubal, T. An inverse mapping theorem in Fréchet-Montel spaces. *Set-Valued Var. Anal. 28*, 1 (2020), 195–208

In the fourth chapter, we provide basics of topology and recall the definition of a quasi-metric space, which is considered in [19, 20]. An extension of Ekeland variational principle to this setting.

The fifth chapter contains extensions of A.D. Ioffe's criterion for metric regularity introduced in [34] for single-valued and set-valued mappings. The criterion is extended to quasi-metric spaces and guarantees nonlinear and directional versions of regularity [12, 25, 30], subregularity [43, 46, 47, 69], and semiregularity. This section is based on:

[17] Cibulka, R., and Roubal, T. Solution stability and path-following for a class of generalized equations. In *Control systems and mathematical methods in economics*, vol. 687 of *Lecture Notes in Econom. and Math. Systems.* Springer, Cham, 2018, pp. 57–80

[1] Available online at https://dspace5.zcu.cz/bitstream/11025/33073/1/RoubalRig.pdf

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Tomáš Roubal

Contents

List of symbols

$\emptyset$	empty set
∞	(positive) infinity
$-\infty$	negative infinity
0	origin
$=$	equals to
$\leq$	is less than or equal to
$<$	is strictly less that
$\in$	is an element of
$\subset$	is a subset of
$\perp$	is orthogonal to
$a := b$	a is defined by b
$a =: b$	b is defined by a
$\mathbb{N}$	set of positive integers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^{m\times n}$	set of matrices with real entries having m rows and n columns
A^T	transpose of a matrix $A \in \mathbb{R}^{m\times n}$
$\mathbb{R}_+$	set of non-negative real numbers
$\mathbb{Q}$	set of rational numbers
$[a,b]$	closed interval with $a < b$
(a,b)	open interval with $a < b$
$[a,b)$	right-open interval with $a < b$
$(a,b]$	left-open interval with $a < b$
$X \times Y$	Cartesian product of sets X and Y
X^n	n-fold Cartesian product of a set X
$X \cup Y$	union of sets X and Y
$X \cap Y$	intersection of sets X and Y
$X \setminus Y$	set difference between X and Y
$X + Y$	set $\{a + b : a \in X \text{ and } b \in Y\}$
$X - Y$	set $X + (-Y)$
$X + x$	set $X + \{x\}$
$X - x$	set $X + \{-x\}$
λX	set $\{\lambda x : x \in X\}$ for $\lambda \in \mathbb{R}$
$\overline{X}$	closure of a set X

$\inf X$	infimum of a set $X \subset \mathbb{R}$
$\min X$	minimum of a set $X \subset \mathbb{R}$
$\sup X$	supremum of a set $X \subset \mathbb{R}$
$\max X$	maximum of a set $X \subset \mathbb{R}$
$\operatorname{int} X$	interior of a set X
$\operatorname{cone} X$	cone generated by a set X
$\operatorname{co} X$	convex hull of a set X
$\overline{\operatorname{co}}\, X$	convex closed hull of a set X

$\operatorname{diag}\{a_1, a_2, \dots, a_m\}$	diagonal matrix with elements $a_1, a_2, \dots, a_m$ on the diagonal
$\operatorname{span}\{v_1, v_2, \dots, v_m\}$	linear span of vectors $v_1, v_2, \dots, v_m$
$\dim X$	dimension of a subspace X
$\ker\ A$	kernel of a matrix A
$\operatorname{rge}\ F$	range of a set-valued mapping F
$\operatorname{dom}\ F$	domain of a set-valued mapping F
$\operatorname{gph}\ F$	graph of a set-valued mapping F

$f : X \longrightarrow Y$	single-valued mapping f from X to Y
$F : X \rightrightarrows Y$	set-valued mapping F from X to Y
$\longmapsto$	maps to
$f \circ g$	composition of single-valued mappings $f : X \longrightarrow Y$ and $g : Z \longrightarrow X$ in the form $Z \ni x \longmapsto f(g(x)) \in Y$
(x_k)	sequence
$\to$	converges to
$\lim_{k\to\infty} x_k$	limit of a sequence (x_k)

$\mathbb{B}_X(x, r)$	open ball with a center $x \in X$ and a radius $r > 0$ in a metric space X
$\mathbb{B}_X[x, r]$	closed ball with a center $x \in X$ and a radius $r > 0$ in a metric space X
$\mathbb{B}_X^{\varphi}(x, r)$	set $\{u \in X : \varphi(x, u) < r\}$, where a function $\varphi : X \times X \longrightarrow [0, \infty]$ is given
$\mathbb{B}_X^{\varphi}[x, r]$	set $\{u \in X : \varphi(x, u) \le r\}$, where a function $\varphi : X \times X \longrightarrow [0, \infty]$ is given
$\mathbb{S}_X$	unit sphere in normed space X
$\mathbb{B}_X$	closed unit ball in normed space X
$\operatorname{dist}(U, x)$	distance of from a point x to a subset U of a metric spaces
$\operatorname{dist}_X^{\varphi}(U, x)$	scalar $\inf_{u\in U} \varphi(u, x)$, where a function $\varphi : X \times X \longrightarrow [0, \infty]$ is given
$\lVert \cdot \rVert$	norm in a normed space
$\mathcal{L}(X, Y)$	space of all linear bounded operators from a Banach space X to into another Banach space Y equipped with standard operator norm
X^*	dual space to a Banach spaces X, i.e., $X^* := \mathcal{L}(X, \mathbb{R})$
$\langle \cdot, \cdot \rangle$	duality pairing
$\partial_B f(x)$	Bouligand generalized Jacobian of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ at $x \in \mathbb{R}^n$
$\partial_C f(x)$	Clarke generalized Jacobian of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ at $x \in \mathbb{R}^n$

Chapter 1

Introduction

1.1 Motivation

Let us consider a single-valued mapping f from X into Y, where X and Y are metric spaces and let $\bar{x} \in X$ be fixed. The mapping f is called open at $\bar{x}$ if the image of every neighborhood of $\bar{x}$ in X is a neighborhood of $f(\bar{x})$ in Y. The mapping f is said to be open if the image of every open set in X is an open set in Y. Suppose for a moment that f is one-to-one taking X onto Y so that there exists the single-valued inverse mapping f^{-1} defined on whole of Y. Then the openness of f at $\bar{x}$ is equivalent to the continuity of f^{-1} at $f(\bar{x})$ which means that the unique solution $x \in X$ of the equation

$$f(x) = y \tag{1.1}$$

is close to $\bar{x}$ whenever $y \in Y$ is sufficiently close to $f(\bar{x})$. Suppose now that f is not one-to-one. Then the solutions of the equation (1.1) may not be determined uniquely and the openness of f at $\bar{x}$ expresses the fact that whenever $y \in Y$ is sufficiently close to $f(\bar{x})$, then there exists a solution $x \in X$ of the equation (1.1) which is close to $\bar{x}$. In this case the inverse mapping f^{-1} is set-valued and we will see later that openness of f is equivalent to a certain kind of continuity of f^{-1}.

A set-valued mapping G from X into Y, denoted by $G : X \rightrightarrows Y$, is determined by a subset of $X \times Y$ called the graph of G denoted by gph G. Then G assigns to a point $x \in X$ a (possibly empty) subset $G(x)$ of Y, which contains all $y \in Y$ such that $(x, y) \in \operatorname{gph} G$ and is called the *image* of x under G or the *value* of G at x. The *domain* of G, denoted by dom G, is the set of points $x \in X$ such that the set $G(x)$ is nonempty, and the *range* of G, denoted by rge G, is the union of all sets $G(x)$ for $x \in \operatorname{dom} G$. Such a mapping G has always the *inverse*, denoted by G^{-1}, which is the set-valued mapping from Y to X such that, for each $(x, y) \in X \times Y$, the point $(y, x) \in \operatorname{gph} G^{-1}$ if and only if $(x, y) \in \operatorname{gph} G$. To emphasize that a mapping from X into Y is single-valued, we use lower-case letters and write $g : X \longrightarrow Y$.

Let a set-valued mapping $F : X \rightrightarrows Y$ and a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ be given. Consider the problem, for a fixed $y \in Y$, to find $x \in X$ such that

$$F(x) \ni y. \tag{1.2}$$

The openness of F at $(\bar{x}, \bar{y})$ means again that, for each neighborhood U of $\bar{x}$ in X, the set $F(U) := \bigcup_{x \in U} F(x)$ is a neighborhood of $\bar{y}$ in Y. In other words, for each $y \in Y$ sufficiently close to $\bar{y}$, there is a solution $x \in X$ of the inclusion (1.2) which is close to $\bar{x}$.

In both cases the openness gives us the existence of a solution but does not say anything about the distance between the solution x and the reference point $\bar{x}$. In order to get such an estimate, we define openness of F *at* $(\bar{x}, \bar{y})$ with a linear rate which means the existence of a constant $c > 0$ such that for

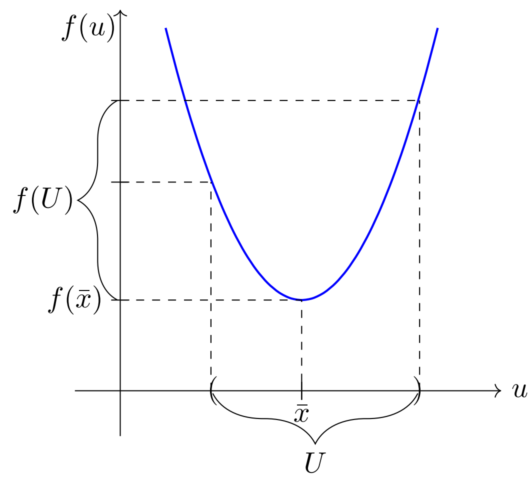

Figure 1.1: The function which is not open at $\bar{x}$.

each $r > 0$ small enough the image of a ball around $\bar{x}$ with the radius r contains a ball around $\bar{y}$ with the radius cr. This property is equivalent to a certain calmness property of the inverse F^{-1}.

We can even request the above property to be satisfied for each point (x, y) close to $(\bar{x}, \bar{y})$, with the same constant c independent of (x, y). This property is called openness *around* $(\bar{x}, \bar{y})$ with a linear rate and is equivalent to a certain kind of Lipschitz property of the inverse mapping F^{-1} called Aubin property. There is the third equivalent property called metric regularity which will be defined later. If X and Y are Banach spaces, then well-known Banach open mapping principle says that *a continuous linear operator from X to Y is open with a linear rate around any reference point if and only if it maps X onto Y.* A generalization of this principle to nonlinear mappings, proved by L.M. Graves, says that *a continuously differentiable mapping f from X to Y is open around a point $\bar{x} \in X$ with a linear rate if and only if its derivative $f'(\bar{x})$ is surjective.*

Now, let $X := \mathbb{R}^n$ and $Y := \mathbb{R}$. Given $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, consider a problem

$$\text{minimize} \quad f(u) \quad \text{subject to} \quad u \in \mathbb{R}^n. \tag{1.3}$$

Let $\bar{x} \in \mathbb{R}^n$ be a solution of (1.3). Then there is a neighborhood U of $\bar{x}$ in $\mathbb{R}^n$ such that $f(U)$ is not a neighborhood of $f(\bar{x})$, hence f is not open at $\bar{x}$, see Figure 1.1. Hence negation of any sufficient condition for openness (or openness with a linear rate) gives us a necessary condition for f to attain its minimum (or maximum) at $\bar{x}$. An example of such condition is Graves theorem. Suppose that f is a smooth function on $\mathbb{R}^n$. The derivative of f at $\bar{x}$ can be represented by the gradient $\nabla f(\bar{x})$ of f at $\bar{x}$ and the linear function $\mathbb{R}^n \ni u \longmapsto \langle \nabla f(\bar{x}), u \rangle$ is not surjective if and only if $\nabla f(\bar{x}) = 0$. So we have derived Euler-Fermat necessary condition.

This idea can be generalized, for example, to a constrained minimization problem in the form:

$$\text{minimize } f(u) \quad \text{subject to} \quad g_i(u) = 0 \quad \text{for} \quad i = 1, \ldots, m, \tag{1.4}$$

where functions $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ are continuously differentiable. Let $\bar{x} \in \mathbb{R}^n$ be a solution of (1.4) and define a mapping $h : \mathbb{R}^n \longrightarrow \mathbb{R}^{m+1}$ by

$$h(u) := (f(u), g_1(u), g_2(u), \ldots, g_m(u))^T \quad \text{for} \quad u \in \mathbb{R}^n. \tag{1.5}$$

Consequently, we have

$$h(\bar{x}) = (f(\bar{x}), 0, 0, \ldots, 0)^T \quad \text{and} \quad \nabla h(\bar{x}) = (\nabla f(\bar{x}), \nabla g_1(\bar{x}), \nabla g_2(\bar{x}), \ldots, \nabla g_m(\bar{x}))^T.$$

Fix any sufficiently small $\varepsilon > 0$ and let

$$y := (f(\bar{x}) - \varepsilon, 0, 0, \ldots, 0)^T.$$

Then there is no $x \in \mathbb{R}^n$ with $h(x) = y$. Indeed, for any such x, we would have $f(x) = f(\bar{x}) - \varepsilon < f(\bar{x})$ and $g_i(x) = 0$ for each $i = 1, \ldots, m$, which contradicts the assumption that $\bar{x}$ solves (1.4). Consequently, h is not open at $\bar{x}$ and, by Graves theorem, the mapping $\mathbb{R}^n \ni u \longmapsto \nabla h(\bar{x})u$ is not surjective. This means that the rows of the Jacobian matrix $\nabla h(\bar{x})$ are linearly dependent. In other words, there are numbers $\lambda_i \in \mathbb{R}$, for $i = 0, 1, \ldots, m$, such that

$$\lambda_0 \nabla f(\bar{x}) + \lambda_1 \nabla g_1(\bar{x}) + \lambda_2 \nabla g_2(\bar{x}) + \cdots + \lambda_m \nabla g_m(\bar{x}) = 0.$$

If all the vectors $\nabla g_i(\bar{x})$, for $i = 1, 2, \ldots, m$, are linearly independent, this is known as the *linear independence constraint qualification condition*, then the previous equality can be rewritten as

$$\nabla f(\bar{x}) + \lambda_1 \nabla g_1(\bar{x}) + \lambda_2 \nabla g_2(\bar{x}) + \cdots + \lambda_m \nabla g_m(\bar{x}) = 0.$$

The numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ are called Lagrange multipliers. We have derived Karush-Kuhn–Tucker necessary conditions for the problem (1.4). An interesting fact is that W. Karush, who derived these conditions in his master thesis in 1939, was a student of Graves, e.g. [23, p. 343].

Further, consider a continuously differentiable mapping $h : \mathbb{R}^n \longrightarrow \mathbb{R}^{m+1}$, a closed convex one $L \subset \mathbb{R}^n$, and a point $\bar{x} \in X$. From the statements in Section 3.2, we can derive the following fact: Suppose that there is $c > 0$ such that

$$\nabla h(\bar{x})(L \cap \mathbb{B}_{\mathbb{R}^n}) \supset c\mathbb{B}_{\mathbb{R}^{m+1}}, \tag{1.6}$$

then for each $c' \in (0, c)$ there is $r > 0$ such that

$$h\big((\bar{x} + L) \cap \mathbb{B}_{\mathbb{R}^n}[\bar{x}, t]\big) \supset \mathbb{B}_{\mathbb{R}^{m+1}}[h(\bar{x}), c't] \quad \text{for each} \quad t \in (0, r]. \tag{1.7}$$

Let us consider the problem of finding a minimum with respect to some closed convex cone $L \subset \mathbb{R}^n$ in the form:

$$\text{minimize} \quad f(u) \quad \text{subject to} \quad u \in L \quad \text{and} \quad g_i(u) = 0 \quad \text{for} \quad i = 1, \ldots, m. \tag{1.8}$$

We derive a necessary condition for a point $\bar{x} \in \mathbb{R}^n$ to solve (1.8). By the similar argument as above the set $h(L)$ is not a neighborhood of $h(\bar{x})$, where h is defined in (1.5). Since L is a convex cone and $\bar{x} \in L$, we have that $h(\bar{x} + L)$ is not a neighborhood of $h(\bar{x})$ and so $h\big((\bar{x} + L) \cap \mathbb{B}_{\mathbb{R}^n}[\bar{x}, r]\big)$ is not a neighborhood of $h(\bar{x})$ for each $r > 0$. Hence (1.7) does not hold, then so (1.6) fails. Hence $\nabla h(\bar{x})(L) \neq \mathbb{R}^{m+1}$, so fix any $y \in \mathbb{R}^{m+1} \setminus \nabla h(\bar{x})(L)$.

Note that $\nabla h(\bar{x})(L)$ is a convex cone, then, by the separation theorem, there is a nonzero $\lambda := (\lambda_0, \lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^{m+1}$ such that

$$\langle \lambda, \nabla h(\bar{x})x \rangle \geq \langle \lambda, y \rangle \quad \text{for each} \quad x \in L. \tag{1.9}$$

This implies that

$$\langle \lambda, \nabla h(\bar{x})x \rangle \geq 0 \quad \text{for each} \quad x \in L.$$

Indeed, on the contrary, assume that there is $x \in L$ such that $\langle \lambda, \nabla h(\bar{x})x \rangle < 0$. Since $\nabla h(\bar{x})(L)$ is a cone, for each $\varepsilon > 0$ we have $\varepsilon x \in L$ and $\langle \lambda, \nabla h(\bar{x})(\varepsilon x) \rangle = \varepsilon \langle \lambda, \nabla h(\bar{x})x \rangle < 0$. Letting $\varepsilon \to \infty$ we get $\varepsilon \langle \lambda, \nabla h(\bar{x})x \rangle \to -\infty$, that contradicts (1.9).

Hence, we derived that

$$\big\langle \nabla h(\bar{x})^T \lambda, x \big\rangle \geq 0 \quad \text{for each} \quad x \in L$$

so

$$\langle \lambda_0 \nabla f(\bar{x}) + \lambda_1 \nabla g_1(\bar{x}) + \cdots + \lambda_m g_m(\bar{x}), x \rangle \geq 0 \quad \text{for each} \quad x \in L.$$

1.2 Regularity of mappings

In this section, we present various regularity properties of a set-valued mapping $F : X \rightrightarrows Y$, that maps from a metric space (X, d) into subsets of a metric space (Y, ϱ).

We focus on three properties called regularity, subregularity, and semiregularity. At the end of this section, we present "stronger" versions of these properties. All of them play a fundamental role in modern variational analysis, non-smooth analysis, and optimization. We will illustrate this on the problems (1.1) and (1.2).

By the term *semiregularity* at the reference point we mean the group of three equivalent properties called metric semiregularity, openness with a linear rate at the reference point, and recession with a linear rate of the inverse. Metric semiregularity was introduced by A.Y. Kruger in [42] in 2009 and can be found also under the name *hemiregularity*, e.g., [2, 26, 65].

Definition 1.2.1 *Let (X, d) and (Y, ϱ) be metric spaces, and a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The mapping F is said to be* metrically semiregular at $(\bar{x}, \bar{y})$ *when there is a constant $\kappa > 0$ along with a neighborhood V of $\bar{y}$ in Y such that*

$$\operatorname{dist}\left(\bar{x}, F^{-1}(y)\right) \leq \kappa \varrho(y, \bar{y}) \quad \text{for every} \quad y \in V. \tag{1.10}$$

The infimum of $\kappa > 0$ for which there exists a neighborhood V of $\bar{y}$ in Y such that (1.10) *holds is called the* semiregularity modulus *of F at $(\bar{x}, \bar{y})$ and is denoted by* $\operatorname{semireg} F(\bar{x}, \bar{y})$.

We use the convention that $\inf \emptyset = \infty$, that is, $\operatorname{semireg} F(\bar{x}, \bar{y}) < \infty$ if and only if F is metrically semiregular at $(\bar{x}, \bar{y})$. For a single-valued mapping $f : X \longrightarrow Y$ we omit the point $\bar{y} = f(\bar{x})$, that is, we write $\operatorname{semireg} f(\bar{x})$ (and the same applies in all the definitions below) and for a linear mapping $A : X \longrightarrow Y$ we omit even the point $\bar{x}$, that is, we write $\operatorname{semireg} A$ (and the same applies for the other properties). Now suppose for a moment that F is metrically semiregular at $(\bar{x}, \bar{y})$. Let $\kappa > \operatorname{semireg} F(\bar{x}, \bar{y})$ be arbitrary. From (1.10), for a fixed $y \in V$, we have

$$\operatorname{dist}\left(\bar{x}, F^{-1}(y)\right) < \infty,$$

that is, the set $F^{-1}(y)$ is nonempty. Moreover, there is a point $x \in X$ with $y \in F(x)$ such that

$$d(\bar{x}, x) \leq \kappa \varrho(y, \bar{y}).$$

Metric semiregularity guarantees the solvability of (1.2) for all $y \in V$ and also the estimate of the distance between the reference point $\bar{x}$ and the solution x. In other words, it guarantees the stability of a solution with respect to small perturbations of the right-hand side.

Metric semiregularity is equivalent to openness with a linear rate at the reference point which can be found under the name *controllability*, e.g., in [23, 26, 32, 33, 35].

Definition 1.2.2 *Let (X, d) and (Y, ϱ) be metric spaces, and a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The mapping F is said to be* open with a linear rate at $(\bar{x}, \bar{y})$ *when there are positive constants c and ε such that*

$$\mathbb{B}_Y[\bar{y}, ct] \subset F(\mathbb{B}_X[\bar{x}, t]) \quad \text{for each} \quad t \in (0, \varepsilon]. \tag{1.11}$$

The supremum of $c > 0$ for which there exists a constant $\varepsilon > 0$ such that (1.11) holds is called the modulus of openness *of F at $(\bar{x}, \bar{y})$ and is denoted by* $\operatorname{lopen} F(\bar{x}, \bar{y})$.

As we work with nonnegative quantities, we use the convention that $\sup \emptyset = 0$, that is, $\operatorname{lopen} F(\bar{x}, \bar{y}) > 0$ if and only if F is open with a linear rate at $(\bar{x}, \bar{y})$.

Recession with a linear rate, introduced by A.D. Ioffe in [35], closes the first group of definitions. Note that this property is sometimes called pseudo-calmness [26] or Lipschitz-lower semicontinuity [42].

Definition 1.2.3 *Let (X, d) and (Y, ϱ) be metric spaces, and a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The mapping F is said to* recede from $\bar{y}$ at $(\bar{x}, \bar{y})$ with a linear rate *when there is a constant $\mu > 0$ along with a neighborhood U of $\bar{x}$ in X such that*

$$\operatorname{dist}\big(\bar{y}, F(x)\big) \leq \mu\, d(\bar{x}, x) \quad \text{for each} \quad x \in U. \tag{1.12}$$

The infimum of $\mu > 0$ for which there exists a neighborhood U of $\bar{x}$ in X such that (1.12) *holds is called the* speed of recession *of F at $(\bar{x}, \bar{y})$ and is denoted by* $\operatorname{recess} F(\bar{x}, \bar{y})$.

The mapping F recedes from $\bar{y}$ at $(\bar{x}, \bar{y})$ with a linear rate if and only if $\operatorname{recess} F(\bar{x}, \bar{y}) < \infty$. If, in addition, the space Y is a vector (linear) space, then for any $\mu > \operatorname{recess} F(\bar{x}, \bar{y})$ there is a neighborhood U of $\bar{x}$ in X such that

$$\bar{y} \in F(x) + \mu\, d(\bar{x}, x) I\!B_Y \quad \text{for each} \quad x \in U.$$

Example 1.2.1 *Consider a single-valued mapping $f : X \longrightarrow Y$ which recedes from $f(\bar{x})$ at $\bar{x}$ with a linear rate. Then for any $\mu > \operatorname{recess} f(\bar{x})$ there is a neighborhood U of $\bar{x}$ in X such that*

$$\varrho\big(f(\bar{x}), f(x)\big) \leq \mu\, d(\bar{x}, x) \quad \text{for each} \quad x \in U.$$

This is the definition of calmness *of f at $\bar{x}$.*

The following theorem guarantees the above mentioned equivalence of metric semiregularity, openness with a linear rate at the reference point, and recession with a linear rate of the inverse, for the proof, see [14, Proposition 2.1].

Theorem 1.2.1 *Let (X, d) and (Y, ϱ) be metric spaces, and a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The following assertions are equivalent:*

(i) *F is metrically semiregular at $(\bar{x}, \bar{y})$;*

(ii) *F is open with a linear rate at $(\bar{x}, \bar{y})$;*

(iii) *F^{-1} recedes from $\bar{x}$ at $(\bar{y}, \bar{x})$ with a linear rate.*

In addition, we have

$$\operatorname{lopen} F(\bar{x}, \bar{y}) \cdot \operatorname{semireg} F(\bar{x}, \bar{y}) = 1 \quad \text{and} \quad \operatorname{semireg} F(\bar{x}, \bar{y}) = \operatorname{recess} F^{-1}(\bar{y}, \bar{x}),$$

under the convention $0 \cdot \infty = \infty \cdot 0 = 1$.

The above statement justifies the following definition.

Definition 1.2.4 *Let (X, d) and (Y, ϱ) be metric spaces, and a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The mapping F is said to be* semiregular at $(\bar{x}, \bar{y})$ *if and only if* $\operatorname{semireg} F(\bar{x}, \bar{y}) < \infty$ *if and only if* $\operatorname{lopen} F(\bar{x}, \bar{y}) > 0$ *if and only if* $\operatorname{recess} F^{-1}(\bar{y}, \bar{x}) < \infty$.

Further, by the term *subregularity* at the reference point we mean the group of three equivalent properties called metric subregularity, pseudo-openness with a linear rate at the reference point, and calmness of the inverse. Metric subregularity is entrenched in the literature [23].

Definition 1.2.5 *Let (X,d) and (Y,ϱ) be metric spaces, and a point $(\bar{x},\bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The mapping F is said to be* metrically subregular at $(\bar{x},\bar{y})$ *when there is a constant $\kappa > 0$ along with a neighborhood U of $\bar{x}$ in X such that*

$$\operatorname{dist}\big(x, F^{-1}(\bar{y})\big) \le \kappa \operatorname{dist}\big(\bar{y}, F(x)\big) \quad \textit{for every} \quad x \in U. \tag{1.13}$$

The infimum of $\kappa > 0$ for which there exists a neighborhood U of $\bar{x}$ in X such that (1.13) *holds is called the* subregularity modulus *of F at $(\bar{x},\bar{y})$ and is denoted by* $\operatorname{subreg} F(\bar{x},\bar{y})$.

The mapping F is metrically subregular at $(\bar{x},\bar{y})$ if and only if $\operatorname{subreg} F(\bar{x},\bar{y}) < \infty$. Note that metric subregularity does not guarantee solvability of (1.1) and (1.2), respectively, as in the case of semiregularity.

Example 1.2.2 *Consider a single-valued mapping $f : X \longrightarrow Y$ which is metrically subregular at a point $\bar{x} \in X$. Then for any $\kappa >$* subreg $f(\bar{x})$ *there is a neighborhood of U of $\bar{x}$ such that for a fixed $x \in U$ there is $u \in X$ such that*

$$\bar{y} = f(u) \quad \textit{and} \quad d\big(x,u\big) \le \kappa\varrho\big(\bar{y}, f(x)\big).$$

In other words, if $x \in U$ is an approximate solution of (1.1) *with $y := \bar{y}$, then we can estimate the distance from x to the solution set $f^{-1}(\bar{y})$ by the residuum $\varrho\big(\bar{y}, f(x)\big)$. The same is true for set-valued mappings.*

The following proposition shows us two more equivalent properties to metric subregularity.

Proposition 1.2.1 *Let (X,d) and (Y,ϱ) be metric spaces, and a point $(\bar{x},\bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The following assertions are equivalent:*

(i) *F is metrically subregular at $(\bar{x},\bar{y})$;*

(ii) *there is a constant $\kappa > 0$ along with a neighborhood $U \times V$ of $(\bar{x},\bar{y})$ in $X \times Y$ such that*

$$\operatorname{dist}\big(x, F^{-1}(\bar{y})\big) \le \kappa \operatorname{dist}(\bar{y}, F(x) \cap V) \quad \textit{for each} \quad x \in U;$$

(iii) *there is a constant $\kappa > 0$ along with a neighborhood U of $\bar{x}$ in X such that*

$$\operatorname{dist}\big(x, F^{-1}(\bar{y})\big) \le \operatorname{dist}_{1,\kappa}((x,\bar{y}), \operatorname{gph} F) \quad \textit{for each} \quad x \in U,$$

where for a subset $A \subset X \times Y$ and a point $(u,v) \in X \times Y$ we define

$$\operatorname{dist}_{1,\kappa}((u,v),A) := \inf\{d(u,u') + \kappa\varrho(v,v') : (u',v') \in A\}. \tag{1.14}$$

The equivalence (i) $\Leftrightarrow$ (ii) was showed in [23, Exercise 3H.4]. The property (iii) is called *graph-subregularity* of F at $(\bar{x},\bar{y})$ and was proved to be equivalent to (i) in [36]. It uses the graph of F instead of the values of F. The mapping $X \times Y \ni (x,y) \longmapsto \operatorname{dist}_{1,\kappa}((x,y), \operatorname{gph} F)$ is Lipschitz continuous whereas the mapping $X \times Y \ni (x,y) \longmapsto \operatorname{dist}(y, F(x))$ may be not even continuous. Therefore sometimes it is convenient to work with the graph-subregularity.

Next property is pseudo-openness that is defined and proved to be equivalent to metric subregularity and calmness in [1].

Definition 1.2.6 *Let (X,d) and (Y,ϱ) be metric spaces, and a point $(\bar{x},\bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The mapping F is said to be* pseudo-open with a linear rate at $(\bar{x},\bar{y})$ *when there are positive constants c and ε along with a neighborhood U of $\bar{x}$ in X such that*

$$\bar{y} \in F(I\!B_X[x,t]) \quad \textit{whenever} \quad x \in U \quad \textit{and} \quad t \in (0,\varepsilon], \ \textit{with } F(x) \cap I\!B_Y[\bar{y}, ct] \neq \emptyset. \tag{1.15}$$

The supremum of $c > 0$ for which there exist a constant $\varepsilon > 0$ and a neighborhood U of $\bar{x}$ in X such that (1.15) holds is called the modulus of pseudo-openness *of F at $(\bar{x},\bar{y})$ and is denoted by* $\operatorname{popen} F(\bar{x},\bar{y})$.

The mapping F is pseudo-open at $(\bar{x}, \bar{y})$ with a linear rate if and only if $\operatorname{popen} F(\bar{x}, \bar{y}) > 0$.

Calmness is entrenched in literature [35, 23] and closes the second group of definitions.

Definition 1.2.7 *Let (X, d) and (Y, ϱ) be metric spaces, and a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The mapping F is said to be* calm at $(\bar{x}, \bar{y})$ *when there is a constant $\mu > 0$ along with a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that*

$$\operatorname{dist}\big(y, F(\bar{x})\big) \leq \mu\, d(x, \bar{x}) \quad \textit{whenever} \quad x \in U \quad \textit{and} \quad y \in F(x) \cap V. \tag{1.16}$$

The infimum of $\mu > 0$ for which there exists a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that (1.16) *holds is called the* calmness modulus *of F at $(\bar{x}, \bar{y})$ and is denoted by* $\operatorname{calm} F(\bar{x}, \bar{y})$.

Hence the mapping F is calm at $(\bar{x}, \bar{y})$ if and only if $\operatorname{calm} F(\bar{x}, \bar{y}) < \infty$. If, in addition, the space Y is a vector space, then for any $\mu > \operatorname{calm} F(\bar{x}, \bar{y})$ there is a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that

$$F(x) \cap V \subset F(\bar{x}) + \mu\, d(x, \bar{x}) I\!\!B_Y \quad \text{for each} \quad x \in U.$$

Example 1.2.3 *Consider a single-valued mapping $f : X \longrightarrow Y$ which is calm at a point $\bar{x} \in X$. Then for any $\mu >$ calm $f(\bar{x})$ there is a neighborhood U of $\bar{x}$ such that*

$$\varrho\big(f(x), f(\bar{x})\big) \leq \mu\, d(x, \bar{x}) \quad \textit{for each} \quad x \in U.$$

In this case, calmness and recession with a linear rate coincide.

The following theorem, established in [42], guarantees the equivalence of metric subregularity, pseudo-openness with a linear rate, and calmness of the inverse.

Theorem 1.2.2 *Let (X, d) and (Y, ϱ) be metric spaces, and a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The following assertions are equivalent:*

(i) *F is metrically subregular at $(\bar{x}, \bar{y})$;*

(ii) *F is pseudo-open with a linear rate at $(\bar{x}, \bar{y})$;*

(iii) *F^{-1} is calm at $(\bar{y}, \bar{x})$.*

In addition, we have

$$\operatorname{popen} F(\bar{x}, \bar{y}) \cdot \operatorname{subreg} F(\bar{x}, \bar{y}) = 1 \quad \textit{and} \quad \operatorname{subreg} F(\bar{x}, \bar{y}) = \operatorname{calm} F^{-1}(\bar{y}, \bar{x}).$$

The above statement justifies the following definition.

Definition 1.2.8 *Let (X, d) and (Y, ϱ) be metric spaces, and a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The mapping F is said to be* subregular at $(\bar{x}, \bar{y})$ *if and only if* subreg $F(\bar{x}, \bar{y}) < \infty$ *if and only if* popen $F(\bar{x}, \bar{y}) > 0$ *if and only if* calm $F^{-1}(\bar{y}, \bar{x}) < \infty$.

We have seen that semiregularity of the mappings appearing in (1.1) or (1.2) gives us solvability of these problems as well as stability of a solution with respect to small perturbations of the right-hand side. On the other hand, subregularity provides an estimate of the error of an approximate solution via the residuum. Now, we present a property which guarantees both the previous ones. By the term *regularity* around the reference point we mean the group of equivalent properties called metric regularity, openness with a linear rate around the reference point, and Aubin property of the inverse.

The name metric regularity was suggested by J.M. Borwein [6] in 1986.

Definition 1.2.9 *Let (X, d) and (Y, ϱ) be metric spaces, and a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The mapping F is said to be* metrically regular *around $(\bar{x}, \bar{y})$ when there is a constant $\kappa > 0$ along with a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that*

$$\operatorname{dist}\left(x, F^{-1}(y)\right) \leq \kappa \operatorname{dist}(y, F(x)) \quad \text{for every} \quad (x, y) \in U \times V. \tag{1.17}$$

The infimum of $\kappa > 0$ for which there exists a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that (1.17) *holds is called the* regularity modulus *of F around $(\bar{x}, \bar{y})$ and is denoted by* $\operatorname{reg} F(\bar{x}, \bar{y})$.

The mapping F is metrically regular at $(\bar{x}, \bar{y})$ if and only if $\operatorname{reg} F(\bar{x}, \bar{y}) < \infty$. In this case, for any $\kappa > \operatorname{reg} F(\bar{x}, \bar{y})$ there is a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that (1.17) holds. Letting $x := \bar{x}$, we get

$$\operatorname{dist}\left(\bar{x}, F^{-1}(y)\right) \leq \kappa \operatorname{dist}(y, F(\bar{x})) \leq \kappa\, \varrho(y, \bar{y}) \quad \text{for every} \quad y \in V.$$

We derived (1.10), hence F is semiregular at $(\bar{x}, \bar{y})$. Further, letting $y := \bar{y}$ in (1.17), we get (1.13), which means F is subregular at $(\bar{x}, \bar{y})$.

There are several equivalent definitions in the literature.

Proposition 1.2.2 *Let (X, d) and (Y, ϱ) be metric spaces, and a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The following assertions are equivalent:*

(i) *F is metrically regular around $(\bar{x}, \bar{y})$;*

(ii) *there is $\kappa > 0$ along with a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that*

$$\operatorname{dist}\left(x, F^{-1}(y)\right) \leq \kappa \operatorname{dist}(y, F(x) \cap V) \quad \text{for each} \quad (x, y) \in U \times V;$$

(iii) *there is $\kappa > 0$ along with a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that*

$$\operatorname{dist}\left(x, F^{-1}(y)\right) \leq \operatorname{dist}_{1,\kappa}((x, y), \operatorname{gph} F) \quad \text{for each} \quad (x, y) \in U \times V,$$

where $\operatorname{dist}_{1,\kappa}$ is defined in (1.14).

The equivalence (i) $\Leftrightarrow$ (ii) was showed in [23, Proposition 5H.1]. The property (iii) is called *graph-regularity at $(\bar{x}, \bar{y})$* in [64], where the equivalence (i) $\Leftrightarrow$ (iii) was proved.

Openness with a linear rate around the reference point is a stronger concept than openness with a linear rate at the reference point defined above.

Definition 1.2.10 *Let (X, d) and (Y, ϱ) be metric spaces, and a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The mapping F is said to be* open with a linear rate *around $(\bar{x}, \bar{y})$ when there are positive constants c and ε along with a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that*

$$I\!B_Y[y, ct] \subset F(I\!B_X[x, t]) \quad \text{whenever} \quad (x, y) \in U \times V, \quad y \in F(x), \quad \text{and} \quad t \in (0, \varepsilon]. \tag{1.18}$$

The supremum of $c > 0$ for which there exist a constant $\varepsilon > 0$ and a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that (1.18) holds is called the modulus of surjection *of F around $(\bar{x}, \bar{y})$ and is denoted by* $\operatorname{sur} F(\bar{x}, \bar{y})$.

The mapping F is open around $(\bar{x}, \bar{y})$ with a linear rate if and only if $\operatorname{sur} F(\bar{x}, \bar{y}) > 0$.

Aubin property, introduced by J.-P. Aubin in [3] under the name pseudo-Lipschitz property, closes the third group of definitions. We can also find a term Lipschitz-like property in literature [44].

Definition 1.2.11 *Let (X, d) and (Y, ϱ) be metric spaces, and $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The mapping F is said to have* Aubin property *around $(\bar{x}, \bar{y})$ when there is a constant $\mu > 0$ along with a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that*

$$\text{dist}\big(y, F(u)\big) \leq \mu\, d(x, u) \quad \textit{whenever} \quad x, u \in U \quad \textit{and} \quad y \in F(x) \cap V. \tag{1.19}$$

The infimum of $\mu > 0$ for which there exists a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that (1.19) *holds is called the Lipschitz modulus of F around $(\bar{x}, \bar{y})$ and is denoted by* $\text{lip}\, F(\bar{x}, \bar{y})$.

The mapping F has Aubin property around $(\bar{x}, \bar{y})$ if and only if $\text{lip}\, F(\bar{x}, \bar{y}) < \infty$. If, in addition, the space Y is a vector space, then for any $\mu > \text{lip}\ F(\bar{x}, \bar{y})$ there is a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that

$$F(x) \cap V \subset F(u) + \mu\, d(x, u) I\!\!B_Y \quad \text{for each} \quad x, u \in U.$$

As in the case of metric regularity and openness with a linear rate around the point, letting $u := \bar{x}$ in (1.19), we conclude that F is calm at $(\bar{x}, \bar{y})$ and, letting $y := \bar{y}$ and $x := \bar{x}$, we conclude that F recedes from $\bar{y}$ at $(\bar{x}, \bar{y})$ with a linear rate.

Example 1.2.4 *Consider a single-valued mapping $f : X \longrightarrow Y$ which has Aubin property around $\bar{x}$. Then for any $\mu > \text{lip}\ f(\bar{x})$ there is a neighborhood U of $\bar{x}$ in X such that*

$$\varrho\big(f(x), f(u)\big) \leq \mu\, d(x, u) \quad \textit{for each} \quad x, u \in U.$$

The last inequality is the definition of Lipschitz continuity of f on U and therefore Aubin property of f around $\bar{x}$ means local Lipschitz continuity of f around $\bar{x}$.

The following theorem guarantees the equivalence of metric regularity, openness with a linear rate around the reference point, and Aubin property of the inverse, and gives us relations among the corresponding moduli. The equivalence of openness with a linear rate and metric regularity was mentioned, probably for the first time, by Dmitruk, Milyutin, and Osmolowski [21] in 1980. In late 80s, Borwein-Zhuang [7] and Penot [48] proved (along with the equivalence with Aubin property) the full statement.

Theorem 1.2.3 *Let (X, d) and (Y, ϱ) be metric spaces, and a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The following assertions are equivalent:*

(i) *F is metrically regular around $(\bar{x}, \bar{y})$;*

(ii) *F is open with a linear rate around $(\bar{x}, \bar{y})$;*

(iii) *F^{-1} has Aubin property around $(\bar{y}, \bar{x})$.*

In addition, we have

$$\text{sur}\, F(\bar{x}, \bar{y}) \cdot \text{reg}\, F(\bar{x}, \bar{y}) = 1 \quad \textit{and} \quad \text{reg}\, F(\bar{x}, \bar{y}) = \text{lip}\, F^{-1}(\bar{y}, \bar{x}).$$

The above statement justifies the following definition.

Definition 1.2.12 *Let (X, d) and (Y, ϱ) be metric spaces, and a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The mapping F is said to be* regular around $(\bar{x}, \bar{y})$ *if and only if* $\text{reg}\ F(\bar{x}, \bar{y}) < \infty$ *if and only if* $\text{sur}\ F(\bar{x}, \bar{y}) > 0$ *if and only if* $\text{lip}\ F^{-1}(\bar{y}, \bar{x}) < \infty$.

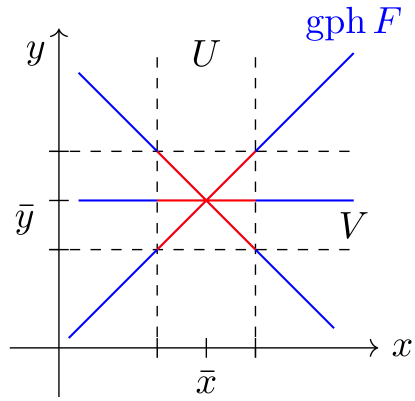

Figure 1.2: A localization (in red) of the set-valued mapping F (in blue).

We close this section by the group of stronger versions of the previous properties. For this purpose we need the notion of a *localization* of a set-valued mapping $F : X \rightrightarrows Y$ around the reference point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, which is any mapping $\widetilde{F} : X \rightrightarrows Y$ such that $\operatorname{gph} \widetilde{F} = \operatorname{gph} F \cap (U \times V)$ for some neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$, see Figure 1.2.

We start with strong semiregularty, e.g. [2].

Definition 1.2.13 *Let (X, d) and (Y, ϱ) be metric spaces, and a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The mapping F is said to be* strongly semiregular at $(\bar{x}, \bar{y})$ *when F is metrically semiregular at $(\bar{x}, \bar{y})$ and F^{-1} has a localization around $(\bar{y}, \bar{x})$ which is nowhere multivalued.*

Let $F : X \rightrightarrows Y$ be strongly semiregular at $(\bar{x}, \bar{y})$. Then for any $\ell > \operatorname{semireg} F(\bar{x}, \bar{y})$ there is a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ such that the mapping $V \ni y \longmapsto F^{-1}(y) \cap U$ is single-valued on V and calm at $\bar{y}$ with the constant ℓ.

Strong subregularity is entrenched in the literature [10].

Definition 1.2.14 *Let (X, d) and (Y, ϱ) be metric spaces, and a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The mapping F is said to be* strongly subregular at $(\bar{x}, \bar{y})$ *when F is subregular at $(\bar{x}, \bar{y})$ and F^{-1} has no localization around $(\bar{y}, \bar{x})$ that is multivalued at $\bar{y}$.*

Let $F : X \rightrightarrows Y$ be strongly subregular at $(\bar{x}, \bar{y})$. Then for any $\ell > \operatorname{subreg} F(\bar{x}, \bar{y})$ there is a neighborhood U of $\bar{x}$ such that

$$d\big(x, \bar{x}\big) \leq \ell \operatorname{dist}(\bar{y}, F(x)) \quad \text{whenever} \quad x \in U,$$

that is, F^{-1} has *isolated calmness property at* $(\bar{y}, \bar{x})$, see [23].

Strong regularity was introduced by S.M. Robinson in [56] for generalized equations. This property is related to the (local) inverse function theorem and the implicit function theorem.

Definition 1.2.15 *Let (X, d) and (Y, ϱ) be metric spaces, and a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. The mapping F is said to be* strongly regular around $(\bar{x}, \bar{y})$ *when F is regular around $(\bar{x}, \bar{y})$ and F^{-1} has a localization around $(\bar{y}, \bar{x})$ which is nowhere multivalued.*

Let $F : X \rightrightarrows Y$ be strongly regular around $(\bar{x}, \bar{y})$. Then for any $\ell > \operatorname{reg} F(\bar{x}, \bar{y})$ there is a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ such that the mapping $V \ni y \longmapsto F^{-1}(y) \cap U$ is single-valued on V and Lipschitz continuous on V with the constant ℓ.

The section closes with several examples.

Example 1.2.5 1) *Let* $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$ *be defined by* $f_1(x) := |x|$ *for* $x \in \mathbb{R}$. *Obviously for each* $y < 0$ *there is no* $x \in \mathbb{R}$ *such* $f_1(x) = y$, *hence* f_1 *is not semiregular at* 0. *On other hand, for each* $x \in \mathbb{R}$ *we have*

$$\operatorname{dist}\left(x, f_1^{-1}(0)\right) = |x| = \operatorname{dist}(0, f_1(x)).$$

Therefore f_1 *is subregular at* 0 *with the constant* 1. *The graph of* f_1 *is in Figure 1.3a;*

2) *Let* $f_2 : \mathbb{R} \longrightarrow \mathbb{R}$ *be defined by*

$$f_2(x) := \begin{cases} x^2 \sin(1/x) & \textit{for} \quad x \neq 0, \\ 0 & \textit{for} \quad x = 0. \end{cases}$$

Then f_2 *is subregular and open at* 0 *but it is not semiregular at* 0. *The graph of* f_2 *is in Figure 1.3b;*

3) *Let* $f_3 : \mathbb{R} \longrightarrow \mathbb{R}$ *be defined by*

$$f_3(x) := \begin{cases} x + x|x \sin(1/x)| & \textit{for} \quad x \neq 0, \\ 0 & \textit{for} \quad x = 0. \end{cases}$$

Then f_3 *is semiregular (not strongly) at* 0 *and strongly subregular at* 0. *This example is from [14] and for the graph of* f_3, *see Figure 1.3c;*

4) *Let* $f_4 : \mathbb{R} \longrightarrow \mathbb{R}$ *be defined by* $f_4(x) := \sqrt[3]{x}$ *for* $x \in \mathbb{R}$. *Then* f_4 *is strongly regular at any* $x \in \mathbb{R}$. *Moreover, the inverse is* $f_4^{-1}(x) = x^3$ *for* $x \in \mathbb{R}$. *The graph of* f_4 *is in Figure 1.3d;*

5) *Let* $f_5 : \mathbb{R} \longrightarrow \mathbb{R}$ *be defined by*

$$f_5(x) := \begin{cases} x & \textit{for} \quad x \in \mathbb{Q}, \\ -x & \textit{for} \quad x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then f_5 *is strongly semiregular at* 0 *and strongly subregular at* 0, *but it is not regular around* 0.

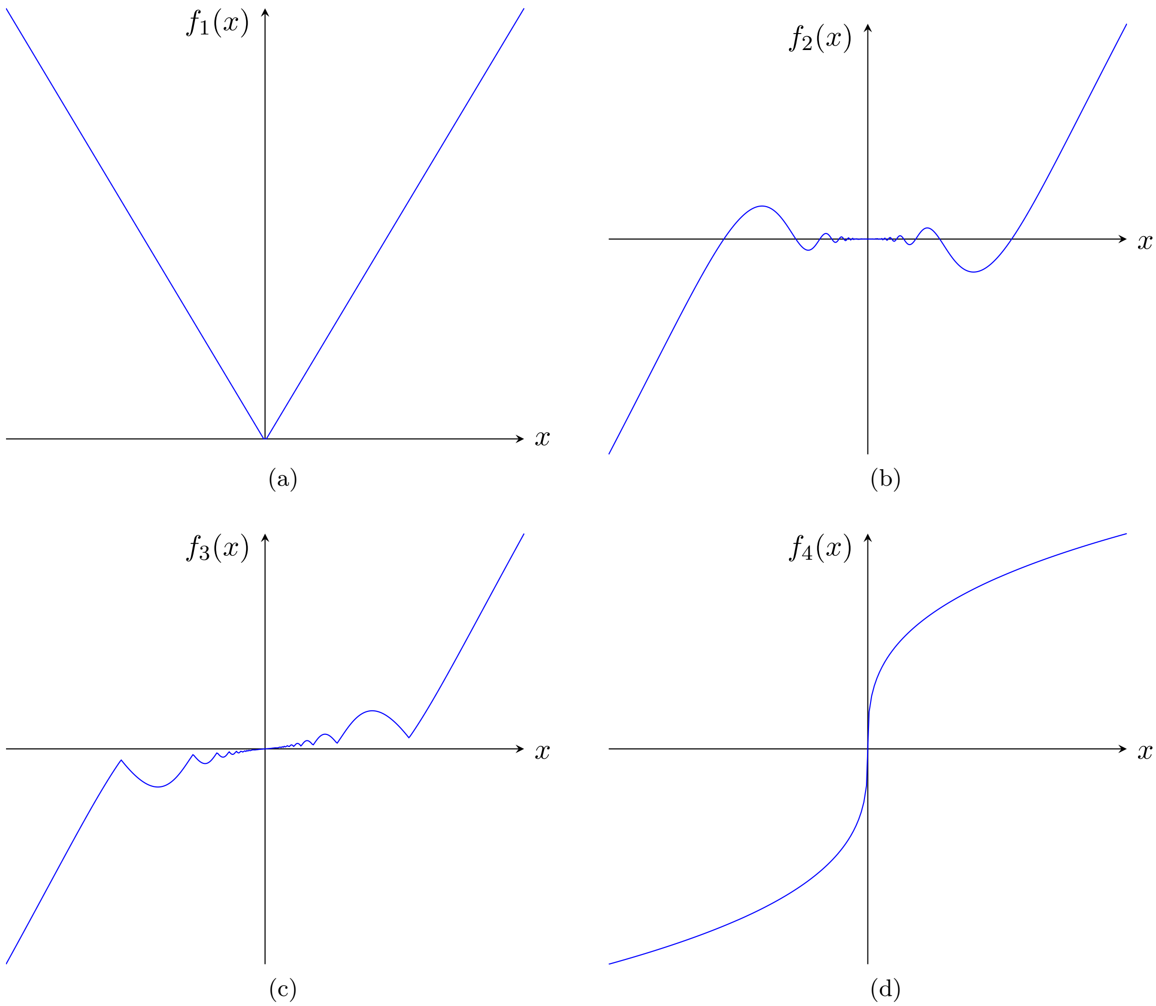

Figure 1.3: Graphs of functions from Example 1.2.5.

Chapter 2

Regularity criteria

In this chapter, we present well-known statements which guarantee regularity, subregularity, semiregularity, and their stronger versions. Also, we present Ioffe criterion for regularity of a mapping and its extensions for subregularity and semiregularity.

2.1 Historical background

We begin with Banach open mapping theorem, which is also known as Banach–Schauder theorem and guarantees regularity of a linear continuous mapping between Banach spaces.

Theorem 2.1.1 (Banach open mapping theorem) *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces, and $A \in \mathcal{L}(X, Y)$ be given. Then the following assertions are equivalent:*

(i) $\operatorname{sur} A > 0$;

(ii) $A(X) = Y$;

(iii) $0 \in \operatorname{int} A(\mathbb{B}_X)$;

(iv) *A is open at* 0;

(v) *the adjoint (dual) operator $A^* : Y^* \longrightarrow X^*$ is injective.*

Moreover, we have

$$\operatorname{sur} A = \operatorname{lopen} A = \sup\{c > 0 : A(\mathbb{B}_X) \supset c\mathbb{B}_Y\} = \inf\{\|A^* y^*\|_{X^*} : y^* \in \mathbb{S}_{Y^*}\}.$$

Note that the constant sur A is also known as *Banach constant* of a linear mapping A.

We emphasize that, in finite dimensions we identify the linear mappings with the corresponding representation matrix with respect to standard canonical bases. The following example shows how to compute Banach constant of a linear mapping in finite dimensional spaces.

Example 2.1.1 *Consider a matrix $A \in \mathbb{R}^{m \times n}$ with $m \leq n$. Then the mapping $\mathbb{R}^m \ni x \longmapsto Ax$ is regular if and only if the rows of A are linearly independent. Moreover,* $\operatorname{sur} A$ *equals to the smallest singular value σ_{min} of A, see Figure 2.1.*

In 1950, L.M. Graves [31] published a sufficient condition for semiregularity of a nonlinear mapping at the reference point, which generalizes Banach open mapping theorem.

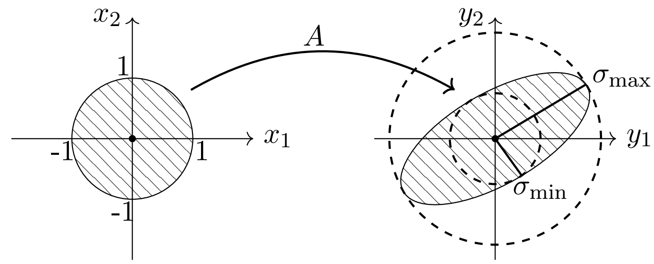

Figure 2.1: Image of a linear mapping, where $\sigma_{\min}$ and $\sigma_{\max}$ are the smallest and the largest singular values of the matrix A, respectively.

Theorem 2.1.2 (Graves theorem) *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $\bar{x} \in X$ be given. Consider a mapping $f : X \longrightarrow Y$ such that there is $A \in \mathcal{L}(X, Y)$ with* $\operatorname{sur} A > \operatorname{lip}(f - A)(\bar{x})$. *Then* $\operatorname{lopen} f(\bar{x}) \geq \operatorname{sur} A - \operatorname{lip}(f - A)(\bar{x}) > 0$.

Twenty two years later, B.H. Pourciau proved sufficient conditions for regularity of a nonlinear mapping, which is Lipschitz continuous in a neighborhood of the reference point, in finite dimensional spaces. For this, he used Clarke generalized Jacobian.

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be Lipschitz continuous in a neighborhood of a point $\bar{x} \in \mathbb{R}^n$. *Bouligand generalized Jacobian* of f at $\bar{x}$, denoted by $\partial_B f(\bar{x})$, consists of all matrices $A \in \mathbb{R}^{m\times n}$ for which there is a sequence (x_k) converging to $\bar{x}$ such that f is differentiable at x_k for each $k \in \mathbb{N}$ and $\nabla f(x_k) \to A$ as $k \to \infty$. *Clarke generalized Jacobian* of f at $\bar{x}$, denoted by $\partial_C f(\bar{x})$, is the convex hull of $\partial_B f(\bar{x})$, that is, $\partial_C f(\bar{x}) := \operatorname{co} \partial_B f(\bar{x})$. Clarke generalized Jacobian satisfies the following: for each $\ell > 0$ there is $\delta > 0$ such that for each $x, u \in \mathbb{B}_{\mathbb{R}^n}[\bar{x}, \delta]$ there is $A \in \partial_C f(\bar{x})$ such that

$$\|f(x) - f(u) - A(x - u)\|_{\mathbb{R}^m} \leq \ell \|x - u\|_{\mathbb{R}^n}.$$

Theorem 2.1.3 *Consider a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, with $m \leq n$, which is Lipschitz continuous on a neighborhood of the point $\bar{x} \in \mathbb{R}^n$. Assume that for each matrix $A \in \partial_C f(\bar{x})$ we have* $\operatorname{sur} A > 0$. *Then* $\operatorname{sur} f(\bar{x}) > 0$.

Another generalization of Banach open mapping theorem was proved by S.M. Robinson [55] and independently by C. Ursescu [66] for set-valued mappings with a closed convex graph. This statement follows, for example, from a constrained version of Banach open mapping theorem applied to the restriction of the canonical projection from $X \times Y$ onto Y to the graph of the mapping, that is, the assignment $\operatorname{gph} F \ni (x, y) \longmapsto y \in Y$.

Theorem 2.1.4 (Robinson–Ursescu theorem) *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $\bar{y} \in Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ having a closed convex graph. Then the following assertions are equivalent:*

(i) $\bar{y} \in \operatorname{int} \operatorname{rge} F$;

(ii) *for each $\bar{x} \in F^{-1}(\bar{y})$, the mapping F is open at $(\bar{x}, \bar{y})$;*

(iii) *for each $\bar{x} \in F^{-1}(\bar{y})$, we have* $\operatorname{sur} F(\bar{x}, \bar{y}) > 0$.

We say that a mapping $f : X \longrightarrow Y$ between Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is *Fréchet differentiable* at a point $\bar{x} \in X$ if there is $A \in \mathcal{L}(X, Y)$ such that $\operatorname{calm}(f - A)(\bar{x}) = 0$, that is, for each $\ell > 0$ there is $\delta > 0$ such that

$$\|f(x) - f(\bar{x}) - A(x - \bar{x})\|_Y \leq \ell \|x - \bar{x}\|_X \quad \text{for each} \quad x \in \mathbb{B}_X(\bar{x}, \delta).$$

Such a mapping A is called the *Fréchet derivative* of f at $\bar{x}$ and denoted by $f'(\bar{x})$. The mapping f is said to be *continuously (Fréchet) differentiable* at $\bar{x}$ if f is Fréchet differentiable on a neighborhood U of $\bar{x}$ in X and the mapping $U \ni x \longmapsto f'(x) \in \mathcal{L}(X,Y)$ is continuous at $\bar{x}$.

In 1970, S.M. Robinson [56] studied the solution stability of the so-called *generalized equation*, which is the problem to find $x \in X$ such that

$$f(x) + F(x) \ni 0,$$

with given mappings $f : X \longrightarrow Y$ and $F : X \rightrightarrows Y$. He proved a sufficient condition for strong regularity in case that f is continuously Fréchet differentiable and F is a *normal cone mapping* N_K associated with a closed convex subset K of X, that is the mapping

$$N_K(x) := \begin{cases} \{x^* \in X^* : \langle x^*, u - x\rangle \le 0 \text{ for each } u \in K\} & \text{for} \quad x \in K \\ \emptyset & \text{otherwise.} \end{cases}$$

More precisely, Robinson proved the implicit function theorem for generalized equations, where $f : P \times X \longrightarrow X^*$ with a parameter space P.

Theorem 2.1.5 (Robinson theorem) *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ and a single-valued mapping $f : X \longrightarrow Y$ which is continuously Fréchet differentiable at $\bar{x}$ and $\bar{y} \in f(\bar{x}) + F(\bar{x})$. If the mapping $f(\bar{x}) + f'(\bar{x})(\cdot - \bar{x}) + F$ is strongly regular around $(\bar{x}, \bar{y})$, then $f + F$ is strongly regular around $(\bar{x}, \bar{y})$.*

In 1996, A.L. Dontchev [22] proved a generalization of Theorem 2.1.2. We need one more definition, we say that a set-valued mapping $F : X \rightrightarrows Y$ has a *locally closed graph* around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if there is a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that the set $\operatorname{gph} F \cap \big(U \times V\big)$ is closed.

Theorem 2.1.6 *Let (X, d) be a complete metric space, (Y, ϱ) be a complete linear metric space with a shift-invariant metric, and a point $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$, with $\bar{y} \in F(\bar{x})$ and a locally closed graph around $(\bar{x}, \bar{y})$, and a single-valued mapping $f : X \longrightarrow Y$ such that $\operatorname{lip} f(\bar{x}) = 0$, that is, for each $\ell > 0$ there is $\delta > 0$ such that*

$$\varrho(f(x), f(u)) \le \ell\, d(x,u) \quad \textit{for each} \quad x, u \in I\!B_X(\bar{x}, \delta). \tag{2.1}$$

Then $\operatorname{sur} F(\bar{x}, \bar{y}) = \operatorname{sur}(f + F)(\bar{x}, f(\bar{x}) + \bar{y})$.

We say that a mapping $f : X \longrightarrow Y$ between Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is *strictly differentiable* at a point $\bar{x} \in X$ if there is $A \in \mathcal{L}(X,Y)$ such that $\operatorname{lip}(f - A)(\bar{x}) = 0$, that is, for each $\ell > 0$ there is $\delta > 0$ such that

$$\|f(x) - f(u) - A(x-u)\|_Y \le \ell \|x - u\|_X \quad \text{for each} \quad x, u \in I\!B_X(\bar{x}, \delta).$$

Such a mapping A is called the *strict derivative* of f at $\bar{x}$. Note that the existence of the strict derivative of f at $\bar{x}$ implies that f is Fréchet differentiable at $\bar{x}$ and Lipschitz continuous in a neighborhood of $\bar{x}$. Clearly, (2.1) means that f is strictly differentiable at $\bar{x}$ and the strict derivative is zero. The following example shows that a strictly differentiable mapping is regular around the reference point if and only if its strict derivative at this point is surjective.

Example 2.1.2 *Let $g : X \longrightarrow Y$ be a single-valued mapping between Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$. Suppose that g is strictly differentiable at $\bar{x} \in X$, then Theorem 2.1.6, with $F := g$ and $f := g(\bar{x}) - g + g'(\bar{x})(\cdot - \bar{x})$, implies that* $\operatorname{sur} g(\bar{x}) = \operatorname{sur}\big(g'(\bar{x})\big)$.

2.2 Ioffe-type criteria

In 1987, M. Fabian and D. Preiss [29, Corollary 1] proved a sufficient condition for semiregularity of both single-valued and set-valued mappings at the reference point via a generalization of Caristi principle. Thirteen years later, A.D. Ioffe [34, Theorem 1b] proved independently the statement in the same spirit containing a necessary and sufficient condition for regularity of a set-valued mapping via Ekeland variational principle.

Theorem 2.2.1 *Let (X, d) be a complete metric space, (Y, ϱ) be a metric space, and $\bar{x} \in X$ be given. Consider a continuous single-valued mapping $f : X \longrightarrow Y$ defined on whole X. Then* $\operatorname{sur} f(\bar{x})$ *equals to the supremum of all $c > 0$ for which there is $r > 0$ such that for any $x \in \mathbb{B}_X[\bar{x}, r]$ and any $y \in \mathbb{B}_Y[f(\bar{x}), r]$, with $f(x) \neq y$, there is $x' \in X$ satisfying*

$$c\, d(x, x') < \varrho(f(x), y) - \varrho(f(x'), y).$$

The statements in the spirit of the previous result will be called *Ioffe-type criteria* and imply set-valued versions, see [34, Proposition 3].

Theorem 2.2.2 *Let (X, d) and (Y, ϱ) be complete metric spaces and $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$ and a locally closed graph around $(\bar{x}, \bar{y})$. Then* $\operatorname{sur} F(\bar{x}, \bar{y})$ *equals to the supremum of all $c > 0$ for which there are $r > 0$ and $\alpha \in (0, 1/c)$ such that for any $x \in \mathbb{B}_X[\bar{x}, r]$, any $v \in \mathbb{B}_Y[\bar{y}, r] \cap F(x)$, and any $y \in \mathbb{B}_Y[\bar{y}, r]$, with $v \neq y$, there is a pair $(x', v') \in \operatorname{gph} F$ such that*

$$c \max\{d(x, x'), \alpha\varrho(v, v')\} < \varrho(v, y) - \varrho(v', y).$$

Applying these criteria we obtain short and easy to read proofs of various regularity statements, e.g., [57, Theorem 2.2.3], [57, Theorem 2.2.4], and [57, Proposition 2.2.1].

An analogy of the previous statement for subregularity of single-valued mappings follows and it is proved by the iterative process in [57, Theorem 2.3.1], which is a modification of the proof from [13].

Theorem 2.2.3 *Let (X, d) be a complete metric space, (Y, ϱ) be a metric space, and $\bar{x} \in X$ be given. Consider a continuous mapping $f : X \longrightarrow Y$ defined on whole X. Then* $\operatorname{popen} f(\bar{x})$ *equals to the supremum of $c > 0$ for which there is $r > 0$ such that for all $x \in \mathbb{B}_X[\bar{x}, r]$, with $f(x) \neq f(\bar{x})$, there is a point $x' \in X$ satisfying*

$$c\, d(x, x') < \varrho(f(x), f(\bar{x})) - \varrho(f(x'), f(\bar{x})).$$

A set-valued version immediately follows from it, see [57, Theorem 2.3.2].

Theorem 2.2.4 *Let (X, d) and (Y, ϱ) be complete metric spaces and $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$ and a locally closed graph around $(\bar{x}, \bar{y})$. Then* $\operatorname{popen} F(\bar{x}, \bar{y})$ *equals to the supremum of all $c > 0$ for which there are $r > 0$ and $\alpha \in (0, 1/c)$ such that for any $x \in \mathbb{B}_X[\bar{x}, r]$ and any $v \in \mathbb{B}_Y[\bar{y}, r] \cap F(x)$, with $v \neq \bar{y}$, there is a pair $(x', v') \in \operatorname{gph} F$ such that*

$$c \max\left\{d(x, x'), \alpha\varrho(v, v')\right\} < \varrho(v, \bar{y}) - \varrho(v', \bar{y}).$$

The sufficiency parts of the criteria for semiregularity of mapping was proved in [14, Proposition 4.1 (i)] via Ekeland variational principle.

Theorem 2.2.5 *Let (X,d) be a complete metric space, (Y,ϱ) be a metric space, and $\bar{x} \in X$ be given. Consider a continuous mapping $f : X \longrightarrow Y$ defined on whole X and a positive number c. Assume that there is $r > 0$ such that for any $x \in \mathbb{B}_X[\bar{x}, r]$ and any $y \in \mathbb{B}_Y[f(\bar{x}), r]$ such that*

$$0 < \varrho(f(x), y) \leq \varrho(f(\bar{x}), y) - c\, d(x, \bar{x})$$

there is $x' \in X$ satisfying

$$c\, d(x, x') < \varrho(f(x), y) - \varrho(f(x'), y).$$

Then $\operatorname{lopen} f(\bar{x}) \geq c$.

The corresponding set-valued version from [14, Proposition 4.2 (i)] follows again immediately.

Theorem 2.2.6 *Let (X,d) and (Y,ϱ) be complete metric spaces and $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$, with $\bar{y} \in F(\bar{x})$ and a locally closed graph around $(\bar{x}, \bar{y})$, and a positive number c. Suppose that there are $r > 0$ and $\alpha \in (0, 1/c)$ such that for any $x \in \mathbb{B}_X[\bar{x}, r]$, any $v \in \mathbb{B}_Y[\bar{y}, r] \cap F(x)$, and any $y \in \mathbb{B}_Y[\bar{y}, r]$ such that*

$$0 < \varrho(v, y) \leq \varrho(\bar{y}, y) - c \max\{d(x, \bar{x}), \alpha\varrho(v, \bar{y})\}$$

there is a pair $(x', v') \in \operatorname{gph} F$ such that

$$c \max\{d(x, x'), \alpha\varrho(v, v')\} < \varrho(v, y) - \varrho(v', y).$$

Then $\operatorname{lopen} F(\bar{x}, \bar{y}) \geq c$.

The necessity part of the criterion for single-valued mappings is formulated in [14, Proposition 4.1 (ii)].

Theorem 2.2.7 *Let (X,d) and (Y,ϱ) be metric spaces, and $\bar{x} \in X$ be given. Consider a continuous mapping $f : X \longrightarrow Y$ defined on whole X and a positive number c. Assume that* $\operatorname{lopen} f(\bar{x}) > 0$*, then for each positive c, with $c <$* $\operatorname{lopen} f(\bar{x})$*, there is $r > 0$ such that for any $x \in \mathbb{B}_X[\bar{x}, r]$ and any $y \in \mathbb{B}_Y[f(\bar{x}), r]$ satisfying*

$$0 < \varrho(f(\bar{x}), y) \leq \varrho(f(x), y) - cd(x, \bar{x})$$

there is a point $x' \in X$ such that

$$\varrho(f(x'), y) < \varrho(f(x), y) - cd(x, x').$$

The corresponding set-valued version from [14, Proposition 4.2 (ii)] follows again immediately.

Theorem 2.2.8 *Let (X,d) and (Y,ϱ) be metric spaces and $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$. Assume that* $\operatorname{lopen} F(\bar{x}, \bar{y}) > 0$*, then for each positive c, with $c <$* $\operatorname{lopen} F(\bar{x}, \bar{y})$*, there are $r > 0$ and $\alpha \in (0, 1/c)$ such that for any $x \in \mathbb{B}_X[\bar{x}, r]$, any $v \in \mathbb{B}_Y[\bar{y}, r] \cap F(x)$, and any $y \in \mathbb{B}_Y[\bar{y}, r]$ such that*

$$0 < \varrho(\bar{y}, y) \leq \varrho(v, y) - c \max\{d(x, \bar{x}), \alpha\varrho(v, \bar{y})\}$$

there is a pair $(x', v') \in \operatorname{gph} F$ such that

$$c \max\{d(x, x'), \alpha\varrho(v, v')\} < \varrho(v, y) - \varrho(v', y).$$

Chapter 3

Constrained semiregularity of single-valued mappings

In this chapter, we study a constrained version of semiregularity (constrained linear openness at the reference point) of a single-valued mapping in the finite-dimensional spaces, meaning that, there are positive c and r such that

$$f(\mathbb{B}_{\mathbb{R}^n}[\bar{x},t] \cap (\bar{x}+L)) \supset \mathbb{B}_{\mathbb{R}^m}[f(\bar{x}),ct] \cap (f(\bar{x})+M) \quad \text{for each} \quad t \in (0,r],$$

where a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, a point $\bar{x} \in \mathbb{R}^n$, nonempty sets $L \subset \mathbb{R}^n$ and $M \subset \mathbb{R}^m$ are given. Furthermore, we study constrained semiregularity of a single-valued mapping perturbed by a constant set-valued mapping in the spirit of [54]. To be specific, we replace the last displayed inclusion by

$$f(\mathbb{B}_{\mathbb{R}^n}[\bar{x},t] \cap (\bar{x}+L)) + D \cap \mathbb{B}_{\mathbb{R}^m} \supset \mathbb{B}_{\mathbb{R}^m}[f(\bar{x}),ct] \cap (f(\bar{x})+M) \quad \text{for each} \quad t \in (0,r],$$

where D is a given nonempty subset of $\mathbb{R}^m$.

In the first section, we study two approximation statements, which guarantee that a fixed set is contained in the range of a nonlinear (single-valued) mapping and by which we can easily prove the results in the following two sections. The second section contains criteria for semiregularity, which are in the spirit of Graves theorem, [14, Theorem 3.4] and [54, Theorem 1]. They rely on the existence of a linear approximation of the nonlinear mapping around the reference point, while criteria in the third section rely on the approximation by a bunch of linear mappings. The last section deals with moduli of constrained semiregularity of linear mappings.

3.1 Ranges of nonlinear mappings

We present the main tools for the following two sections. The first one is based on certain kind of approximation by one single-valued mapping.

Proposition 3.1.1 *Consider a nonempty compact convex set $\Omega \subset \mathbb{R}^n$, nonempty sets Γ, $\Xi \subset \mathbb{R}^m$, and mappings f, $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ which are both continuous on Ω. Assume that*

$$g(\Omega) \supset \Gamma + \Xi, \tag{3.1}$$

that for each $v \in \Gamma + \Xi$ the set $g^{-1}(v)$ is convex, and that

$$g(u) - f(u) \in \Xi \quad \textit{for each} \quad u \in \Omega. \tag{3.2}$$

Then $f(\Omega) \supset \Gamma$.

Proof. Pick an arbitrary $z \in \Gamma$. By (3.2), we have

$$z + g(u) - f(u) \in \Gamma + \Xi \quad \text{for each} \quad u \in \Omega. \tag{3.3}$$

We are showing that the assumptions of Kakutani fixed point theorem [40] (see Theorem A.3.1) hold for $\Psi : \Omega \rightrightarrows \Omega$ defined by

$$\Psi(u) := \{x \in \Omega : \, g(x) = z + g(u) - f(u)\} \quad \text{for} \quad u \in \Omega.$$

Fix an arbitrary $u \in \Omega$. Combining (3.1) and (3.3), we get that $\Psi(u) \neq \emptyset$. Pick arbitrary $\widetilde{x}, \widehat{x} \in \Psi(u)$ and $\lambda \in (0,1)$. Since both $\widetilde{x}$ and $\widehat{x}$ lie in the convex set $\Omega \cap g^{-1}(z + g(u) - f(u))$, so does $x := \lambda\widetilde{x} + (1-\lambda)\widehat{x}$. Thus $x \in \Psi(u)$.

To show that $\operatorname{gph}\Psi$ is closed in $\mathbb{R}^n \times \mathbb{R}^n$, pick any sequence $\big((u_k, x_k)\big)$ in $\operatorname{gph}\Psi$ converging to $(u,x) \in \mathbb{R}^n \times \mathbb{R}^n$. Then $u_k \in \Omega$ and $x_k \in \Psi(u_k) \subset \Omega$ for each $k \in \mathbb{N}$. Since Ω is closed, we have u, $x \in \Omega$. As $g(x_k) = z + g(u_k) - f(u_k)$ for each $k \in \mathbb{N}$, the continuity of f and g implies that

$$g(x) = \lim_{k\to\infty} g(x_k) = \lim_{k\to\infty} \big(z + g(u_k) - f(u_k)\big) = z + g(u) - f(u).$$

Therefore $x \in \Psi(u)$, that is, $(u,x) \in \operatorname{gph}\Psi$.

Kakutani fixed point theorem yields a point $x \in \Omega$ such that $x \in \Psi(x)$. Hence $g(x) = z + g(x) - f(x)$, which means that $z = f(x) \in f(\Omega)$. ■

A prominent example of the approximating mapping is a linear one.

Example 3.1.1 Let $A \in \mathbb{R}^{m\times n}$ be a matrix. The mapping $\mathbb{R}^n \ni x \longmapsto g(x) := Ax$ is continuous on $\mathbb{R}^n$ and the set $g^{-1}(y)$ is convex for each $y \in \mathbb{R}^m$.

Note that [15, Theorem 5] is an extension of the previous statement, when g is an affine mapping, to Fréchet spaces[1] (complete metrizable locally convex topological vector spaces, e.g. [38, p. 109] and [59, p. 49]).

Using Rådström cancellation rule [53] one can easily formulate a "converse" statement where the roles of mappings are interchanged.

Proposition 3.1.2 *Consider a matrix $A \in \mathbb{R}^{m\times n}$, a nonempty compact convex set $\Omega \subset \mathbb{R}^n$, and nonempty sets Γ, $\Xi \subset \mathbb{R}^m$ with Ξ being bounded. Let a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be such that*

$$f(\Omega) \supset \Gamma + \Xi \quad \textit{and} \quad f(u) - Au \in \Xi \quad \textit{for each} \quad u \in \Omega.$$

Then $A(\Omega) \supset \Gamma$.

Proof. Given an arbitrary $u \in \Omega$, we have $f(u) \in Au + \Xi$. Consequently,

$$\Gamma + \Xi \subset f(\Omega) \subset A(\Omega) + \Xi.$$

Since $A(\Omega)$ is a closed (even compact) convex set and Ξ is a nonempty bounded set, [53, Lemma 1] (cf. Lemma A.3.4) yields that $\Gamma \subset A(\Omega)$. ■

Now, we focus on the approximation of the (nonlinear) single-valued mapping by a bunch of linear mappings. Michael selection theorem [28, Theorem 7.53] (cf. Theorem A.3.5) and Kakutani fixed point theorem (cf. Theorem A.3.1) provide an easy and straightforward proof. For this, we need the notion of lower semicontinuity of a set-valued mapping.

Let Ω be a subset of $\mathbb{R}^n$. We say that a set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^{m\times n}$ is *lower semiconinuous on* Ω if for each $u \in \Omega$ and each open set $\mathcal{O}$ in $\mathbb{R}^{m\times n}$, with $\Phi(u) \cap \mathcal{O} \neq \emptyset$, there is a neighborhood U of u in Ω such that $\Phi(x) \cap \mathcal{O} \neq \emptyset$ for each $x \in U$.

[1] Do not confuse this with the space defined in Definition 4.1.5 (ii).

Proposition 3.1.3 *Consider a nonempty compact convex set $\Omega \subset \mathbb{R}^n$, a nonempty closed convex set $\mathcal{T} \subset \mathbb{R}^{m\times n}$, a nonempty open convex set $\Xi \subset \mathbb{R}^m$, a nonempty set $\Gamma \subset \mathbb{R}^m$, and a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ which is continuous on Ω. Assume that:*

(i) *for each $A \in \mathcal{T}$ we have $A(\Omega) \supset \Gamma + \overline{\Xi}$;*

(ii) *for each $u \in \Omega$ there is $A \in \mathcal{T}$ such that $Au - f(u) \in \Xi$.*

Then $f(\Omega) \supset \Gamma$.

Proof. We are going to show that the assumptions of Theorem A.3.5 hold for $\Phi : \Omega \rightrightarrows \mathcal{T}$ defined by

$$\Phi(u) := \big\{A \in \mathcal{T} : Au - f(u) \in \overline{\Xi}\big\} \quad \text{for} \quad u \in \Omega.$$

Fix any $u \in \Omega$. Then $\Phi(u) \neq \emptyset$ by (ii). Pick arbitrary $\widetilde{A}$, $\widehat{A} \in \Phi(u)$ and $\lambda \in (0,1)$. The sets $\mathcal{T}$ and $\overline{\Xi}$ are convex, therefore $A := (1-\lambda)\widetilde{A} + \lambda\widehat{A} \in \mathcal{T}$ and $Au - f(u) = (1-\lambda)\big(\widetilde{A}u - f(u)\big) + \lambda\big(\widehat{A}u - f(u)\big) \in (1-\lambda)\overline{\Xi} + \lambda\overline{\Xi} \subset \overline{\Xi}$. Thus $A \in \Phi(u)$. Pick any sequence (A_k) in $\Phi(u)$ converging to some $A \in \mathbb{R}^{m\times n}$. Since $\mathcal{T}$ and $\overline{\Xi}$ are closed sets we have $A \in \mathcal{T}$ and

$$Au - f(u) = \lim_{k\to\infty} \big(A_k u - f(u)\big) \in \overline{\Xi}.$$

Hence $A \in \Phi(u)$. Summarizing, the set $\Phi(u)$ is nonempty closed convex.

Now, we are showing that Φ is lower semicontinuous at u. Let $\mathcal{O}$ be an open set in $\mathbb{R}^{m\times n}$ such that $\Phi(u) \cap \mathcal{O} \neq \emptyset$. Fix any $\widehat{A} \in \Phi(u) \cap \mathcal{O}$. According to (ii) there is $\widetilde{A} \in \mathcal{T}$ such that $\widetilde{A}u - f(u) \in \Xi$. Thus $\widetilde{A} \in \Phi(u)$. As the open set $\mathcal{O}$ contains $\widehat{A}$ and $\Phi(u)$ is convex, there is $\lambda \in (0,1)$ such that $A := (1-\lambda)\widehat{A} + \lambda\widetilde{A} \in \Phi(u) \cap \mathcal{O}$. Since $\widehat{A}u - f(u) \in \overline{\Xi}$ and $\widetilde{A}u - f(u) \in \Xi$, the line segment principle (Theorem A.3.3) says that

$$\Xi \ni (1-\lambda)\big(\widehat{A}u - f(u)\big) + \lambda\big(\widetilde{A}u - f(u)\big) = Au - f(u).$$

As Ξ is open, the continuity of A and f yields a neighborhood U of u in Ω such that for each $x \in U$ we have $Ax - f(x) \in \Xi$; thus $A \in \Phi(x) \cap \mathcal{O}$.

Applying Theorem A.3.5, we find a continuous mapping $s : \Omega \longrightarrow \mathcal{T}$ such that $s(u) \in \Phi(u)$ for each $u \in \Omega$. Pick an arbitrary $z \in \Gamma$. Then

$$z + s(u)u - f(u) \in \Gamma + \overline{\Xi} \quad \text{for each} \quad u \in \Omega. \tag{3.4}$$

We are showing that the assumptions of Kakutani fixed point theorem hold for $\Psi : \Omega \rightrightarrows \Omega$ defined by

$$\Psi(u) := \{x \in \Omega : \; s(u)x = z + s(u)u - f(u)\} \quad \text{for} \quad u \in \Omega.$$

Fix an arbitrary $u \in \Omega$. Then $\Psi(u) \neq \emptyset$ by (3.4) and (i). Pick arbitrary $\widetilde{x}$, $\widehat{x} \in \Psi(u)$ and $\lambda \in (0,1)$. Both $\widetilde{x}$ and $\widehat{x}$ lie in the convex set Ω, hence so does $x := \lambda\widetilde{x} + (1-\lambda)\widehat{x}$. Then $s(u)x = \lambda s(u)\widetilde{x} + (1-\lambda)s(u)\widehat{x} = z + s(u)u - f(u)$. So $x \in \Psi(u)$.
To show that $\operatorname{gph}\Psi$ is closed, pick any sequence $\big((u_k, x_k)\big)$ in $\operatorname{gph}\Psi$ converging to $(u,x) \in \mathbb{R}^n \times \mathbb{R}^n$. Then $u_k \in \Omega$ and $x_k \in \Psi(u_k) \subset \Omega$ for each $k \in \mathbb{N}$. As Ω is closed, we have u, $x \in \Omega$. As $s(u_k)x_k = z + s(u_k)u_k - f(u_k)$ for each $k \in \mathbb{N}$, the continuity of f and s implies that

$$s(u)x = \lim_{k\to\infty} s(u_k)x_k = \lim_{k\to\infty} \big(z + s(u_k)u_k - f(u_k)\big) = z + s(u)u - f(u).$$

Therefore $x \in \Psi(u)$, that is, $(u,x) \in \operatorname{gph}\Psi$.

Kakutani fixed point theorem yields a point $x \in \Omega$ such that $x \in \Psi(x)$. Hence $s(x)x = z + s(x)x - f(x)$, which means that $z = f(x) \in f(\Omega)$. ■

Note that the approximation by a bunch of linear mappings is useful in the case, when we have not a "good" single-valued mapping at hand. For example, such a case occurs when we consider a nondifferentiable Lipschitz continuous function around the reference point. Then Clarke generalized Jacobian at the reference point can be considered as an approximation bunch of linear mappings. Let us apply the previous statement to get an easy proof of Theorem 2.1.3.

Proof of Theorem 2.1.3. Since $\mathcal{T} := \partial_C f(\bar{x})$ is a compact set and the mapping $A \longmapsto \operatorname{sur} A$ is continuous, there are $c > 0$ and $\ell > 0$ such that for each $A \in \mathcal{T}$ we have

$$A(\mathbb{B}_{\mathbb{R}^n}) \supset (c+\ell)\mathbb{B}_{\mathbb{R}^m}. \tag{3.5}$$

Further, there is $r > 0$ such that for each $x, u \in \mathbb{B}_{\mathbb{R}^n}[\bar{x}, 2r]$, with $x \neq u$, there is $A \in \mathcal{T}$ such that

$$\|f(x) - f(u) - A(x-u)\|_{\mathbb{R}^m} < \ell \|x - u\|_{\mathbb{R}^n}. \tag{3.6}$$

Fix any $t \in (0, r]$ and any $x \in \mathbb{B}_{\mathbb{R}^n}[\bar{x}, r]$. Let

$$\Omega := t\mathbb{B}_{\mathbb{R}^n}, \quad \Xi := \mathbb{B}_{\mathbb{R}^m}(0, \ell t), \quad \text{and} \quad \Gamma := ct\mathbb{B}_{\mathbb{R}^m}.$$

Define the mapping $\widetilde{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by $\widetilde{f}(u) := f(u + x) - f(x)$ for $u \in \Omega$.

Then, by (3.5), for each $A \in \mathcal{T}$, we have

$$A(t\mathbb{B}_{\mathbb{R}^n}) \supset (c+\ell)t\mathbb{B}_{\mathbb{R}^m} \supset ct\mathbb{B}_{\mathbb{R}^m} + \ell t\mathbb{B}_{\mathbb{R}^m} = \Gamma + \overline{\Xi}.$$

For each nonzero $u \in \Omega$, by (3.6), with $u := x$ and $x := x + u$, there is $A \in \mathcal{T}$ such that

$$Au - \widetilde{f}(u) = Au - f(u + x) + f(x) \in \mathbb{B}_{\mathbb{R}^m}(0, \ell t) = \Xi.$$

If $u = 0$, then the previous inclusion is trivial. Proposition 3.1.3, with $f := \widetilde{f}$, yields that $\widetilde{f}(\Omega) \supset \Gamma$. Hence, $f(\mathbb{B}_{\mathbb{R}^n}[x, t]) \supset f(x + \Omega) = \widetilde{f}(\Omega) + f(x) \supset \Gamma + f(x) = \mathbb{B}_{\mathbb{R}^m}[f(x), ct]$. ■

3.2 Semiregularity from single-matrix-approximations

We formulate criteria, based on a linear approximation, for constrained semiregularity of a (nonlinear) mapping defined on a (locally) convex and closed set.

Theorem 3.2.1 *Let r, ε, ℓ, and c be positive constants, $L \subset \mathbb{R}^n$ be such that the set $L \cap \varepsilon\mathbb{B}_{\mathbb{R}^n}$ is closed, convex, and contains the origin, and $M \subset \mathbb{R}^m$ be such that there is a cone $C \subset \mathbb{R}^m$ for which*

$$M \cap (c+\ell)\varepsilon\mathbb{B}_{\mathbb{R}^m} = C \cap (c+\ell)\varepsilon\mathbb{B}_{\mathbb{R}^m}. \tag{3.7}$$

Consider a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, which is continuous on $(\bar{x} + L) \cap \mathbb{B}_{\mathbb{R}^n}[\bar{x}, r]$, and assume that there is a matrix $A \in \mathbb{R}^{m \times n}$ such that

$$A(L \cap \varepsilon\mathbb{B}_{\mathbb{R}^n}) \supset M \cap (c+\ell)\varepsilon\mathbb{B}_{\mathbb{R}^m} \tag{3.8}$$

and

$$Au - f(\bar{x} + u) + f(\bar{x}) \in M \cap \ell\,\|u\|\mathbb{B}_{\mathbb{R}^m} \quad \text{for each} \quad u \in L \cap r\mathbb{B}_{\mathbb{R}^n}. \tag{3.9}$$

Then, for each $t \in \big(0, \min\{r, \varepsilon\}\big]$, we have

$$\begin{aligned} f\big((\bar{x} + L) \cap \mathbb{B}_{\mathbb{R}^n}[\bar{x}, t]\big) &\supset f(\bar{x}) + \tfrac{c}{c+\ell} A(L \cap t\mathbb{B}_{\mathbb{R}^n}) &(3.10)\\ &\supset \big(f(\bar{x}) + M\big) \cap \mathbb{B}_{\mathbb{R}^m}\big[f(\bar{x}), ct\big]. &(3.11) \end{aligned}$$

Proof. First, observe that, for each $\alpha \in (0,1]$ and each $\beta \in [0,(c+\ell)\varepsilon]$, we have

$$\alpha(M \cap \beta I\!B_{\mathbb{R}^m}) = M \cap \alpha\beta I\!B_{\mathbb{R}^m}. \tag{3.12}$$

Indeed, fix any such α and β. Since C is the cone, we have $\alpha(C \cap \beta I\!B_{\mathbb{R}^m}) = C \cap \alpha\beta I\!B_{\mathbb{R}^m}$. The choice of α and β along with (3.7) yields that

$$\begin{aligned}
\alpha(M \cap \beta I\!B_{\mathbb{R}^m}) &= \alpha(M \cap (c+\ell)\varepsilon I\!B_{\mathbb{R}^m} \cap \beta I\!B_{\mathbb{R}^m}) \\
&= \alpha(C \cap (c+\ell)\varepsilon I\!B_{\mathbb{R}^m} \cap \beta I\!B_{\mathbb{R}^m}) = \alpha(C \cap \beta I\!B_{\mathbb{R}^m}) \\
&= C \cap \alpha\beta I\!B_{\mathbb{R}^m} = C \cap (c+\ell)\varepsilon I\!B_{\mathbb{R}^m} \cap \alpha\beta I\!B_{\mathbb{R}^m} \\
&= M \cap (c+\ell)\varepsilon I\!B_{\mathbb{R}^m} \cap \alpha\beta I\!B_{\mathbb{R}^m} = M \cap \alpha\beta I\!B_{\mathbb{R}^m}.
\end{aligned}$$

Further, without any loss of generality assume that $f(\bar{x}) = 0$ and $\bar{x} = 0$. Fix an arbitrary $t \in \big(0, \min\{r,\varepsilon\}\big]$. Let

$$\Omega := L \cap t I\!B_{\mathbb{R}^n}, \quad \Gamma := \tfrac{c}{c+\ell} A(\Omega), \quad \text{and} \quad \Xi := M \cap \ell t I\!B_{\mathbb{R}^m}.$$

The convex set $L \cap \varepsilon I\!B_{\mathbb{R}^n}$ contains the origin and $t \le \varepsilon$, therefore

$$\begin{aligned}
\tfrac{t}{\varepsilon}(L \cap \varepsilon I\!B_{\mathbb{R}^n}) &= \tfrac{t}{\varepsilon}\big((L \cap \varepsilon I\!B_{\mathbb{R}^n}) \cap \varepsilon I\!B_{\mathbb{R}^n}\big) \subset \big(\tfrac{t}{\varepsilon}(L \cap \varepsilon I\!B_{\mathbb{R}^n})\big) \cap t I\!B_{\mathbb{R}^n} \\
&\subset (L \cap \varepsilon I\!B_{\mathbb{R}^n}) \cap t I\!B_{\mathbb{R}^n} = L \cap t I\!B_{\mathbb{R}^n} = \Omega.
\end{aligned}$$

Hence the positive homogeneity of A, (3.8), and (3.12), with $\alpha := t/\varepsilon$ and $\beta := (c+\ell)\varepsilon$, imply that

$$A(\Omega) \supset A\big(\tfrac{t}{\varepsilon}(L \cap \varepsilon I\!B_{\mathbb{R}^n})\big) \supset \tfrac{t}{\varepsilon}\big(M \cap (c+\ell)\varepsilon I\!B_{\mathbb{R}^m}\big) = M \cap (c+\ell) t I\!B_{\mathbb{R}^m}.$$

Using (3.12), with $\beta := (c+\ell)t$ and α equal to $c/(c+\ell)$ and $\ell/(c+\ell)$, respectively, we get that

$$\Gamma \supset M \cap ct I\!B_{\mathbb{R}^m} \quad \text{and} \quad \tfrac{\ell}{c+\ell} A(\Omega) \supset \Xi.$$

As $A(\Omega)$ is a convex set, we conclude that

$$A(\Omega) \supset \tfrac{c}{c+\ell} A(\Omega) + \tfrac{\ell}{c+\ell} A(\Omega) \supset \Gamma + \Xi.$$

Remembering that $t \le r$, we have $\Omega = L \cap t I\!B_{\mathbb{R}^n} \subset L \cap r I\!B_{\mathbb{R}^n}$. Thus (3.9) implies that

$$Au - f(u) \in M \cap \ell t I\!B_{\mathbb{R}^m} = \Xi \quad \text{for each} \quad u \in \Omega.$$

Proposition 3.1.1 yields that $f(\Omega) \supset \Gamma \supset M \cap ct I\!B_{\mathbb{R}^m}$. ■

In the case of cones, one can simplify the assumptions slightly.

Theorem 3.2.2 *Let r, ℓ, and c be positive constants, L be a closed convex cone in $\mathbb{R}^n$, and M be a cone in $\mathbb{R}^m$. Consider a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, which is continuous on $(\bar{x} + L) \cap I\!B_{\mathbb{R}^n}[\bar{x}, r]$, and assume that there is a matrix $A \in \mathbb{R}^{m \times n}$ such that*

$$A(L \cap I\!B_{\mathbb{R}^n}) \supset M \cap (c+\ell) I\!B_{\mathbb{R}^m} \tag{3.13}$$

and (3.9) *is satisfied. Then inclusions* (3.10)-(3.11) *hold true for each $t \in (0, r]$.*

Proof. Since both L and M are the cones, the positive homogeneity of A and (3.13) imply that

$$\begin{aligned} A(L \cap r\mathbb{B}_{\mathbb{R}^n}) &= A\big(r(L \cap \mathbb{B}_{\mathbb{R}^n})\big) = rA(L \cap \mathbb{B}_{\mathbb{R}^n}) \supset r\big(M \cap (c+\ell)\mathbb{B}_{\mathbb{R}^m}\big) \\ &= M \cap (c+\ell)r\mathbb{B}_{\mathbb{R}^m}. \end{aligned}$$

Therefore the assumptions of Theorem 3.2.1, with $\varepsilon := r$ and $C := M$, are satisfied. ■

The above statement contains quantitative and slightly more general versions of [14, Theorem 3.4], where $L := \mathbb{R}^n$ and $M := \mathbb{R}^m$, and of [18, Theorem 29], where L is a subspace of $\mathbb{R}^n$ and $M := A(L)$. The proof is elementary, that is, without any reference to the singular value decomposition, to the minimal time function, etc. Applying Theorem 3.2.1 we also get conditions for the semiregularity of the sum of nonlinear and linear mappings.

Theorem 3.2.3 *Let r, ε, ℓ, and c be positive constants, let $L \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^p$ be such that the sets $L \cap \varepsilon\mathbb{B}_{\mathbb{R}^n}$ and $D \cap \varepsilon\mathbb{B}_{\mathbb{R}^p}$ are closed, convex, and contain the origin, and let $M \subset \mathbb{R}^m$ be such that* (3.7) *holds for a cone $C \subset \mathbb{R}^m$. Consider a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, which is continuous on $(\bar{x}+L)\cap\mathbb{B}_{\mathbb{R}^n}[\bar{x}, r]$, and assume that there are matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times p}$ such that*

$$A(L \cap \varepsilon\mathbb{B}_{\mathbb{R}^n}) + B\big(D \cap \varepsilon\mathbb{B}_{\mathbb{R}^p}\big) \supset M \cap (c+\ell)\varepsilon\mathbb{B}_{\mathbb{R}^m} \tag{3.14}$$

and (3.9) *is satisfied. Then, for each $t \in \big(0, \min\{r, \varepsilon\}\big]$, we have*

$$f\big((\bar{x}+L) \cap \mathbb{B}_{\mathbb{R}^n}[\bar{x}, t]\big) + B(D \cap t\mathbb{B}_{\mathbb{R}^p}) \supset \big(f(\bar{x}) + M\big) \cap \mathbb{B}_{\mathbb{R}^m}\big[f(\bar{x}), ct\big].$$

Proof. Without any loss of generality, assume that $f(\bar{x}) = 0$ and $\bar{x} = 0$. Let $\widetilde{A} := (A, B) \in \mathbb{R}^{m\times(n+p)}$ and let

$$\widetilde{f}(u, v) := f(u) + Bv \quad \text{for} \quad u \in L \cap r\mathbb{B}_{\mathbb{R}^n} \quad \text{and} \quad v \in \mathbb{R}^p.$$

Given arbitrary $u \in L \cap r\mathbb{B}_{\mathbb{R}^n}$ and $v \in \mathbb{R}^p$, the inclusion (3.9) implies that

$$\widetilde{A}(u, v) - \widetilde{f}(u, v) = Au - f(u) \in M \cap \ell\,\|u\|\mathbb{B}_{\mathbb{R}^m}.$$

Clearly, (3.14) says that

$$\widetilde{A}\big((L \cap \varepsilon\mathbb{B}_{\mathbb{R}^n}) \times (D \cap \varepsilon\mathbb{B}_{\mathbb{R}^p})\big) \supset M \cap (c+\ell)\varepsilon\mathbb{B}_{\mathbb{R}^m}.$$

Theorem 3.2.1, with $A := \widetilde{A}$, $f := \widetilde{f}$, and $L := L \times D$, implies the conclusion. ■

We can easily get a statement in the spirit of Robinson [54, Theorem 1].

Corollary 3.2.1 *Let r, ℓ, $\varepsilon > 0$, and c be positive constants, let $L \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^p$ be such that the sets $L \cap \mathbb{B}_{\mathbb{R}^n}$ and $D \cap \mathbb{B}_{\mathbb{R}^p}$ are closed, convex, and contain the origin, and a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be continuous on $(\bar{x}+L) \cap \mathbb{B}_{\mathbb{R}^n}[\bar{x}, r]$. Assume that there is a matrix $A \in \mathbb{R}^{m\times n}$ such that*

$$A(L \cap \varepsilon\mathbb{B}_{\mathbb{R}^n}) + D \cap \varepsilon\mathbb{B}_{\mathbb{R}^m} \supset (c+\ell)\varepsilon\mathbb{B}_{\mathbb{R}^m}$$

and that (3.9)*, with $M := \mathbb{R}^m$, is satisfied. Then, for each $t \in \big(0, \min\{r, \varepsilon\}\big]$ we have*

$$f\big((\bar{x}+L) \cap \mathbb{B}_{\mathbb{R}^n}[\bar{x}, t]\big) + D \cap t\mathbb{B}_{\mathbb{R}^m} \supset \mathbb{B}_{\mathbb{R}^m}\big[f(\bar{x}), ct\big].$$

Proof. Apply Theorem 3.2.3 with $p := m$ and B being the identity. ■

3.3 Semiregularity from multiple-matrix-approximations

We formulate sufficient conditions for (constrained) semiregularity, which are based on approximation of a (nonlinear) mapping by a bunch of linear mappings.

Proposition 3.3.1 *Let r, ε, ℓ, and c be positive constants, and $L \subset \mathbb{R}^n$ be such that the set $L \cap \varepsilon \mathbb{B}_{\mathbb{R}^n}$ is closed, convex, and contains the origin. Consider a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, which is continuous on $(\bar{x} + L) \cap \mathbb{B}_{\mathbb{R}^n}[\bar{x}, r]$ and a closed convex set $\mathcal{T} \subset \mathbb{R}^{m \times n}$. Assume that*

(i) *for each matrix $A \in \mathcal{T}$ we have $A(L \cap \varepsilon \mathbb{B}_{\mathbb{R}^n}) \supset (c + \ell)\varepsilon \mathbb{B}_{\mathbb{R}^m}$;*

(ii) *for each nonzero $u \in L \cap r\mathbb{B}_{\mathbb{R}^n}$ there is a matrix $A \in \mathcal{T}$ such that*

$$Au - f(\bar{x} + u) + f(\bar{x}) \in \mathbb{B}_{\mathbb{R}^m}(0, \ell \, \|u\|).$$

Put $r' = \min\{r, \varepsilon\}$. Then, for each $t \in (0, r']$, we have

$$f\big((\bar{x} + L) \cap \mathbb{B}_{\mathbb{R}^n}[\bar{x}, t]\big) \supset \mathbb{B}_{\mathbb{R}^m}\big[f(\bar{x}), ct\big]. \tag{3.15}$$

Equivalently,

$$\operatorname{dist}(\bar{x}, f^{-1}(y) \cap (\bar{x} + L)) \le \tfrac{1}{c}\|y - f(\bar{x})\| \quad \text{for each} \quad y \in \mathbb{B}_{\mathbb{R}^m}[f(\bar{x}), cr'].$$

If, in addition, each matrix $A \in \mathcal{T}$ is nonsingular, then for $\bar{y} := f(\bar{x})$, we have

$$f^{-1}(\bar{y}) \cap \mathbb{B}_{\mathbb{R}^n}[\bar{x}, r'] \cap (\bar{x} + L) = \{\bar{x}\}. \tag{3.16}$$

Proof. Without any loss of generality assume that $f(\bar{x}) = 0$ and $\bar{x} = 0$. Fix an arbitrary $t \in \big(0, r'\big]$. Let

$$\Omega := L \cap t\mathbb{B}_{\mathbb{R}^n}, \quad \Gamma := ct\mathbb{B}_{\mathbb{R}^m}, \quad \text{and} \quad \Xi := \mathbb{B}_{\mathbb{R}^m}(0, \ell t).$$

For each $A \in \mathcal{T}$, similarly as in the proof of Theorem 3.2.1, we get that

$$A(\Omega) \supset \Gamma + \overline{\Xi}.$$

Remembering that $t \le r$, (ii) implies that for each nonzero $u \in \Omega$ there is $A \in \mathcal{T}$ such that

$$Au - f(u) \in \mathbb{B}_{\mathbb{R}^m}(0, \ell t) = \Xi.$$

Clearly, the last inclusion holds if $u = 0$. Proposition 3.1.3 yields that $f(\Omega) \supset \Gamma$.

Further, fix any nonzero $y \in cr'\mathbb{B}_{\mathbb{R}^m}$. Let $t := \frac{\|y\|}{c}$, then $0 < t \le r'$. By (3.15), there is $x \in f^{-1}(y) \cap L$ such that

$$\tfrac{1}{c}\|y\| = t \ge \|x\| \ge \operatorname{dist}(0, f^{-1}(y) \cap L).$$

Clearly, when $y = 0$ the previous inequality holds.

To show (3.16), suppose on the contrary that there is $x \in f^{-1}(0) \cap r'\mathbb{B}_{\mathbb{R}^n} \cap L$ such that $x \neq 0$. Then $f(x) = 0$ and $x \in L \cap r'\mathbb{B}_{\mathbb{R}^n}$. By (ii), with $u := x$, there is $A \in \mathcal{T}$ such that

$$0 < \|x\| \le \|A^{-1}\| \|Ax - f(x)\| < \ell \|A^{-1}\| \|x\| \le \tfrac{\ell}{c+\ell}\|x\| < \|x\|,$$

a contradiction. Note that the penultimate inequality holds, by (i), since for each $A \in \mathcal{T}$ we have

$$A(\varepsilon\mathbb{B}_{\mathbb{R}^n}) \supset A(L \cap \varepsilon\mathbb{B}_{\mathbb{R}^n}) \supset (c + \ell)\varepsilon\mathbb{B}_{\mathbb{R}^m}.$$

■

From the previous result, we are able to prove an analogy of Corollary 3.2.1.

Corollary 3.3.1 *Let r, ε, ℓ, and c be positive constants, and let $L \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^p$ be such that the sets $L \cap \varepsilon I\!B_{\mathbb{R}^n}$ and $D \cap \varepsilon I\!B_{\mathbb{R}^p}$ are closed, convex, and contain the origin. Consider a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, which is continuous on $(\bar{x} + L) \cap I\!B_{\mathbb{R}^n}[\bar{x}, r]$, and a closed convex set $\mathcal{T} \subset \mathbb{R}^{m\times n}$. Assume that*

(i) *for each matrix $A \in \mathcal{T}$ we have $A(L \cap \varepsilon I\!B_{\mathbb{R}^n}) + D \cap \varepsilon I\!B_{\mathbb{R}^m} \supset (c+\ell)\varepsilon I\!B_{\mathbb{R}^m}$;*

(ii) *for each nonzero $u \in L \cap r I\!B_{\mathbb{R}^n}$ there is $A \in \mathcal{T}$ such that*

$$Au - f(\bar{x} + u) + f(\bar{x}) \in I\!B_{\mathbb{R}^m}(0, \ell\, \|u\|).$$

Then, for each $t \in (0, \min\{r, \varepsilon\}]$, we have

$$f\big((\bar{x} + L) \cap I\!B_{\mathbb{R}^n}[\bar{x}, t]\big) + D \cap t I\!B_{\mathbb{R}^m} \supset I\!B_{\mathbb{R}^m}\big[f(\bar{x}), ct\big].$$

Proof. Without any loss of generality assume that $f(\bar{x}) = 0$ and $\bar{x} = 0$. Let $\widetilde{\mathcal{T}} := \{(A\,,\, I) \in \mathbb{R}^{m\times(n+m)} : A \in \mathcal{T}\}$ and let

$$\widetilde{f}(u, v) := f(u) + v \quad \text{for} \quad u \in L \cap r I\!B_{\mathbb{R}^n} \quad \text{and} \quad v \in \mathbb{R}^m.$$

For each nonzero $u \in L \cap r I\!B_{\mathbb{R}^n}$ and $v \in \mathbb{R}^m$, (ii) implies that there is $\widetilde{A} \in \widetilde{\mathcal{T}}$ such that

$$\widetilde{A}(u, v) - \widetilde{f}(u, v) = Au - f(u) \in I\!B_{\mathbb{R}^m}(0, \ell\|u\|).$$

If $u = 0$, then the previous inclusion holds for each $v \in \mathbb{R}^n$. Clearly, (i) says that

$$\widetilde{A}\big((L \cap \varepsilon I\!B_{\mathbb{R}^n}) \times (D \cap \varepsilon I\!B_{\mathbb{R}^m})\big) \supset (c+\ell)\varepsilon I\!B_{\mathbb{R}^m}.$$

Proposition 3.3.1, with $A := \widetilde{A}$, $f := \widetilde{f}$, $\mathcal{T} := \widetilde{\mathcal{T}}$, and $L := L \times D$, implies the conclusion. ■

One can ask how we can satisfy the inclusion in Proposition 3.3.1 (i) and Corollary 3.3.1 (i), which is uniform with respect to the elements of $\mathcal{T}$. We need the notion of a recession cone.

For a set $\mathcal{T} \subset \mathbb{R}^{m\times n}$ the *recession cone* of $\mathcal{T}$, denoted by $\mathcal{T}_\infty$, is defined by

$$\mathcal{T}_\infty := \{A \in \mathbb{R}^{m\times n} : A = \lim_{k\to\infty} t_k A_k \text{ for some } (A_k) \text{ in } \mathcal{T} \text{ and } (t_k) \text{ in } (0,\infty) \text{ with } t_k \downarrow 0 \text{ as } k \to \infty\}.$$

If $\mathcal{T}$ is bounded, then $\mathcal{T}_\infty = \{0\}$. The following lemma is a slight generalization of [39, Lemma 3.1.1].

Proposition 3.3.2 *Let $\mathcal{T} \subset \mathbb{R}^{m\times n}$, with $m \le n$, be a closed set and $L \subset \mathbb{R}^n$ be a closed convex set. Suppose that for each $A \in \mathcal{T} \cup \big(\mathcal{T}_\infty \setminus \{0\}\big)$ we have $0 \in \operatorname{int} A(L)$. Then there is $c > 0$ such that for each $A \in \mathcal{T}$ we have*

$$A(L) \supset c I\!B_{\mathbb{R}^m}.$$

Proof. On the contrary, suppose that there are sequences (A_k) in $\mathcal{T}$ and (v_k) in $\mathbb{R}^m$ such that

$$v_k \notin A_k(L) \quad \text{and} \quad v_k \in \tfrac{1}{k} I\!B_{\mathbb{R}^m} \quad \text{for each} \quad k \in \mathbb{N}.$$

Obviously, (v_k) converges to the origin. Fix any $k \in \mathbb{N}$. Note that $A_k(L)$ is a convex set, then, by the separation theorem (see Theorem A.3.2 with $X := v_k$ and $Y := A_k(L)$), there is $\xi_k \in \mathbb{R}^m$ such that $\|\xi_k\| = 1$ and

$$\langle \xi_k, v_k\rangle \le \langle \xi_k, A_k x\rangle \quad \text{for each} \quad x \in L.$$

Since the sequence (ξ_k) is bounded, we can assume that it converges to some ξ in $\mathbb{R}^m$ with $\|\xi\| = 1$. If the sequence (A_k) is bounded, we can assume that it converges to some $A \in \mathcal{T}$. Fix any $x \in L$, then

$$0 = \langle \xi, 0 \rangle = \lim_{k\to\infty} \langle \xi_k, v_k \rangle \leq \lim_{k\to\infty} \langle \xi_k, A_k x \rangle = \langle \xi, Ax \rangle ,$$

a contradiction because $0 \in \operatorname{int} A(L)$.

If (A_k) is unbounded, we can assume that $\lim_{k\to\infty} \|A_k\| = \infty$ and that the sequence $\left(\frac{A_k}{\|A_k\|}\right)$ converges to some $A \in \mathcal{T}_\infty$. Therefore, for each $x \in L$ we have

$$0 = \lim_{k\to\infty} \tfrac{1}{\|A_k\|} \langle \xi_k, v_k \rangle \leq \lim_{k\to\infty} \left\langle \xi_k, \tfrac{A_k}{\|A_k\|} x \right\rangle = \langle \xi, Ax \rangle .$$

This is a contradiction because $0 \in \operatorname{int} A(L)$. ■

Corollary 3.3.2 *Let $\mathcal{T} \subset \mathbb{R}^{m\times n}$, with $m \leq n$, be a closed set and let $L \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^m$ be closed convex sets and contain the origin. Suppose that for each $A \in \mathcal{T} \cup \big(\mathcal{T}_\infty \setminus \{0\}\big)$ we have $0 \in \operatorname{int} \big(A(L) + D\big)$. Then there is $c > 0$ such that for each $A \in \mathcal{T}$ we have*

$$A(L) + D \supset c\mathbb{B}_{\mathbb{R}^m}.$$

Proof. Define $\widetilde{\mathcal{T}} := \{(A\,,\, I) \in \mathbb{R}^{m\times(n+m)} : A \in \mathcal{T}\}$. Thus $\widetilde{\mathcal{T}}_\infty = \mathcal{T}_\infty \times \{0\}$ by [39, Lemma 1.5.1(viii)]. For each $\widetilde{A} \in \widetilde{\mathcal{T}} \cup \big(\widetilde{\mathcal{T}}_\infty \setminus \{0\}\big)$ we have $0 \in \operatorname{int} \widetilde{A}(L \times D)$. Then, by Proposition 3.3.2, with $\mathcal{T} := \widetilde{\mathcal{T}}$ and $L := L \times D$, there is $c > 0$ such that for each $\widetilde{A} \in \widetilde{\mathcal{T}}$ we have

$$\widetilde{A}(L \times D) = A(L) + D \supset c\mathbb{B}_{\mathbb{R}^m},$$

where $\widetilde{A} = \big(A\,,\, I\big)$ and $A \in \mathcal{T}$. ■

We conclude this section by a modification of [39, Proposition 3.1.6], where a bunch of matrices is an image of a set-valued mapping at the reference point. For this statement, we need the notion of upper semicontinuity of a set-valued mapping. We say that a set-valued mapping $\mathcal{H} : \mathbb{R}^n \rightrightarrows \mathbb{R}^{m\times n}$ is *upper semicontinuous at* $\bar{x} \in \mathbb{R}^n$ if for each open set $\mathcal{O} \subset \mathbb{R}^{m\times n}$, with $\mathcal{H}(\bar{x}) \subset \mathcal{O}$, there is $\delta > 0$ such that

$$\mathcal{H}(\mathbb{B}_{\mathbb{R}^n}[\bar{x}, \delta]) \subset \mathcal{O}.$$

Proposition 3.3.3 *Let $\bar{x} \in \mathbb{R}^n$ be given and let $L \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^m$ be closed convex sets containing the origin. Suppose that $\mathcal{H} : \mathbb{R}^n \rightrightarrows \mathbb{R}^{m\times n}$ is upper semicontinuous at $\bar{x}$ and there are $c > 0$, $\ell > 0$, and $\varepsilon > 0$ such that for each $A \in \overline{\operatorname{co}}\, \mathcal{H}(\bar{x})$ we have*

$$A(L \cap \varepsilon\mathbb{B}_{\mathbb{R}^n}) + D \cap \varepsilon\mathbb{B}_{\mathbb{R}^m} \supset (c + \ell)\varepsilon\mathbb{B}_{\mathbb{R}^m}.$$

Then there is $\delta > 0$ such that for each $A \in \overline{\operatorname{co}}\, \mathcal{H}(\mathbb{B}_{\mathbb{R}^n}[\bar{x}, \delta])$ we have

$$A(L \cap \varepsilon\mathbb{B}_{\mathbb{R}^n}) + D \cap \varepsilon\mathbb{B}_{\mathbb{R}^m} \supset c\varepsilon\mathbb{B}_{\mathbb{R}^m}. \tag{3.17}$$

Proof. Fix any $A \in \overline{\operatorname{co}}\, \mathcal{H}(\bar{x}) + \ell\mathbb{B}_{\mathbb{R}^{m\times n}}$. Then there are $\widetilde{A} \in \overline{\operatorname{co}}\, \mathcal{H}(\bar{x})$ and $\widehat{A} \in \ell\mathbb{B}_{\mathbb{R}^{m\times n}}$ such that $A = \widetilde{A} + \widehat{A}$. Thus for each $u \in \varepsilon\mathbb{B}_{\mathbb{R}^n}$ we have

$$Au - \widetilde{A}u = \widehat{A}u \in \ell\|u\|\mathbb{B}_{\mathbb{R}^m}.$$

Then, by Corollary 3.2.1, with $f := A$, $A := \widetilde{A}$, $r := \varepsilon$, and $\bar{x} := 0$, we get

$$A(L \cap \varepsilon\mathbb{B}_{\mathbb{R}^n}) + D \cap \varepsilon\mathbb{B}_{\mathbb{R}^m} \supset c\varepsilon\mathbb{B}_{\mathbb{R}^m}.$$

Further, since $\mathcal{H}$ is upper semicontinuous at $\bar{x}$, there is $\delta > 0$ such that

$$\mathcal{H}(I\!B_{\mathbb{R}^n}[\bar{x},\delta]) \subset \mathcal{H}(\bar{x}) + \ell I\!B_{\mathbb{R}^{m\times n}} \subset \overline{\text{co}}\,\mathcal{H}(\bar{x}) + \ell I\!B_{\mathbb{R}^{m\times n}}.$$

The set on the right hand side is convex and closed because it is the sum of a closed convex set and a compact convex set (see Lemma A.3.2). Hence

$$\overline{\text{co}}\,\mathcal{H}(I\!B_{\mathbb{R}^n}[\bar{x},\delta]) \quad \subset \quad \overline{\text{co}}\,\mathcal{H}(\bar{x}) + \ell I\!B_{\mathbb{R}^{m\times n}}.$$

■

At the end of the day, we achieve the following consequence. We need the notion of the *line segment* $[x,u] := \{x + \lambda(u-x) : \lambda \in [0,1]\}$ for $x, u \in \mathbb{R}^n$ and for $\mathcal{T} \subset \mathbb{R}^{m\times n}$ we define $\mathcal{T}x := \{Ax : \quad A \in \mathcal{T}\}$ for $x \in \mathbb{R}^n$.

Corollary 3.3.3 *Let $r > 0$ and $\varepsilon > 0$ be given and let $L \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^m$ be such that the sets $L \cap \varepsilon I\!B_{\mathbb{R}^n}$ and $D \cap \varepsilon I\!B_{\mathbb{R}^m}$ are closed, convex, and contain the origin. Consider a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, which is continuous on $(\bar{x}+L) \cap I\!B_{\mathbb{R}^n}[\bar{x},r]$, and a set-valued mapping $\mathcal{H} : \mathbb{R}^n \rightrightarrows \mathbb{R}^{m\times n}$, which is defined on $(\bar{x}+L) \cap I\!B_{\mathbb{R}^n}[\bar{x},r]$ and upper semicontinuous at $\bar{x}$. Assume that*

(i) *for each matrix $A \in \overline{\text{co}}(\mathcal{H}(\bar{x})) \cup \big(\overline{\text{co}}(\mathcal{H}(\bar{x}))_\infty \setminus \{0\}\big)$ we have $0 \in \text{int}\,\big(A(L \cap \varepsilon I\!B_{\mathbb{R}^n}) + D \cap \varepsilon I\!B_{\mathbb{R}^m}\big)$;*

(ii) *for each $u \in L \cap r I\!B_{\mathbb{R}^n}$, we have*

$$f(\bar{x}+u) - f(\bar{x}) \in \overline{\text{co}}\big(\mathcal{H}([\bar{x},\bar{x}+u])u\big).$$

Then there are $\delta \in (0, \min\{r,\varepsilon\}]$ and $c > 0$ such that, for each $t \in (0,\delta]$, we have

$$f\big((\bar{x}+L) \cap I\!B_{\mathbb{R}^n}[\bar{x},t]\big) + D \cap t I\!B_{\mathbb{R}^m} \supset I\!B_{\mathbb{R}^m}\big[f(\bar{x}), ct\big].$$

Proof. Without any loss of generality assume that $f(\bar{x}) = 0$ and $\bar{x} = 0$. By (i), Corollary 3.3.2, with $L := L \cap \varepsilon I\!B_{\mathbb{R}^n}$, $D := D \cap \varepsilon I\!B_{\mathbb{R}^m}$, and $\mathcal{T} := \overline{\text{co}}\,\mathcal{H}(0)$, implies that there are $c > 0$ and $\ell > 0$ such that for each $A \in \overline{\text{co}}\,\mathcal{H}(0)$ we have

$$A(L \cap \varepsilon I\!B_{\mathbb{R}^n}) + D \cap \varepsilon I\!B_{\mathbb{R}^m} \supset (c+2\ell)\varepsilon I\!B_{\mathbb{R}^m}.$$

Therefore assumptions of Proposition 3.3.3, with $c := c+\ell$, are satisfied; therefore there is $\delta \in (0, \min\{r,\varepsilon\}]$ such that for each $A \in \overline{\text{co}}\,\mathcal{H}(\delta I\!B_{\mathbb{R}^n})$ the inclusion (3.17), with $c := c+\ell$, holds Clearly, $\overline{\text{co}}\,\mathcal{H}(\delta I\!B_{\mathbb{R}^n}) \supset \overline{\text{co}}\,\mathcal{H}(L \cap \delta I\!B_{\mathbb{R}^n})$.

Further, fix any nonzero $u \in L \cap \delta I\!B_{\mathbb{R}^n}$, then, by (ii), we have

$$f(u) \in \overline{\text{co}}\big(\mathcal{H}([0,u])(u)\big) \subset \text{co}\big(\mathcal{H}([0,u])\big)u + I\!B_{\mathbb{R}^m}(0,\ell\|u\|).$$

Hence there is $A \in \text{co}\,\mathcal{H}([0,u]) \subset \overline{\text{co}}\,\mathcal{H}(L \cap \delta I\!B_{\mathbb{R}^n})$ such that

$$Au - f(u) \in I\!B_{\mathbb{R}^m}(0,\ell\|u\|).$$

Corollary 3.3.1, with $\mathcal{T} := \overline{\text{co}}\,\mathcal{H}(L \cap r I\!B_{\mathbb{R}^n})$ and $r := \delta$, implies the conclusion. ■

3.4 Directional (semi)regularity of linear mappings

In this section, we focus on the modulus of constrained (semi)regularity of a linear mapping, where the constraints are given subspaces or cones. As it was mentioned in Example 2.1.1, when the matrix $A \in \mathbb{R}^{m\times n}$, with $m \le n$, has a full rank, then we have

$$A(\mathbb{B}_{\mathbb{R}^n}) \supset \mathbb{B}_{\mathbb{R}^m}[0,c],$$

where c is the smallest singular value of the matrix A and that value is the modulus of regularity of A. Hence, the singular value decomposition (SVD) is proper manner, how to compute the modulus of regularity for a linear mapping. The question is how to find the modulus of constrained (semi)regularity, that is, we want to find a positive c, as large as possible, such that

$$A(\mathbb{B}_{\mathbb{R}^n} \cap L) \supset \mathbb{B}_{\mathbb{R}^m}[0,c] \cap M,$$

where the sets $L \subset \mathbb{R}^n$ and $M \subset \mathbb{R}^m$ are the given constraints.

At first, we show that SVD is useful in the case of regularity with subspace constraints, that is, when the sets L and M are subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. See Section A.1 in Appendix for the basics of the singular value decomposition and Moore–Penrose inverse.

We need the notion of the orthogonal projection onto subspace. Let M be a non-trivial subspace of $\mathbb{R}^n$. By the term non-trivial, we mean that a subspace or a cone does not contain only the origin. We say that the linear mapping $\mathbb{R}^n \ni x \longmapsto P(x) \in \mathbb{R}^n$ is the *orthogonal projection onto* M if $\operatorname{rge} P = M$, $Px = x$ for each $x \in M$, $P^2 = P$, and $P = P^T$. Moreover, $\|P\| = 1$. Note that we use a notation $\|\cdot\|$ for the spectral norm for a matrix.

In the following three statements, we use the fact that a linear mapping given by a matrix A is a bijection between $\operatorname{rge} A^T$ and $\operatorname{rge} A$, see Remark A.1.2.

Lemma 3.4.1 *Let $A \in \mathbb{R}^{m\times n}$ be given. If* $\operatorname{rge} A$ *is a non-trivial subspace, then*

$$A(\mathbb{B}_{\mathbb{R}^n} \cap L) \supset \mathbb{B}_{\mathbb{R}^m}[0,c] \cap M, \tag{3.18}$$

where $L := \operatorname{rge} A^T$, $M := \operatorname{rge} A$, and c is the smallest singular value of A.

Proof. Lemma A.1.1, implies that there are numbers $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_j > 0$, orthonormal vectors $v_1, v_2, \ldots, v_j$ in $\mathbb{R}^n$, and orthonormal vectors $u_1, u_2, \ldots, u_j$ in $\mathbb{R}^m$ such that

$$A^T A v_i = \sigma_i^2 v_i \quad \text{and} \quad A v_i = \sigma_i u_i \quad \text{for each} \quad i = 1, 2, \ldots, j, \tag{3.19}$$

where $j = \dim \operatorname{rge} A$. Since $\operatorname{rge} A$ is non-trivial, we have $j > 0$. We have $L = \operatorname{span}\{v_1, v_2, \ldots, v_j\}$ and $M = \operatorname{span}\{u_1, u_2, \ldots, u_j\}$. By Remark A.1.2, A is a bijection between L and M.

Further, fix any $y \in \mathbb{B}_{\mathbb{R}^m}[0,\sigma_j] \cap M$. Then there is $x \in L$ such that $Ax = y$. Find numbers $a_1, a_2, \ldots, a_j$ such that $x = a_1 v_1 + a_2 v_2 + \cdots + a_j v_j$. Then, by (3.19), we get that

$$y = Ax = \sum_{i=1}^{j} A(a_i v_i) = \sum_{i=1}^{j} a_i A v_i = \sum_{i=1}^{j} a_i \sigma_i u_i.$$

This implies that $(a_1\sigma_1)^2 + (a_2\sigma_2)^2 + \cdots + (a_j\sigma_j)^2 = \|y\|^2 \le \sigma_j^2$. Therefore $a_1^2 + a_2^2 + \cdots + a_j^2 \le 1$; hence $x \in \mathbb{B}_{\mathbb{R}^n} \cap L$. Letting $c := \sigma_j$, we conclude that (3.18) holds. ■

Now, we consider a given subspace as the constraint in the domain space of a linear mapping.

Proposition 3.4.1 *Let L be a subspace of $\mathbb{R}^n$ and $P \in \mathbb{R}^{n\times n}$ be the orthogonal projection onto L. Consider a linear mapping $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and the set $M := A(L)$. If M is a non-trivial subspace, then*

$$A(\mathbb{B}_{\mathbb{R}^n} \cap L) = A(\mathbb{B}_{\mathbb{R}^n} \cap \widetilde{L}) \supset \mathbb{B}_{\mathbb{R}^m}[0, c] \cap M,$$

where $\widetilde{L} := (AP)^\dagger(M) \subset L$ and c is the smallest singular value of the matrix AP.

Proof. Clearly, $\text{rge}(AP) = A(L) = M$ and $\text{rge}(AP)^T = \text{rge}(PA^T) = P(\text{rge } A^T)$. Then $\text{rge}\,(AP)^\dagger = P(\text{rge } A^T)$ and $\text{rge}\,\big((AP)^\dagger\big)^T = M$; therefore $\widetilde{L} = (AP)^\dagger(M) = \text{rge}\,(AP)^\dagger = P(\text{rge } A^T) \subset L$.

Lemma 3.4.1, with $A := AP$, implies that

$$AP(\mathbb{B}_{\mathbb{R}^n} \cap \widetilde{L}) \supset \mathbb{B}_{\mathbb{R}^n}[0, c] \cap M,$$

where c is the smallest singular value of the matrix AP. Since $\text{rge}(AP)^T = \widetilde{L}$ and $\widetilde{L} \subset L$, we have $A(\mathbb{B}_{\mathbb{R}^n} \cap L) = AP(\mathbb{B}_{\mathbb{R}^n} \cap L) = AP(\mathbb{B}_{\mathbb{R}^n} \cap \widetilde{L}) = A(\mathbb{B}_{\mathbb{R}^n} \cap \widetilde{L})$. ■

In the following examples, we use the method for finding singular values and vectors described in Remark A.1.1.

Example 3.4.1 *Consider a matrix*

$$A := \begin{pmatrix} -1 & -1 \\ 2 & -2 \end{pmatrix}$$

and a subspace $L := \text{span}\{(0, 1)^T\}$. Then the matrix P of the orthogonal projection onto the subspace L has the form

$$P := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and we have

$$AP = \begin{pmatrix} 0 & -1 \\ 0 & -2 \end{pmatrix}.$$

Then the number $\sqrt{5}$ is the only singular value of the matrix AP and the vector $(-1/\sqrt{5}, -2/\sqrt{5})^T$ is the corresponding left singular vector; therefore

$$A(\mathbb{B}_{\mathbb{R}^2} \cap L) \supset \mathbb{B}_{\mathbb{R}^2}[0, \sqrt{5}] \cap M,$$

where $M := \text{span}\{(-1/\sqrt{5}, -2/\sqrt{5})^T\}$.

Further, we consider a given subspace as the constraint in the range of a linear mapping.

Proposition 3.4.2 *Let $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear mapping, a set $M \subset \mathbb{R}^m$ be a non-trivial subspace of* rge A, *and $Q \in \mathbb{R}^{m\times m}$ be the orthogonal projection onto M. Then*

$$A(\mathbb{B}_{\mathbb{R}^n} \cap L) \supset \mathbb{B}_{\mathbb{R}^m}[0, c] \cap M,$$

where $L := A^\dagger(M)$ and c equals to the smallest singular value of the matrix $(A^\dagger Q)^\dagger$.

Proof. Note that $\text{rge } A^\dagger = \text{rge } A^T$ and $\text{rge}(A^\dagger)^T = \text{rge } A$. Since $M \subset \text{rge } A$, we have $\text{rge}(A^\dagger Q)^T = \text{rge}(Q(A^\dagger)^T) = M$ and then $\text{rge}(A^\dagger Q) = A^\dagger(M) = L$. Then $\text{rge}\,\big(A^\dagger Q\big)^\dagger = M$ and $\text{rge}\,\big(\big(A^\dagger Q\big)^\dagger\big)^T = L$. Lemma 3.4.1, with $A := (A^\dagger Q)^\dagger$, implies that

$$(A^\dagger Q)^\dagger(\mathbb{B}_{\mathbb{R}^n} \cap L) \supset \mathbb{B}_{\mathbb{R}^m}[0, c] \cap M,$$

where c is the smallest singular value of the matrix $(A^\dagger Q)^\dagger$. Since $(A^\dagger Q)(A^\dagger Q)^\dagger$ and $AA^\dagger$ are the matrices of the orthogonal projection onto L and rge A, respectively, the last inclusion implies that

$$A(\mathbb{B}_{\mathbb{R}^n} \cap L) = A(A^\dagger Q)(A^\dagger Q)^\dagger(\mathbb{B}_{\mathbb{R}^n} \cap L) \supset AA^\dagger Q(\mathbb{B}_{\mathbb{R}^m}[0, c] \cap M) = \mathbb{B}_{\mathbb{R}^m}[0, c] \cap M.$$

■

Example 3.4.2 *Consider a matrix*

$$A := \begin{pmatrix} -1 & -1 \\ 2 & -2 \end{pmatrix}$$

and a subspace $M := \operatorname{span}\{(0,1)^T\}$*. Then the matrix* Q *of the orthogonal projection onto the subspace* M *has form*

$$Q := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and we have[2]

$$(A^\dagger Q)^\dagger = \begin{pmatrix} 0 & 0 \\ 2 & -2 \end{pmatrix}.$$

Then the number $2\sqrt{2}$ *is the only singular value of the matrix* $(A^\dagger Q)^\dagger$ *and the vector* $(-1/\sqrt{2}, 1/\sqrt{2})^T$ *is the corresponding right singular vector; therefore*

$$A(\mathbb{B}_{\mathbb{R}^2} \cap L) \supset \mathbb{B}_{\mathbb{R}^2}[0, 2\sqrt{2}] \cap M,$$

where $L := \operatorname{span}\{(-1/\sqrt{2}, 1/\sqrt{2})^T\}$*.*

Now, we combine the both previous cases.

Proposition 3.4.3 *Let* $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *be a linear mapping,* L *be a subspace of* $\mathbb{R}^n$*, and* M *be a nontrivial subspace of* $A(L)$*,* $P \in \mathbb{R}^{n\times n}$ *be the orthogonal projection onto* L*, and* $Q \in \mathbb{R}^{m\times m}$ *be the orthogonal projection onto* M*. Then*

$$A(\mathbb{B}_{\mathbb{R}^n} \cap \widetilde{L}) \supset \mathbb{B}_{\mathbb{R}^m}[0, c] \cap M,$$

where $\widetilde{L} := (AP)^\dagger(M) \subset L$ *and* c *equals to the smallest singular value of the matrix* $\big((AP)^\dagger Q\big)^\dagger$*.*

Proof. Clearly, $\operatorname{rge}(AP)^T = \operatorname{rge}(PA^T) = P(\operatorname{rge} A^T)$ and $\operatorname{rge}(AP) = A(L)$. Thus $\operatorname{rge}(AP)^\dagger = P(\operatorname{rge} A^T)$ and $\operatorname{rge}((AP)^\dagger)^T = A(L)$. Hence $\operatorname{rge}((AP)^\dagger Q)^T = \operatorname{rge}(Q((AP)^\dagger)^T) = Q(A(L)) = M$ and then $\operatorname{rge}((AP)^\dagger Q) = (AP)^\dagger(M)$; so $\operatorname{rge}(((AP)^\dagger Q)^\dagger)^T = (AP)^\dagger(M) = \widetilde{L}$ and $\operatorname{rge}((AP)^\dagger Q)^\dagger = M$. Note that since $\operatorname{rge}(AP)^\dagger \subset L$, then

$$\widetilde{L} = \operatorname{rge}(((AP)^\dagger Q)^\dagger)^T = \operatorname{rge}((AP)^\dagger Q) = ((AP)^\dagger Q)(\mathbb{R}^m) = (AP)^\dagger(M) \subset L.$$

Lemma 3.4.1, with $A := \big((AP)^\dagger Q\big)^\dagger$, implies that

$$\big((AP)^\dagger Q\big)^\dagger(\mathbb{B}_{\mathbb{R}^n} \cap \widetilde{L}) \supset \mathbb{B}_{\mathbb{R}^m}[0, c] \cap M,$$

where c is the smallest singular value of the matrix $\big((AP)^\dagger Q\big)^\dagger$. Since $(AP)^\dagger Q\big((AP)^\dagger Q\big)^\dagger$ and $AP(AP)^\dagger$ are matrices of the orthogonal projection onto $\widetilde{L}$ and $A(L)$, respectively, the last inclusion implies

$$AP(\mathbb{B}_{\mathbb{R}^n} \cap \widetilde{L}) = AP(AP)^\dagger Q\big((AP)^\dagger Q\big)^\dagger(\mathbb{B}_{\mathbb{R}^n} \cap \widetilde{L}) \supset AP(AP)^\dagger Q(\mathbb{B}_{\mathbb{R}^m}[0, c] \cap M) = \mathbb{B}_{\mathbb{R}^m}[0, c] \cap M.$$

Since $\widetilde{L} \subset L$, we have $AP(\mathbb{B}_{\mathbb{R}^n} \cap \widetilde{L}) = A(\mathbb{B}_{\mathbb{R}^n} \cap \widetilde{L})$, by this we finish the proof. ■

Further, we focus on the constrained (semi)regularity, where the constraints are given by cones. To find the modulus, we use a generalization of the eigenvalues. See Section A.2 for a (very brief) introduction to K-eigenvalues problem.

The following proposition gives us the manner how to compute the modulus with a given constraint by convex cone in the domain space.

[2] The matrix was computed by Wolfram Mathematica 11 using the function PseudoInverse.

Proposition 3.4.4 *Let $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear mapping and $K \subset \operatorname{rge} A^T$ be a non-trivial closed convex cone. Consider a set $M := A(K)$. Then*

$$A(I\!B_{\mathbb{R}^n} \cap K) \supset I\!B_{\mathbb{R}^m}[0, c] \cap M, \tag{3.20}$$

where $c := \lambda^{1/2}$ is positive and λ is the smallest K-eigenvalue of the matrix $A^T A$.

Proof. Let $c := \min_{x \in K \cap \mathbb{S}_{\mathbb{R}^n}} \|Ax\|$, then (3.20) holds. Since $K \subset \operatorname{rge} A^T$ and by Remark A.1.2, we have $c > 0$. To show that (3.20) holds, fix any $y \in I\!B_{\mathbb{R}^m}[0, c] \cap M \subset \operatorname{rge} A$. If $y = 0$, then (3.20) trivially holds. If not, then, by Lemma 3.4.1, there is nonzero $x \in K \subset \operatorname{rge} A^T$ such that $Ax = y$. Clearly, $\frac{\|Ax\|}{\|x\|} \geq c$, hence $c \geq \|y\| = \|Ax\| \geq c\|x\|$; therefore $x \in I\!B_{\mathbb{R}^n} \cap K$ and (3.20) holds.

Further, let $x \in K \cap \mathbb{S}_{\mathbb{R}^n}$, be such that $c = \|Ax\|$. Then the function $1/2\|A(\cdot)\|^2$ has a minimum at x with respect to $x \in K \cap \mathbb{S}_{\mathbb{R}^n}$. Then, by Lemma A.2.1, we have

$$\|Ax\|^2 = \lambda,$$

where λ is the smallest K-eigenvalue of $A^T A$. Hence $c = \lambda^{1/2}$. ■

The following example shows how to find the smallest K-eigenvalue of $A^T A$ for a given matrix A and a given cone $K \subset \operatorname{rge} A^T$.

Example 3.4.3 *Consider a matrix*

$$A := \begin{pmatrix} -1 & -9 \\ 9 & 1 \end{pmatrix}$$

and a cone $K := \mathbb{R}^2_+$. Then we have

$$A^T A = \begin{pmatrix} 82 & 18 \\ 18 & 82 \end{pmatrix}.$$

We are finding K-eigenvalues of the matrix $A^T A$, that is, we want to find $\lambda \in \mathbb{R}$ and $u \in \mathbb{R}^2$, with $u \neq 0$, such that

$$0 \leq (A^T A - \lambda I)u \perp u \geq 0.$$

Let us note that $K^* = \mathbb{R}^2_+$. The problem is equivalent to find $\lambda \in \mathbb{R}$ and non-negative numbers x, y, with $x \neq 0$ or $y \neq 0$, such that

$$(\lambda - 82)(x^2 + y^2) - 36xy = 0, \quad (82 - \lambda)x + 18y \geq 0, \quad \text{and} \quad 18x + (82 - \lambda)y \geq 0. \tag{3.21}$$

The previous problem is satisfied if one of the following holds:

(i) $\lambda = 82, x = 0,$ *and* $y > 0$*;*

(ii) $\lambda = 82, x > 0,$ *and* $y = 0$*;*

(iii) $\lambda = 100, x > 0,$ *and* $y = x$*.*

Note that, if $\lambda \in (0, 82) \cap (100, \infty)$, then the first equality in (3.21) is satisfied if and only if $x = 0$ and $y = 0$. If $\lambda \in (82, 100)$, then (3.21) holds if and only if $x = 0$ and $y = 0$.

Hence $\sigma_K(A^T A) = \{82, 100\}$ and we conclude that

$$A(I\!B_{\mathbb{R}^2} \cap K) \supset I\!B_{\mathbb{R}^2}\left[0, \sqrt{82}\right] \cap M,$$

where $M := A(K)$.

Suppose that we have a cone as a constraint in the range of some linear mapping in the hand. The following proposition shows how to find a cone, which is mapped onto the cone in the range. Moreover, it gives us how to compute the modulus.

Proposition 3.4.5 *Let $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear mapping and $K \subset \operatorname{rge} A$ be a non-trivial closed convex cone. Consider a set $L := A^\dagger(K)$. Then*

$$A(\mathbb{B}_{\mathbb{R}^n} \cap L) \supset \mathbb{B}_{\mathbb{R}^m}[0, 1/c] \cap K,$$

where $c := (-\lambda)^{1/2}$ is positive and λ is the smallest K-eigenvalue of the matrix $-(A^\dagger)^T A^\dagger$.

Proof. Let $c := \max_{y \in K \cap \mathbb{S}_{\mathbb{R}^m}} \|A^\dagger y\|$. Since $K \subset \operatorname{rge} A$ and by Remark A.1.2, we have $c > 0$. Then we have

$$\mathbb{B}_{\mathbb{R}^n}[0, c] \cap L \supset A^\dagger(\mathbb{B}_{\mathbb{R}^m} \cap K).$$

Multiplying by $1/c$ and then by A from the left, we get

$$A(\mathbb{B}_{\mathbb{R}^n} \cap L) \supset \mathbb{B}_{\mathbb{R}^m}[0, 1/c] \cap K.$$

Let us note that the previous inclusion holds because A maps $\operatorname{rge} A^\dagger$ onto $\operatorname{rge} A$. Now, find $y \in K \cap \mathbb{S}_{\mathbb{R}^m}$ such that $\|A^\dagger y\| = c$. Then $c^2 = -\min_{y \in K \cap \mathbb{S}_{\mathbb{R}^m}} \big(- \|A^\dagger y\|^2 \big)$. Hence, by Lemma A.2.1, with $A := A^\dagger$, $n := m$, and $m := n$, we have $-\min_{y \in K \cap \mathbb{S}_{\mathbb{R}^m}} \big(- \|A^\dagger y\|^2 \big) = -\min \sigma_K(-(A^\dagger)^T A^\dagger)$, therefore $c = \big(-\min \sigma_K(-(A^\dagger)^T A^\dagger)\big)^{1/2}$. ■

Chapter 4

Topological spaces

In the first section, we present the basics of topology. The second section contains the definition of topology given by a function $\varphi : X \times X \longrightarrow [0, \infty]$ in the spirit of quasi-metric spaces defined in [19] and [20]. A generalization of Ekeland variational principle to the extended quasi-metric spaces is given in the last section.

4.1 Background of topology

We present a brief topological background, which is necessary for our purposes. For a deeper insight, we refer to [24, 58, 62]. We follow the notions from [20, 58, 62, 68]. For the readers' convenience, we recall the notation and basic concepts.

Definition 4.1.1 *Let X be a nonempty set. The* topology τ *on X is a family of subsets of X such that:*

(i) *the sets $\emptyset$ and X are members of τ;*

(ii) *an intersection of any two members of τ is also a member of τ;*

(iii) *a union of any members of τ is also a member of τ.*

The couple (X, τ) consisting of the set X and the topology τ on X is called the topological space*.*

Let us point out that a topology on a set is not unique, so several topologies can be defined on a given set.

Example 4.1.1 *Let X be a nonempty set. Let $\tau_1 := \{\emptyset, X\}$ and $\tau_2 := 2^X$ (the power set of the set X), then τ_1 and τ_2 define topologies on X. The topology τ_1 is called the* indiscrete topology *and the topology τ_2 is called the* discrete topology*.*

Then the following proposition gives us how to define the topology on the Cartesians product of the topological spaces.

Proposition 4.1.1 *Let (X, τ_X) and (Y, τ_Y) be topological spaces, and $\tau_{X\times Y}$ be the family of all unions of sets of the form $U \times V$, where $U \in \tau_X$ and $V \in \tau_Y$. Then the pair $(X \times Y, \tau_{X\times Y})$ is a topological space.*

The topology $\tau_{X\times Y}$ is called the *product topology on $X \times Y$* and the space $(X \times Y, \tau_{X\times Y})$ is called the *product space* of (X, τ_X) and (Y, τ_Y).

Next, we define the notion of open and closed set.

Definition 4.1.2 *Let (X, τ) be a topological space. We say that each set in τ is* open *and we say that a set $A \subset X$ is* closed *if the set $X \setminus A$ is open. For arbitrary $x \in X$, we say that a set $U \subset X$ is a* neighborhood *of x if $x \in U$ and there is a open set $V \subset U$ with $x \in V$.*

We can write τ-open, τ-closed, and τ-neighborhood to specify that properties are in the topology τ.

Several properties of a topology can be verified by properties of some system of subsets of τ or sometimes, it is proper to define a topology by a system of subsets. For these reasons, we define a basis for a topology.

Definition 4.1.3 *Let (X, τ) be a topological space. A family of sets $\mathcal{B} \subset \tau$ is called a* basis *for the topology τ if each open set is the union of members of $\mathcal{B}$.*

Example 4.1.2 *Let $X := \mathbb{R}$. The family of sets τ we define the following way: the set $U \in \tau$ if for each $x \in U$ there is $r > 0$ such that $(x - r, x + r) \subset U$. Then (X, τ) is a topological space and the family of sets $\mathcal{B} := \{(x - r, x + r) : x \in X \text{ and } r > 0\}$ is a basis for the topology τ.*

For our purposes, we use the notion of a convergence of a sequence.

Definition 4.1.4 *Let (X, τ) be a topological space. We say that a sequence (x_k) in X is* convergent *(or* convergent in the topology τ*) if there is $x \in X$ such that for each neighborhood U of x there is $k_0 \in \mathbb{N}$ such that for each $k > k_0$ we have $x_k \in U$; and the point x is called the* limit *of the sequence (x_k).*

We can also write that a sequence (x_k) is τ-convergent. If x is a limit of sequence (x_k), thus we write either the sequence (x_k) converges to x (in the topology τ), $x_k \to x$ as $k \to \infty$ or $\lim_{k\to\infty} x_k = x$. This definition of convergence corresponds to a convergence of sequence in metric, Banach, and Hilbert spaces, etc.

Remark 4.1.1 *Let (X, τ) be a topological space and $U \subset X$ be a nonempty closed set. Consider a sequence (x_k) in U. If (x_k) converges to some $x \in X$, then $x \in U$.*

Indeed, on the contrary, suppose that $x \in X \setminus U$. Then $X \setminus U$ is an open set and a neighborhood of x. Thus, since (x_k) converges to x, there is $k_0 \in \mathbb{N}$ such that $x_k \in X \setminus U$ for each $k > k_0$, a contradiction.

For the separation of two distinct points, there are several separation axioms in the literature. This is very important for the uniqueness of the limit of a sequence. Note that such axioms are called Tychonoff separation axioms.

Definition 4.1.5 *Let (X, τ) be a topological space. The space (X, τ) is said to be:*

(i) T_0 space *(or Kolmogorov space) if for each $x, y \in X$, with $x \neq y$, there is a neighborhood U of x such that $y \notin U$ or a neighborhood V of y such that $x \notin V$;*

(ii) T_1 space *(or Fréchet space) if for each $x, y \in X$, with $x \neq y$, there are neighborhoods U of x and V of y such that $y \notin U$ and $x \notin V$;*

(iii) T_2 space *(or Hausdorff space) if for each $x, y \in X$, with $x \neq y$, there are disjoint neighborhoods U of x and V of y.*

The meaning of these axioms can be seen in Figure 4.1. Of course, there are more separation axioms, see [20, p. 6], but for our purposes the previous three ones are enough. The following examples show that a limit of a sequence in T_0 and T_1 space is not necessarily unique.

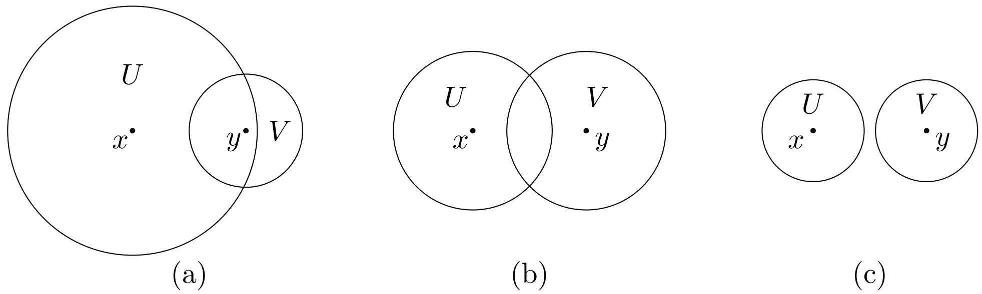

Figure 4.1: Neighborhoods in (a) T_0 space, (b) T_1 space, and (c) T_2 space.

Example 4.1.3 (i) *Let $X := \mathbb{R}$. We define the topology τ on X the following way: $\emptyset, X \in \tau$, and for each $a \in \mathbb{R}$, τ contains the set $(-\infty, a)$. Then the couple (X, τ) is T_0 space, but not T_1 space. Moreover, a limit of a sequence is not necessarily unique.*

Indeed, fix any $x, y \in X$ with $y < x$. By definition of τ, for each $a, b \in X$, with $y < b < x < a$, we have $y \in (-\infty, a)$, but $x \notin (-\infty, b)$. Hence (X, τ) is T_0 space, but not T_1 space.

Consider the sequence $\big((-1)^k\big)$. Fix any $x \geq 1$ and choose any neighborhood U of x, then there is $a > x$ such that $(-1)^k \in (-\infty, a) \subset U$ for each $k \in \mathbb{N}$. Hence the sequence $\big((-1)^k\big)$ converges to every $x \in \langle 1, \infty)$.

(ii) *Let $X := \mathbb{R}$. We define the topology τ on X the following way: $\emptyset, X \in \tau$, and τ contains all sets $U \subset \mathbb{R}$ such that the set $X \setminus U$ is bounded, that is, there are $a, b \in X$, with $a < b$, such that for each $x \in X \setminus U$ we have $a \leq x \leq b$. Then the couple (X, τ) is T_1 space but not T_2 space. Nonetheless a limit of a sequence is not necessarily unique.*

Indeed, fix any $x, y \in X$ with $x \neq y$. Fix any $U, V \in \tau$, such that $x \in U$ and $y \in V$. Then $y \notin U \setminus \{y\} \in \tau$ and $x \notin V \setminus \{x\} \in \tau$. Further, by the definition of τ, there is a big enough $z \in X$ such that $z \in U \cap V$. Hence the topological space is T_1 space but not T_2 space.

Consider the sequence (k). Fix any $x \in X$ and any neighborhood U of x. Then there is an integer j such that $j > |x|$ such that $V := (-\infty, -j) \cup \{x\} \cup (j, \infty) \subset U$. Clearly, $V \in \tau$ and for each $k > j$ we get $k \in V$. In conclusion, the sequence (k) converges to every $x \in X$.

In a Hausdorff space, the limit of a sequence is unique, see [62, Proposition 11.4].

Proposition 4.1.2 *Let (X, τ) be a topological space, which is T_2 space and (x_k) be a sequence in X. If the sequence (x_k) is convergent, then its limit is unique.*

The continuity for a single-valued mapping g between two topological spaces can be defined as follows.

Definition 4.1.6 *Let (X, τ_X) and (Y, τ_Y) be topological spaces, $g : X \longrightarrow Y$ be a single-valued mapping, and $x \in X$ be given. A single-valued mapping $g : X \longrightarrow Y$ is continuous at x if for each τ_Y-neighborhood V of $g(x)$, there is a τ_X-neighborhood U of x such that $g(U) \subset V$.*

We say that $g : X \longrightarrow Y$ is *continuous* if g is continuous at each $x \in X$. We also write that g is τ_X-to-τ_Y-continuous (at x) to specify topologies involved. Also, we need the notion of a sequential continuity.

Definition 4.1.7 *Let (X, τ_X) and (Y, τ_Y) be topological spaces, $g : X \longrightarrow Y$ be a single-valued mapping, and $x \in X$ be given. We say that g is* sequentially continuous *at x if for each sequence (x_k) in X, which converges to x in the topology τ_X, the sequence $(g(x_k))$ converges to $g(x)$ in the topology τ_Y.*

Recall of the notion of the first-countable topological space.

Definition 4.1.8 *Let (X, τ) be a topological space. The space is said to be* first-countable *if for each $x \in X$ there is a sequence of its neighborhoods U_k for $k \in \mathbb{N}$ such that for each neighborhood U of x there is $j \in \mathbb{N}$ such that $U_j \subset U$.*

In the first-countable topological space, the continuity and the sequential continuity coincide, see [68, pp. 73].

Proposition 4.1.3 *Let (X, τ_X) and (Y, τ_Y) be topological spaces, $g : X \longrightarrow Y$ be a single-valued mapping, and $x \in X$ be given. Suppose that (X, τ_X) is a first-countable space, then g is continuous at $x \in X$ if and only if g is sequentially continuous at x.*

Let (X, τ) be a topological space. Consider a function f which maps from the set X to the set $[-\infty, \infty]$. This function may attain the value ∞ or $-\infty$ at some points and such functions have several applications, in particular, in the constrained minimization. The domain of such an f is defined by

$$\operatorname{dom} f := \{x \in X : |f(x)| < \infty\}.$$

We also need the notion of semicontinuity of f on a topological spaces, see [49, Definition 1.14].

Definition 4.1.9 *Let (X, τ) be a topological space, a function $f : X \longrightarrow [-\infty, \infty]$, and a point $\bar{x} \in X$ be given. We say that f is:*

(i) upper semicontinuous at $\bar{x}$ *if for any $\varepsilon > f(\bar{x})$ there is a neighborhood U of $\bar{x}$ such that $f(x) < \varepsilon$ for each $x \in U$;*

(ii) lower semicontinuous at $\bar{x}$ *if for any $\varepsilon < f(\bar{x})$ there is a neighborhood U of $\bar{x}$ such that $f(x) > \varepsilon$ for each $x \in U$;*

(iii) upper semicontinuous *if f is upper semicontinuous at every $x \in X$;*

(iv) lower semicontinuous *if f is lower semicontinuous at every $x \in X$.*

We can also write that f is τ-upper semicontinuous and τ-lower semicontinuous. Note that, if f is upper semicontinuous or lower semicontinuous, then $-f$ is lower semicontinuous or upper semicontinuous, respectively.

On the set $[-\infty, \infty]$, we consider the topology from the following example.

Example 4.1.4 *Let $X := [-\infty, \infty]$. We define a family of sets $\mathcal{B}$ the following way: $(x - a, x + a) \in \mathcal{B}$, $(a, \infty] \in \mathcal{B}$, and $[-\infty, a) \in \mathcal{B}$ for each $a > 0$ and each $x \in (-\infty, \infty)$. The family of sets τ we define the following way: the set $U \in \tau$ if for each $x \in U$ there is $V \in \mathcal{B}$ such that $x \in V \subset U$. Then the pair (X, τ) is a topological space and the family of sets $\mathcal{B}$ is a basis for (X, τ).*

Hence we can write that a sequence (x_k) in $[-\infty, \infty]$ converges to some $x \in \mathbb{R}$ if for each $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$, such that for each $k > k_0$ we have $|x_k - x| < \varepsilon$, a sequence (x_k) in $[-\infty, \infty]$ has the limit ∞ if for each $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$, such that for each $k > k_0$ we have $x_k > 1/\varepsilon$, or a sequence (x_k) in $[-\infty, \infty]$ has the limit $-\infty$ if for each $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$, such that for each $k > k_0$ we have $x_k < -1/\varepsilon$.

In the first-countable topological space, the upper and lower semicontinuity can be defined via sequences, cf. [50, Lemma 2.2].

Proposition 4.1.4 *Let (X, τ) be a first-countable topological space and a function $f : X \longrightarrow [-\infty, \infty]$ be given. Then the following holds:*

(i) *f is upper semicontinuous at $\bar{x}$ if and only if for each sequence (x_k) converging to $\bar{x}$ we have*

$$\limsup_{k\to\infty} f(x_k) \le f(\bar{x});$$

(ii) *f is lower semicontinuous at $\bar{x}$ if and only if for each sequence (x_k) converging to $\bar{x}$ we have*

$$\liminf_{k\to\infty} f(x_k) \ge f(\bar{x});$$

where

$$\limsup_{k\to\infty} y_k := \lim_{k\to\infty} \big(\sup_{k\le j}\{y_j\}\big) \quad \text{and} \quad \liminf_{k\to\infty} y_k := \lim_{k\to\infty} \big(\inf_{k\le j}\{y_j\}\big),$$

for any sequence (y_k) in $[-\infty, \infty]$.

4.2 Quasi-metric spaces

In this section, we present the topological space defined in [19] and [20, Section 1.1.2], that is, when the topology is defined by a function $\varphi : X \times X \longrightarrow [0, \infty]$ for a nonempty set X, which has similar properties as a metric, except for the symmetry of the distance, that is, there are $x, u \in X$ such that $\varphi(x, u) \ne \varphi(u, x)$, in general. A lack of the symmetry may cause that a convergent sequence does not have a unique limit. This type of topological spaces is called (extended) quasi-(semi)metric spaces (see Remark 4.2.1) in the literature.

Since the terminology used by various authors in (extended) quasi-metric spaces is not unified, we postulate the following set of axioms without giving them particular names.

Definition 4.2.1 *Let X be a nonempty set and a function $\varphi : X \times X \longrightarrow [0, \infty]$ be given. We say that φ has property:*

($\mathcal{A}_1$) *provided that $\varphi(x, x) = 0$ for each $x \in X$;*

($\mathcal{A}_2$) *provided that $\varphi(x, y) \le \varphi(x, z) + \varphi(z, y)$ whenever $x, y, z \in X$;*

($\mathcal{A}_3$) *provided that $\varphi(x, y) > 0$ whenever $x, y \in X$ are distinct;*

($\mathcal{A}_4$) *provided that for each (x_k) in X such that for each $\varepsilon > 0$ there is an index $k_0 = k_0(\varepsilon)$ such that for each $k, j \in \mathbb{N}$, with $k_0 \le k < j$, we have $\varphi(x_j, x_k) < \varepsilon$; there is a point $u \in X$ such that $\varphi(u, x_k) \to 0$ as $k \to \infty$.*

The *conjugate* of φ is the function $\overline{\varphi} : X \times X \longrightarrow [0, \infty]$ defined by $\overline{\varphi}(x, u) := \varphi(u, x)$ for $x, u \in X$. When φ has some of the properties $(\mathcal{A}_1) - (\mathcal{A}_3)$, then the function $\overline{\varphi}$ has the same property.

Let X be a nonempty set. For $x \in X$, $r > 0$, and a mapping $\varphi : X \times X \longrightarrow [0, \infty]$, having the property $(\mathcal{A}_1)$, we define the open ball and the closed ball, determined by φ, in the form

$$I\!B^{\varphi}_X(x, r) := \{u \in X : \varphi(x, u) < r\} \quad \text{and} \quad I\!B^{\varphi}_X[x, r] := \{u \in X : \varphi(x, u) \le r\},$$

respectively. Define a system of sets τ_φ on X by

$$\tau_\varphi := \{U \subset X : \text{ for each } x \in U \text{ there is } r > 0 \text{ such that } I\!B^{\varphi}_X(x, r) \subset U\}.$$

The question is when τ_φ defines a topology on X. The answer is easy.

Proposition 4.2.1 *Let X be a nonempty set and a mapping $\varphi : X \times X \longrightarrow [0, \infty]$ be given. Then the following is true:*

(i) *if φ has the property $(\mathcal{A}_1)$, then τ_φ is a topology on X;*

(ii) *if φ has the property $(\mathcal{A}_1)$ and*

$$\varphi(x, y) = 0 \quad \text{and} \quad \varphi(y, x) = 0 \quad \text{implies} \quad x = y, \quad \text{for each} \quad x, y \in X, \tag{4.1}$$

then (X, τ_φ) is a T_0 space;

(iii) *if φ has the properties $(\mathcal{A}_1)$ and $(\mathcal{A}_3)$, then (X, τ_φ) is a T_1 space.*

Proof. At first, we are showing that (i) holds. Clearly, $\emptyset, X$ are elements of τ_φ. Let U and V be from τ_φ. If $U \cap V = \emptyset$, then obviously $U \cap V \in \tau_\varphi$. If not, fix any $x \in U \cap V$. Since U and V are τ_φ-open sets, there are positive r_1 and r_2 such that $I\!\!B^\varphi_X(x, r_1) \subset U$ and $I\!\!B^\varphi_X(x, r_2) \subset V$. Let $r := \min\{r_1, r_2\}$, then $I\!\!B^\varphi_X(x, r) \subset U \cap V$; hence $U \cap V \in \tau_\varphi$.

Let E be any indexing set. Let U_i be an element of τ_φ for $i \in E$. Fix any $x \in \cup_{i \in E} U_i$. Then there is $j \in E$ such that $x \in U_j$ and there is $r > 0$ such that $I\!\!B^\varphi_X(x, r) \subset U_j$. Then $I\!\!B^\varphi_X(x, r) \subset U_j \subset \cup_{i \in E} U_i$.

Further, we are showing that (ii) holds. Fix any $x, y \in X$ with $x \neq y$. Thus, without any loss of generality, we can assume that $\varphi(x, y) > 0$. Let $r := \min\{\varphi(x, y), 1\}$, then $y \notin I\!\!B^\varphi_X(x, r)$.

To show (iii), fix any $x, y \in X$ with $x \neq y$. Let $r_1 := \min\{\varphi(x, y), 1\}$ and $r_2 := \min\{\varphi(y, x), 1\}$, then $y \notin I\!\!B^\varphi_X(x, r_1)$ and $x \notin I\!\!B^\varphi_X(y, r_2)$. ∎

Therefore, the family of sets $\{I\!\!B^\varphi_X(x, r) : \quad x \in X \quad \text{and} \quad r > 0\}$ is a basis for the topology τ_φ.

Note that, if φ has the properties $(\mathcal{A}_1) - (\mathcal{A}_2)$, then for each $x \in X$ and each $r > 0$, the sets $I\!\!B^\varphi_X(x, r)$ and $I\!\!B^\varphi_X[x, r]$ are the τ_φ-open set and the $\tau_{\overline{\varphi}}$-closed set, respectively.

By the definition of the convergence we can write that a sequence (x_k) converges to x in X with respect to the topology τ_φ if and only if $\varphi(x, x_k) \to 0$ as $k \to \infty$. Equivalently, a sequence (x_k) converges to x in X if for each $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ such that for each index $k > k_0$ we have $\varphi(x, x_k) < \varepsilon$.

Remark 4.2.1 *Let us comment on the terminology from [19]. Let X be a nonempty set and a function $\varphi : X \times X \longrightarrow [0, \infty)$ be given. Any function φ having properties $(\mathcal{A}_1) - (\mathcal{A}_2)$ is called a quasi-semimetric and the pair (X, φ) is called a quasi-semimetric space. Moreover, if φ satisfies (4.1), then φ is called a quasi-metric and the pair (X, φ) is called a quasi-metric space. The property $(\mathcal{A}_4)$ means that the space (X, τ_φ) is right φ-K-complete.*

Remark 4.2.2 *Let X be a nonempty set and $\varphi : X \times X \longrightarrow [0, \infty]$ satisfy $(\mathcal{A}_1)$. Then the space (X, τ_φ) is first-countable.*

Indeed, fix any $x \in X$ and a family of neighborhoods $\{I\!\!B^\varphi_X(x, 1/k) : k \in \mathbb{N}\}$. By the definition of τ_φ for each neighborhood U of x there is $r > 0$ such that $I\!\!B^\varphi_X(x, r) \subset U$. Then there is $k_0 \in \mathbb{N}$ such that $I\!\!B^\varphi_X(x, 1/k) \subset I\!\!B^\varphi_X(x, r)$ for each $k \in \mathbb{N}$ with $k_0 < k$.

Remark 4.2.3 *Let a constant $\alpha > 0$, nonempty sets X and Y, and functions $\varphi : X \times X \longrightarrow [0, \infty]$ and $\varrho : Y \times Y \longrightarrow [0, \infty]$ be given. If both φ and ϱ have some of properties $(\mathcal{A}_1) - (\mathcal{A}_4)$, then the function $\omega : (X \times Y)^2 \longrightarrow [0, \infty]$, defined by*

$$\omega\big((x, y), (u, v)\big) := \max\{\varphi(x, u), \alpha\varrho(y, v)\} \quad \text{for} \quad (x, y), (u, v) \in X \times Y, \tag{4.2}$$

has the same property. Moreover, if the family of sets τ_ω defines a topology on $X \times Y$, then it is the product topology on $X \times Y$.

Further, we focus on semicontinuity properties of the function φ.

Lemma 4.2.1 *Let X be a nonempty set and a function $\varphi : X \times X \longrightarrow [0,\infty]$ having the properties $(\mathcal{A}_1) - (\mathcal{A}_2)$ be given. Then for each $u \in X$ the function $X \ni x \longmapsto \varphi(x,u)$ is τ_φ-lower semicontinuous and the function $X \ni x \longmapsto \varphi(u,x)$ is τ_φ-upper semicontinuous.*

Proof. Fix any $x \in X$ and any $u \in X$. Take any sequence (x_k) in X such that $\varphi(x,x_k) \to 0$ as $k \to \infty$. Then

$$\liminf_{k\to\infty} \varphi(x_k,u) = \liminf_{k\to\infty} \big(\varphi(x,x_k) + \varphi(x_k,u)\big) \geq \liminf_{k\to\infty} \varphi(x,u) = \varphi(x,u)$$

and

$$\limsup_{k\to\infty} \varphi(u,x_k) \leq \limsup_{k\to\infty} \big(\varphi(u,x) + \varphi(x,x_k)\big) = \varphi(u,x).$$

■

Let X be a nonempty set. A function $\varphi : X \times X \longrightarrow [0,\infty)$ defines a metric on X if φ has the properties $(\mathcal{A}_1) - (\mathcal{A}_3)$ and satisfies

$$\varphi(x,u) = \varphi(u,x) = \overline{\varphi}(x,u) \quad \text{for each} \quad x,u \in X.$$

In the case, that φ has the property $(\mathcal{A}_4)$, then the metric space (X,φ) is complete. If φ has only properties $(\mathcal{A}_1) - (\mathcal{A}_3)$, then the function $X \times X \ni (x,u) \longmapsto \max\{\varphi(x,u), \overline{\varphi}(x,u)\}$ defines the metric on X.

Of course, any (positive multiple of) metric has the properties $(\mathcal{A}_1) - (\mathcal{A}_3)$.

Let us present an example (cf. [12, 25]) of the function φ having the properties $(\mathcal{A}_1) - (\mathcal{A}_4)$.

Example 4.2.1 *Let $(X, \|\cdot\|_X)$ be a normed space and L be a nonempty subset of $\mathbb{S}_X$. The* directional minimal time function with respect to L *is the function $X \times X \ni (x,u) \longmapsto T_L(x,u) \in [0,\infty]$ defined by*

$$T_L(x,u) := \inf\{t \geq 0 : u - x \in tL\} \quad \text{for} \quad (x,u) \in X \times X.$$

Clearly, if for some $x,u \in X$ we have $T_L(x,u) < \infty$ (which is equivalent to $u - x \in \operatorname{cone} L$), then

$$T_{-L}(u,x) = T_L(x,u) = \|u - x\|_X.$$

The open ball and the closed ball is given by

$$\mathbb{B}_X^{T_L}(x,r) = \mathbb{B}_X(x,r) \cap (x + \operatorname{cone} L) \quad \text{and} \quad \mathbb{B}_X^{T_L}[x,r] = \mathbb{B}_X[x,r] \cap (x + \operatorname{cone} L),$$

respectively. Moreover, we have

(i) *T_L has the properties $(\mathcal{A}_1)$ and $(\mathcal{A}_3)$;*

(ii) *if L is closed, then the function T_L is lower semicontinuous;*

(iii) *if* cone *L is convex, then T_L has the property $(\mathcal{A}_2)$;*

(iv) *if $(X, \|\cdot\|_X)$ is a Banach space and L is closed, then T_L has the property $(\mathcal{A}_4)$.*

Indeed, it is easy to see that (i) *holds. If L is closed, we are showing that for each $x,u \in X$, with $T_L(x,u) < \infty$, we have $u - x \in T_L(x,u)L$. Suppose that $x = u$, then $T_L(x,u) = 0$ and $u - x = 0 \in 0L = T_L(x,u)L$. Further, suppose that $x \neq u$, then there is a sequence (t_k) in $[0,\infty)$, with $t_k \downarrow T_L(x,u)$ as*

$k \to \infty$, such that $u - x \in t_k L$ for each $k \in \mathbb{N}$. Thus $\frac{1}{t_k}(u - x) \in L$ for each $k \in \mathbb{N}$ and since L is closed, we have $\frac{1}{T_L(x,u)}(u - x) \in L$.

To see (ii)*, fix any $(x,u) \in X \times X$ and choose any sequence $\big((x_k,u_k)\big)$ in $X \times X$ such that $(x_k,u_k) \to (x,u)$ as $k \to \infty$. If $\liminf_{k\to\infty} T_L(x_k,u_k) = \infty$, then the conclusion is clear. Suppose that $\alpha := \liminf_{k\to\infty} T_L(x_k,u_k) < \infty$. We can assume that $\lim_{k\to\infty} T_L(x_k,u_k) = \alpha$. Then there is $k_0 \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, with $k > k_0$, we have $T_L(x_k,u_k) < \infty$. Since L is closed, we have $u_k - x_k \in T_L(x_k,u_k)L$ for each $k > k_0$. Hence $u - x \in \alpha L$ and so $T_L(x,u) \le \alpha = \liminf_{k\to\infty} T_L(x_k,u_k)$.*

Further, we are proving that (iii) *holds. To see $(\mathcal{A}_2)$, fix any $x,y,z \in X$. If $T_L(x,z) = \infty$ or $T_L(z,y) = \infty$, then $(\mathcal{A}_2)$ is obvious. Assume that the values are finite. Let $t_1 := T_L(x,z)$ and $t_2 := T_L(z,y)$. Then there are $\tilde{x}, \hat{x} \in L$ such that $z - x = t_1\tilde{x}$ and $y - z = t_2\hat{x}$. Then*

$$y - x = y - z - (x - z) = t_1\tilde{x} + t_2\hat{x}.$$

If $t_1 = t_2 = 0$, then $y - x \in 0L$ and $T_L(x,y) = 0$; hence the desired inequality holds. If not, we get

$$y - x = t_1\tilde{x} + t_2\hat{x} = (t_1 + t_2)\left(\tfrac{t_1}{t_1+t_2}\tilde{x} + \tfrac{t_2}{t_1+t_2}\hat{x}\right),$$

and, by convexity of $\operatorname{cone} L$*, there are $w \in L$ and $\alpha \in (0,1]$ such that*

$$\alpha w = \tfrac{t_1}{t_1+t_2}\tilde{x} + \tfrac{t_2}{t_1+t_2}\hat{x}.$$

We conclude that

$$T_L(x,y) \le \alpha(t_1 + t_2) \le t_1 + t_2 = T_L(x,z) + T_L(z,y).$$

To show (iv)*, fix any sequence (x_k) in X such that for each $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ such that for each $k,j \in \mathbb{N}$, with $k_0 \le k < j$ we have $T_L(x_j,x_k) < \varepsilon$. Since (x_k) is a Cauchy sequence in the Banach space, there is $x \in X$ such that $\|x_k - x\|_X \to 0$ as $k \to \infty$. Fix $\varepsilon > 0$ and find $k_0 \in \mathbb{N}$ such that for each $k,j \in \mathbb{N}$, with $k_0 \le k < j$, we have $x_k - x_j + x \in x + \operatorname{cone} L$. Letting $j \to \infty$, by the closeness of L, we get $x_k \in x + \operatorname{cone} L$ for each $k \in \mathbb{N}$ with $k > k_0$. Hence*

$$T_L(x,x_k) = \|x - x_k\| \to 0 \quad \textit{as} \quad k \to \infty.$$

Note that, if a function $f : X \longrightarrow (-\infty,\infty]$ is lower semicontinuous, then it is also τ_{T_L}-lower semicontinuous, but not vice versa.

4.3 Ekeland variational principle

The statement known as Ekeland variational principle was introduced and proved by I. Ekeland in [27] in 1974. It states that there is an approximated solution of some minimization problem, so it has a lot of applications in optimization theory. We present an extension of this principle to the space defined in Section 4.2.

We start with the original statement in a complete metric space.

Corollary 4.3.1 (Ekeland variational principle) *Let (X,d) be a complete metric space and $\bar{x} \in X, \varepsilon > 0$, and $\lambda > 0$ be given. Consider a proper lower semicontinuous function $f : X \longrightarrow (0,\infty]$, which is bounded from below, such that $f(\bar{x}) < \inf f(X) + \varepsilon$. Then there is $u \in X$ such that*

(i) *$\lambda d(u,\bar{x}) \le \varepsilon$;*

(ii) *$\lambda d(\bar{x},u) \le f(\bar{x}) - f(u)$;*

(iii) $f(u) < f(x) + \lambda d(u,x)$ *whenever* $x \in X \setminus \{u\}$.

In 1977, J.D. Weston proved, in [67], that if for each lower-semicontinuous function the conclusion of Ekeland variational principle holds, then the metric space is necessarily complete.

Proposition 4.3.1 *Let (X,d) be a metric space. Suppose that for each lower semicontinuous function $f : X \longrightarrow \mathbb{R}$, which is bounded from below, and each $\lambda > 0$ there is $u \in X$ such that*

$$f(u) \le f(x) + \lambda d(u,x) \quad \text{whenever} \quad x \in X.$$

Then the metric space (X,d) is complete.

S. Cobzas extended Ekeland variational principle to T_1 quasi-metric spaces in [19, Theorem 2.4]. We present this statement in our setting.

Theorem 4.3.1 *Let X be a nonempty set, a function $\varphi : X \times X \longrightarrow [0,\infty]$ have the properties $(\mathcal{A}_1) - (\mathcal{A}_4)$, and $\bar{x} \in X$ be given. Consider a τ_φ-lower semicontinuous function $f : X \longrightarrow [0,\infty]$ such that $f(\bar{x}) < \infty$. Then there exists a point $u \in X$ such that $f(u) + \varphi(u,\bar{x}) \le f(\bar{x})$ and*

$$f(u) < f(x) + \varphi(x,u) \quad \text{whenever} \quad x \in X \setminus \{u\}.$$

Proof. We are using a standard iterative approach. We are constructing inductively a sequence $x_1, x_2, \ldots$ in $\operatorname{dom} f$. Let $x_1 := \bar{x}$. If $x_k \in \operatorname{dom} f$ is already defined for $k \in \mathbb{N}$, find $x_{k+1} \in X$ such that

$$f(x_{k+1}) + \varphi(x_{k+1}, x_k) \le f(x_k) \quad \text{and} \quad f(x_{k+1}) < i_k + 1/k, \tag{4.3}$$

where

$$i_k := \inf\{f(x') : x' \in X \quad \text{and} \quad f(x') + \varphi(x', x_k) \le f(x_k)\}.$$

Note that $0 \le i_k \le f(x_k) < \infty$ hence a point x_{k+1} exists and lies necessarily in $\operatorname{dom} f$ because we have $f(x_{k+1}) \le f(x_{k+1}) + \varphi(x_{k+1}, x_k) \le f(x_k) < \infty$.

The first inequality in (4.3) implies that the sequence $\big(f(x_k)\big)$ is decreasing and bounded from below, so $\ell := \lim_{k\to\infty} f(x_k)$ exists and is finite. Moreover, for all $1 \le k < j$, we have

$$\begin{aligned} 0 \le \varphi(x_j, x_k) &\le \varphi(x_j, x_{j-1}) + \cdots + \varphi(x_{k+1}, x_k) \\ &\le \big(f(x_{j-1}) - f(x_j)\big) + \cdots + \big(f(x_k) - f(x_{k+1})\big) \le f(x_k) - f(x_j). \end{aligned} \tag{4.4}$$

Hence for each $\varepsilon > 0$ there is an index $k_0 = k_0(\varepsilon)$ such that for each $k, j \in \mathbb{N}$ with $k_0 \le k < j$ we have $\varphi(x_j, x_k) < \varepsilon$. By $(\mathcal{A}_4)$, there is $u \in X$ such that $\varphi(u, x_k) \to 0$ as $k \to \infty$. Given $k \in \mathbb{N}$, as both f and $\varphi(\cdot, x_k)$ are τ_φ-lower semicontinuous, so is $f + \varphi(\cdot, x_k)$, and thus (4.4) implies that

$$f(u) + \varphi(u, x_k) \le \liminf_{p\to\infty} \big(\varphi(x_{k+p}, x_k) + f(x_{k+p})\big) \le f(x_k). \tag{4.5}$$

Taking $k = 1$ and remembering that $x_1 = \bar{x}$, we obtain the first inequality in the conclusion.

Suppose that there is $x \in X$ such that

$$x \ne u \quad \text{and} \quad f(x) + \varphi(x,u) \le f(u).$$

This, $(\mathcal{A}_2)$, and (4.5) imply that, for each $k \in \mathbb{N}$ we have

$$f(x) + \varphi(x, x_k) \le f(x) + \varphi(x,u) + \varphi(u, x_k) \le f(u) + \varphi(u, x_k) \le f(x_k).$$

The definition of (i_k) implies that

$$f(x) < f(x) + \varphi(x,u) \le f(u) \le \liminf_{k\to\infty} f(x_{k+1}) = \liminf_{k\to\infty} \big(f(x_{k+1}) - 1/k\big) \le \liminf_{k\to\infty} i_k \le f(x),$$

a contradiction. ■

Remark 4.3.1 *Note that the above statement implies Corollary 4.3.1. Indeed, consider a complete metric space* (X,d)*, a point* $\bar{x} \in X$*, along with a lower semicontinuous and bounded from below function* $f : X \longrightarrow (-\infty,\infty]$ *such that* $f(\bar{x}) \le \inf f(X) + \varepsilon < \infty$ *for some* $\varepsilon > 0$*. Let* $\lambda > 0$ *be arbitrary. Applying Theorem 4.3.1 with* $f := \frac{1}{\varepsilon}(f - \inf f(X))$ *and* $\varphi := \frac{\lambda}{\varepsilon}\, d$*, we conclude that there is a point* $u := u(\varepsilon,\lambda) \in X$ *such that* $\frac{1}{\varepsilon} f(u) + \frac{\lambda}{\varepsilon}\, d(u,\bar{x}) \le \frac{1}{\varepsilon} f(\bar{x})$ *and* $\frac{1}{\varepsilon} f(u) < \frac{1}{\varepsilon} f(x) + \frac{\lambda}{\varepsilon}\, d(x,u)$ *for each* $x \in X \setminus \{u\}$*. Hence* $f(u) + \lambda\, d(u,\bar{x}) \le f(\bar{x})$ *and* $f(u) < f(x) + \lambda\, d(x,u)$ *for each* $x \in X \setminus \{u\}$*. Further, necessarily,* $\lambda\, d(u,\bar{x}) \le f(\bar{x}) - f(u) \le f(\bar{x}) - \inf f(X) \le \varepsilon$*.*

In [19, Proposition 2.9], the author derived an extension of Proposition 4.3.1 to T_1 quasi-metric spaces. We present similar statement in our setting. We need to define the following property.

Let X be a nonempty set and a function $\varphi : X \times X \longrightarrow [0,\infty]$ be given. We say that φ has property:

($\mathcal{A}_4'$) provided that for each (x_k) in X such that for each $\varepsilon > 0$ there is an index $k_0 = k_0(\varepsilon)$ such that for each $k, j \in \mathbb{N}$, with $k_0 \le k < j$, we have $\varphi(x_j, x_k) < \varepsilon$; there is a point $u \in X$ such that

$$\overline{\varphi}(u, x_k) \to 0 \quad \text{as} \quad k \to \infty.$$

Note that sequences in $(\mathcal{A}_4')$ converge in the topology $\tau_{\overline{\varphi}}$, while sequences (x_k) satisfying $(\mathcal{A}_4)$ converge in the topology τ_{φ}.

Proposition 4.3.2 *Let* X *be a nonempty set and a function* $\varphi : X \times X \longrightarrow [0,\infty]$ *have the properties* $(\mathcal{A}_1) - (\mathcal{A}_3)$*. Suppose that for each proper* $\tau_{\overline{\varphi}}$*-lower semicontinuous function* $h : X \longrightarrow [0,\infty]$ *there is* $u \in \operatorname{dom} h$ *such that*

$$h(u) \le h(x) + \varphi(x,u) \quad \textit{for each} \quad x \in X.$$

Then φ *has the property* $(\mathcal{A}_4')$*.*

Proof. We are following the proof of the second part of the main theorem in [67].

On the contrary, suppose that there is a sequence (x_k) in X such that for each $\varepsilon > 0$ there is an index $k_0 = k_0(\varepsilon)$ such that for each $k, j \in \mathbb{N}$, with $k_0 \le k < j$, we have $\varphi(x_j, x_k) < \varepsilon$, but there is no $x \in X$ such that $\varphi(x_k, x) \to 0$ as $k \to \infty$.

Let $h : X \longrightarrow [0,\infty]$ be defined by $h(x) := 2 \limsup_{k\to\infty} \varphi(x_k, x)$ for $x \in X$. Note that $h(x) > 0$ for each $x \in X$. If not, then there is $x \in X$ such that $\limsup_{k\to\infty} \varphi(x_k, x) = 0$ and since h has nonnegative values, we have $\liminf_{k\to\infty} \varphi(x_k, x) = 0$ hence $\lim_{k\to\infty} \varphi(x_k, x) = 0$; this is a contradiction.

We are showing that h is proper. To show this we are finding $x \in X$ such that $\varphi(x_k, x) < 2$ for almost all $k \in \mathbb{N}$. Find an index k_0 such that for each $k, j \in \mathbb{N}$, with $k_0 \le k < j$, we have $\varphi(x_j, x_k) < 1$. Let $x := x_{k_0}$, then for each $k, j \in \mathbb{N}$, with $k_0 < k < j$, we have

$$\varphi(x_j, x) \le \varphi(x_j, x_k) + \varphi(x_k, x) = \varphi(x_j, x_k) + \varphi(x_k, x_{k_0}) < 2;$$

therefore $\varphi(x_k, x) < 2$ for almost all $k \in \mathbb{N}$ and $h(x) \le 4 < \infty$, hence $\operatorname{dom} h \ne \emptyset$.

To show that h is $\tau_{\overline{\varphi}}$-lower semicontinuous, fix any $u \in X$ and choose any sequence (u_k) such that $\overline{\varphi}(u, u_k) = \varphi(u_k, u) \to 0$ as $k \to \infty$. Then for each $k \in \mathbb{N}$ we have

$$\begin{aligned} \tfrac{1}{2} h(u) &= \limsup_{j\to\infty} \varphi(x_j, u) \le \limsup_{j\to\infty} \big(\varphi(x_j, u_k) + \varphi(u_k, u)\big) \\ &= \limsup_{j\to\infty} \varphi(x_j, u_k) + \varphi(u_k, u) = \tfrac{1}{2} h(u_k) + \varphi(u_k, u). \end{aligned}$$

Applying $\liminf_{k\to\infty}$ on both sides of the previous inequality, we get

$$\tfrac{1}{2} h(u) = \liminf_{k\to\infty} \big(\tfrac{1}{2} h(u_k) + \varphi(u_k, u)\big) \le \tfrac{1}{2} \liminf_{k\to\infty} h(u_k) + \lim_{k\to\infty} \varphi(u_k, u) = \tfrac{1}{2} \liminf_{k\to\infty} h(u_k).$$

Hence h is $\tau_{\overline{\varphi}}$-lower semicontinuous on X.

Now, we are showing that $\lim_{k\to\infty} h(x_k) = 0$. To see this, fix any $\varepsilon > 0$, find an index $k_0 = k_0(\varepsilon)$ such that for each $k, j \in \mathbb{N}$, with $k_0 \le k < j$, we have $\varphi(x_j, x_k) < \varepsilon/4$. Then for each $k > k_0$ we have

$$h(x_k) = 2 \limsup_{j\to\infty} \varphi(x_j, x_k) \le \varepsilon/2 < \varepsilon.$$

Further, by the assumptions, there is $u \in \operatorname{dom} h$ such that

$$h(u) \le h(x) + \varphi(x, u) \quad \text{for each} \quad x \in X. \tag{4.6}$$

For each $k \in \mathbb{N}$, let $x := x_k$ in (4.6); apply $\limsup_{k\to\infty}$ on both sides, to get

$$h(u) \le \limsup_{k\to\infty} \big(h(x_k) + \varphi(x_k, u)\big) \le \lim_{k\to\infty} h(x_k) + \limsup_{k\to\infty} \varphi(x_k, u) = 1/2h(u) < \infty,$$

therefore $h(u) = 0$, a contradiction. ■

In [5], the authors derived an extension of Ekeland variational principle in $\mathbb{R}^n$ involving a function $f : D \times D \longrightarrow \mathbb{R}$ with D being a closed subset of $\mathbb{R}^n$, and $\varphi(x, u) := \|x - u\|$ for $x, u \in \mathbb{R}^n$.

However, it seems that [5, Theorem 2.1] follows easily from the usual version of the principle (Corollary 4.3.1) similarly as the following slight extension of it.

Corollary 4.3.2 *Let X be nonempty set and a function $\varphi : X \times X \longrightarrow [0, \infty]$ having the properties $(\mathcal{A}_1) - (\mathcal{A}_4)$ and $\bar{x} \in X$ be given. Consider a function $h : X \times X \longrightarrow (-\infty, \infty]$ such that*

(i) *$h(\bar{x}, \cdot)$ is bounded from below and τ_φ-lower semicontinuous;*

(ii) *$h(\bar{x}, \bar{x}) = 0$;*

(iii) *$h(\bar{x}, x) \le h(\bar{x}, u) + h(u, x)$ for each $x, u \in X$.*

Then there exists a point $u \in X$ such that $\varphi(u, \bar{x}) + h(\bar{x}, u) \le 0$ and

$$\varphi(x, u) + h(u, x) > 0 \quad \textit{whenever} \quad x \in X \setminus \{u\}.$$

Proof. Let $\ell := \inf_{x\in X} h(\bar{x}, x)$, then $-\infty < \ell \le 0 < \infty$. By Theorem 4.3.1, with $f := h(\bar{x}, \cdot) - \ell$, there is $u \in X$ such that

$$h(\bar{x}, u) - \ell + \varphi(u, \bar{x}) = f(u) + \varphi(u, \bar{x}) \le f(\bar{x}) = -\ell$$

and for each $x \in X \setminus \{u\}$ we have

$$h(\bar{x}, u) - \ell = f(u) < f(x) + \varphi(x, u) = h(\bar{x}, x) - \ell + \varphi(x, u) \overset{\text{(iii)}}{\le} h(\bar{x}, u) + h(u, x) - \ell + \varphi(x, u),$$

adding ℓ and $\ell - h(\bar{x}, u)$, respectively, we finish the proof. ■

Finally, Theorem 4.3.1 allows us easily prove an extension of Elementary error bound theorem, see [37, Lemma 2.42], [51, Theorem 2] and cf. [29, Lemma 1], in the same spirit.

Corollary 4.3.3 *Let X be nonempty set and a function $\varphi : X \times X \longrightarrow [0, \infty]$ having the properties $(\mathcal{A}_1) - (\mathcal{A}_4)$ and $\bar{x} \in X$ be given. Consider a τ_φ-lower semicontinuous function $f : X \longrightarrow [0, \infty]$ such that $0 < f(\bar{x}) < \infty$. Assume that for each $x \in X$ satisfying*

$$0 < f(x) \le f(\bar{x}) - \varphi(x, \bar{x}) \tag{4.7}$$

there is $x' \in X$ such that

$$f(x') + \varphi(x', x) < f(x).$$

Then there is $u \in X$ such that $f(u) = 0$ and $\varphi(u, \bar{x}) \le f(\bar{x})$.

Proof. By Theorem 4.3.1, there is $u \in X$ such that $f(u) + \varphi(u, \bar{x}) \leq f(\bar{x})$ and

$$f(u) < f(x) + \varphi(x, u) \quad \text{whenever} \quad x \in X \setminus \{u\}. \tag{4.8}$$

We are showing that $f(u) = 0$. On the contrary, we assume that $f(u) > 0$. Then (4.7), with $x := u$, holds and by the assumptions there is $x' \in X$ such that

$$f(x') + \varphi(x', u) < f(u).$$

Combining the last inequality and (4.8), with $x := x'$, we get $f(u) < f(x') + \varphi(x', u) < f(u)$, a contradiction. Hence $f(u) = 0$ and $\varphi(u, \bar{x}) \leq f(\bar{x})$. ■

Chapter 5

Openness in quasi-metric spaces

We extend Ioffe-type criteria (cf. Theorem 2.2.1, Theorem 2.2.3, and Theorem 2.2.5, and their set-valued counterparts) onto the topological space defined in Section 4.2. This general setting allows us to derive some extensions of metric (sub)regularity occurring in the literature, such as nonlinear and directional versions of regularity and subregularity. Ekeland variational principle, specifically Theorem 4.3.1, seems to be very useful for proving these criteria.

For readers' convenience, we re-attach the axioms from Definition 4.2.1.

Let X be a nonempty set and a function $\varphi : X \times X \longrightarrow [0, \infty]$ be given. We say that φ has property:

($\mathcal{A}_1$) provided that $\varphi(x, x) = 0$ for each $x \in X$;

($\mathcal{A}_2$) provided that $\varphi(x, y) \le \varphi(x, z) + \varphi(z, y)$ whenever $x, y, z \in X$;

($\mathcal{A}_3$) provided that $\varphi(x, y) > 0$ whenever $x, y \in X$ are distinct;

($\mathcal{A}_4$) provided that for each (x_k) in X such that for each $\varepsilon > 0$ there is an index $k_0 = k_0(\varepsilon)$ such that for each $k, j \in \mathbb{N}$, with $k_0 \le k < j$, we have $\varphi(x_j, x_k) < \varepsilon$; there is a point $u \in X$ such that $\varphi(u, x_k) \to 0$ as $k \to \infty$.

Throughout this chapter, we always suppose that the sets X and Y are nonempty and a point $(\bar{x}, \bar{y}) \in X \times Y$ is given, and we consider functions $\varphi, \gamma : X \times X \longrightarrow [0, \infty]$ and $\psi, \varrho : Y \times Y \longrightarrow [0, \infty]$, which always have the property ($\mathcal{A}_1$).

In [41, Theorem 2] and [21, Theorem 1.5 and Theorem 1.6] there is considered a function φ, which has the property ($\mathcal{A}_2$) and $\overline{\varphi}$ has the property ($\mathcal{A}_4$), to prove generalizations of Banach open mapping theorem for a set-valued mapping and generalizations of Lyusternik theorem, respectively.

5.1 Nonlinear and directional regularity

We study three very general properties of a set-valued mapping. The first property is a generalized version of the openness with a linear rate around the reference point (cf. Definition 1.2.10).

Definition 5.1.1 *Let a set-valued mapping $F : X \rightrightarrows Y$, with $\bar{y} \in F(\bar{x})$, be given. The mapping F is said to be* $(\varphi, \gamma, \psi, \varrho)$-open around $(\bar{x}, \bar{y})$ *if there is $r > 0$ such that*

(5.1) $\quad F\left(\mathbb{B}^{\varphi}_X[x, t]\right) \supset \mathbb{B}^{\psi}_Y[v, t]$ *whenever* $x \in \mathbb{B}^{\gamma}_X[\bar{x}, r]$, $v \in \mathbb{B}^{\varrho}_Y[\bar{y}, r] \cap F(x)$, *and* $t \in (0, r]$.

Recall, for a set $U \subset X$ and a function φ, that the distance function is given by

$$X \ni x \longmapsto \operatorname{dist}^{\varphi}_X(x, U) := \inf\{\varphi(x, u) : \quad \text{for} \quad u \in U\} \in [0, \infty].$$

Note that we only write $\operatorname{dist}(x, U)$, when φ is a metric on X compatible with the topology on X.

Remark 5.1.1 *When a function φ has the property ($\mathcal{A}_2$) and a set $U \subset X$, then the distance function for each $x, u \in X$ satisfies*

$$\mathrm{dist}_X^{\varphi}(u, U) \le \mathrm{dist}_X^{\varphi}(x, U) + \varphi(u, x).$$

Indeed, fix any $x, u \in X$ and $\varepsilon > 0$. If either $\mathrm{dist}_X^{\varphi}(x, U)$ or $\varphi(u, x)$ are infinite, then the inequality holds. If not, then find $y \in U$ such that

$$\varphi(x, y) \le \mathrm{dist}_X^{\varphi}(x, U) + \varepsilon.$$

Then

$$\mathrm{dist}_X^{\varphi}(u, U) \le \varphi(u, y) \le \varphi(u, x) + \varphi(x, y) \le \varphi(u, x) + \mathrm{dist}_X^{\varphi}(x, U) + \varepsilon.$$

Letting $\varepsilon \to 0$, we get the desired inequality.

The following lemma gives us a relation between the openness from Definition 5.1.1 and a corresponding generalization of metric regularity (cf. Definition 1.2.9).

Lemma 5.1.1 *Let a function ψ have the property ($\mathcal{A}_3$) and a constant $r > 0$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$, with $\bar{y} \in F(\bar{x})$, satisfying* (5.1) *with $r := 2r$. Then for any $x \in I\!B_X^{\gamma}[\bar{x}, 2r]$, any $v \in I\!B_Y^{\varrho}[\bar{y}, 2r] \cap F(x)$, and any $y \in I\!B_Y^{\varrho}[\bar{y}, 2r]$, with $\psi(v, y) \le 2r$, we have*

$$\mathrm{dist}_X^{\varphi}(x, F^{-1}(y)) \le \psi(v, y). \tag{5.2}$$

In addition, we have:

(i) *if ϱ has the property ($\mathcal{A}_2$) and $\varrho(y, v) \le \psi(v, y)$ for each $y, v \in Y$, then for any $x \in I\!B_X^{\gamma}[\bar{x}, r]$ and any $y \in I\!B_Y^{\varrho}[\bar{y}, r]$ such that*

$$\mathrm{dist}_Y^{\overline{\psi}}(y, F(x)) < r,$$

we have

$$\mathrm{dist}_X^{\varphi}(x, F^{-1}(y)) \le \mathrm{dist}_Y^{\overline{\psi}}(y, F(x)). \tag{5.3}$$

(ii) *if φ and ϱ have the property ($\mathcal{A}_2$), $\varphi(x, \bar{x}) \le \gamma(\bar{x}, x)$ for each $x \in X$, and $\varrho = \psi$, then for any $x \in I\!B_X^{\gamma}[\bar{x}, r/2]$ and any $y \in I\!B_Y^{\varrho}[\bar{y}, r/2]$, we have*

$$\mathrm{dist}_X^{\varphi}(x, F^{-1}(y)) \le \mathrm{dist}_Y^{\overline{\varrho}}(y, F(x)). \tag{5.4}$$

Proof. Fix any $x \in I\!B_X^{\gamma}[\bar{x}, 2r]$, any $v \in I\!B_Y^{\varrho}[\bar{y}, 2r] \cap F(x)$, and any $y \in I\!B_Y^{\varrho}[\bar{y}, 2r]$ with $\psi(v, y) \le 2r$. If $y = v$, then (5.2) holds. Assume that $y \ne v$. Let $t := \psi(v, y) > 0$, therefore $t \le 2r$.

Further, by (5.1), there is $x' \in F^{-1}(y)$ such that

$$\mathrm{dist}_X^{\varphi}(x, F^{-1}(y)) \le \varphi(x, x') \le t = \psi(v, y).$$

Therefore we get (5.2).

To see (i), assume that ϱ has the property ($\mathcal{A}_2$) and that $\varrho(y, v) \le \psi(v, y)$ for each $y, v \in Y$. Fix any $x \in I\!B_X^{\gamma}[\bar{x}, r]$ and any $y \in I\!B_Y^{\varrho}[\bar{y}, r]$ such that $\mathrm{dist}_Y^{\overline{\psi}}(y, F(x)) < r$. Then for each $v \in F(x)$ such that $\psi(v, y) < r$, we have

$$\varrho(\bar{y}, v) \le \varrho(\bar{y}, y) + \varrho(y, v) \le r + \psi(v, y) < 2r;$$

hence $v \in I\!B_Y^{\varrho}[\bar{y}, 2r] \cap F(x)$. Then for such x, y, and for each $v \in F(x)$, with $\psi(v, y) < r$, (5.2) holds. Since $F(x) \cap I\!B_Y^{\overline{\psi}}(y, r) \ne \emptyset$, then

$$\mathrm{dist}_Y^{\overline{\psi}}(y, F(x)) = \mathrm{dist}_Y^{\overline{\psi}}(y, F(x) \cap I\!B_Y^{\overline{\psi}}(y, r)).$$

Thus taking an infimum of $\overline{\psi}(y,v)$ over $v \in F(x)$ on the right-hand side of the (5.2), we get (5.3).

To see (ii), assume that φ has the property $(\mathcal{A}_2)$, ϱ has the property $(\mathcal{A}_2)$, $\varphi(x,\bar{x}) \le \gamma(\bar{x},x)$ for each $x \in X$, and $\varrho = \psi$. Fix any $x \in I\!B_X^{\gamma}[\bar{x}, r/2]$, any $y \in I\!B_Y^{\varrho}[\bar{y}, r/2]$, and any $v \in F(x)$. If $\varrho(v,y) \le r$, then

$$\varrho(\bar{y},v) \le \varrho(\bar{y},y) + \varrho(y,v) \le r/2 + r < 2r.$$

For such x, y, and v, we get

$$\operatorname{dist}_X^{\varphi}(x, F^{-1}(y)) \le \overline{\varrho}(y,v). \tag{5.5}$$

Now, suppose that $\varrho(v,y) > r$, then, by Remark 5.1.1 and (5.2), with $v := \bar{y}$ and $x := \bar{x}$, we have

$$\operatorname{dist}_X^{\varphi}(x, F^{-1}(y)) \le \operatorname{dist}_X^{\varphi}(\bar{x}, F^{-1}(y)) + \varphi(x,\bar{x}) \le \varrho(\bar{y},y) + \gamma(\bar{x},x) \le r < \varrho(v,y) = \overline{\varrho}(y,v).$$

Therefore for each $x \in I\!B_X^{\gamma}[\bar{x}, r/2]$, each $y \in I\!B_Y^{\varrho}[\bar{y}, r/2]$, and each $v \in F(x)$ we ge (5.5).

For such fixed x and y, taking an infimum over $v \in F(x)$ on the right hand side, we get (5.4). ■

In the following examples, we use the directional minimal time function T_L, for its definition, see Example 4.2.1 and, for $x \in X$ and $U \subset X$, by $T_L(x,U)$ we mean

$$T_L(x,U) := \operatorname{dist}_X^{T_L}(x,U).$$

We are able to deduce from (5.1) some types of nonlinear and directional versions of metric regularity and openness of a mapping occurring in the literature.

Example 5.1.1 *Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$.*

(i) *Let (X,d) and (Y,ϱ) be metric spaces. Consider a continuous stricly increasing function $\phi : [0,\infty] \longrightarrow [0,\infty]$ with $\phi(0) = 0$. Suppose that there is $r > 0$ such that* (5.1)*, with $\varphi := \gamma := d$ and $\psi := \phi \circ \varrho$, holds. For any $y \in Y$ and any $t \in (0, \phi(r)]$ we have*

$$I\!B_Y^{\phi\circ\varrho}[y,t] = \{v \in Y : \phi(\varrho(y,v)) \le t\} = \{v \in Y : \varrho(y,v) \le \phi^{-1}(t)\} = I\!B_Y^{\varrho}\left[y, \phi^{-1}(t)\right]. \tag{5.6}$$

Thus we have a nonlinear version of the openness around the reference point in the form: for any $x \in I\!B_X[\bar{x}, r]$ and any $v \in I\!B_Y[\bar{y}, r] \cap F(x)$ we have

$$F\left(I\!B_X[x,t]\right) \supset I\!B_Y[v, \phi^{-1}(t)] \quad \text{for each} \quad t \in (0, \min\{r, \phi(r)\}].$$

(ii) *Let (X,d) and (Y,ϱ) be metric spaces. Suppose that there are $r > 0$, $\kappa > 0$, and $k \ge 1$ such that* (5.1)*, with $r := 4r$, $\varphi := \gamma := d$, and $\psi := \kappa(\varrho)^{1/k}$, holds.*

By Lemma 5.1.1(ii), with $\varphi := \gamma := d$, $\psi := \varrho := \kappa(\varrho)^{1/k}$, and $r := 2r$, see (5.6)*, with $\phi := \kappa(\cdot)^{1/k}$ and $t := r$, and Lemma A.3.3, we get the following property: for any $x \in I\!B_X[\bar{x}, r]$ and any $y \in I\!B_Y\left[\bar{y}, (r/\kappa)^k\right]$ we have*

$$\operatorname{dist}(x, F^{-1}(y)) \le \kappa \operatorname{dist}(y, F(x))^{1/k}.$$

Such an F is said to be metrically regular of order k at $(\bar{x},\bar{y})$ in [30, Definition 1.2].

(iii) *Let (X,d) be a metric space, $(Y, \|\cdot\|_Y)$ be a normed space, and M be a convex compact subset of Y. Note that* cone M *is convex set. Let $K := \mathbb{S}_Y \cap \operatorname{cone} M$, then $\operatorname{cone} M = \operatorname{cone} K$. Suppose that there are $r > 0$, $\kappa > 0$, and $k \ge 1$ such that* (5.1)*, with $\varphi := \gamma := d$ and $\varrho(y,v) := \|y - v\|_Y$ and $\psi(y,v) := \kappa T_K(y,v)^{1/k}$ for $y, v \in Y$, holds.*

By Lemma 5.1.1, with $r := r/2$, $\varphi := \gamma := d$, *and* $\varrho(y,v) := \|y-v\|_Y$ *and* $\psi(y,v) := \kappa T_K(y,v)^{1/k}$ *for* $y, v \in Y$, *we have that for any* $x \in I\!B_X[\bar{x}, r]$, *any* $v \in I\!B_Y[\bar{y}, r] \cap F(x)$, *and any* $y \in I\!B_Y[\bar{y}, r]$, *with* $\kappa T_K(v,y)^{1/k} \leq r$, *we have*

$$\operatorname{dist}(x, F^{-1}(y)) \leq \kappa T_K(v,y)^{1/k} = \kappa\|v-y\|_Y^{1/k}.$$

Let $r' := \frac{1}{2}\min\{r, (r/\kappa)^k\}$, *then we get property: for any* $x \in I\!B_X[\bar{x}, r']$ *and any* $v \in I\!B_Y[\bar{y}, r'] \cap F(x)$ *we have*

$$\operatorname{dist}(x, F^{-1}(y)) \leq \kappa\|v-y\|_Y^{1/k} \quad \textit{for each} \quad y \in I\!B_Y[\bar{y}, r'] \cap \big(v + \operatorname{cone} M\big).$$

Such an F *is said to be metrically regular of order* k *at* $(\bar{x}, \bar{y})$ *with respect to* M *in [30, Definition 5.1].*

(iv) *Let* $(X, \|\cdot\|_X)$, *and* $(Y, \|\cdot\|_Y)$ *be normed spaces,* M *be a nonempty subset of* $\mathbb{S}_Y$, *and* L *be a nonempty subset of* $\mathbb{S}_X$. *Suppose that there are* $r > 0$ *and* $\kappa > 0$ *such that* (5.1), *with* $r := 2r$, $\varphi(x,u) := \frac{1}{\kappa}T_L(x,u)$ *and* $\gamma(x,u) := \|x-u\|_X$ *for* $x, u \in X$, *and* $\varrho(y,v) := \|y-v\|_Y$ *and* $\psi(y,v) := T_{-M}(y,v)$ *for* $y, v \in Y$, *holds. Then for any* $x \in I\!B_X[\bar{x}, 2r]$ *and any* $v \in I\!B_Y[\bar{y}, 2r] \cap F(x)$ *we have*

$$F\big(I\!B_X[x,t] \cap (x + \operatorname{cone} L)\big) \supset I\!B_Y\left[v, \kappa^{-1}t\right] \cap (v - \operatorname{cone} M) \quad \textit{for each} \quad t \in (0, 2\kappa r].$$

By Lemma 5.1.1(i), *with* $\varphi(x,u) := \frac{1}{\kappa}T_L(x,u)$ *and* $\gamma(x,u) := \|x-u\|_X$ *for* $x, u \in X$, *and* $\varrho(y,v) := \|y-v\|_Y$ *and* $\psi(y,v) := T_{-M}(y,v)$ *for* $y, v \in Y$, *we get the following property: for any* $x \in I\!B_X[\bar{x}, r]$ *and any* $y \in I\!B_Y[\bar{y}, r]$, *with* $T_M(y, F(x)) < r$, *we have*

$$T_L(x, F^{-1}(y)) \leq \kappa T_M(y, F(x)).$$

Such an F *is said to be directionally metrically regular around* $(\bar{x}, \bar{y})$ *with respect to* L *and* M *with the constant* κ *in [12, Definition 1] and [25, Definition 2.2].*

Further, we study a generalized version of the pseudo-openness with a linear rate at the reference point (cf. Definition 1.2.6).

Definition 5.1.2 *Let a set-valued mapping* $F : X \rightrightarrows Y$, *with* $\bar{y} \in F(\bar{x})$, *be given. The mapping* F *is said to be* $(\varphi, \gamma, \varrho)$*-pseudo-open at* $(\bar{x}, \bar{y})$, *if there is* $r > 0$ *such that*

(5.7) $$F\big(I\!B_X^{\varphi}[x,t]\big) \ni \bar{y} \quad \textit{whenever} \quad x \in I\!B_X^{\gamma}[\bar{x}, r] \quad \textit{and} \quad t \in (0, r] \textit{ with } I\!B_Y^{\varrho}[\bar{y}, t] \cap F(x) \neq \emptyset.$$

The following lemma derives a relation between $(\varphi, \gamma, \varrho)$-pseudo-openness and corresponding generalized version of metric subregularity (cf. Definition 1.2.5).

Lemma 5.1.2 *Let a function* $\varrho : Y \times Y \longrightarrow [0, \infty]$ *have the property* $(\mathcal{A}_3)$ *and a constant* $r > 0$ *be given. Consider a set-valued mapping* $F : X \rightrightarrows Y$, *with* $\bar{y} \in F(\bar{x})$, *satisfying* (5.7).

Then for any $x \in I\!B_X^{\gamma}[\bar{x}, r]$ *and any* $v \in I\!B_Y^{\varrho}[\bar{y}, r] \cap F(x)$ *we have*

(5.8) $$\operatorname{dist}_X^{\varphi}(x, F^{-1}(\bar{y})) \leq \varrho(\bar{y}, v).$$

Moreover, if $\varphi(x, \bar{x}) \leq \gamma(\bar{x}, x)$ *for each* $x \in I\!B_X^{\gamma}[\bar{x}, r]$, *then for any* $x \in I\!B_X^{\gamma}[\bar{x}, r]$ *we have*

(5.9) $$\operatorname{dist}_X^{\varphi}(x, F^{-1}(\bar{y})) \leq \operatorname{dist}_Y^{\varrho}(\bar{y}, F(x)).$$

Proof. To see this, fix any $x \in I\!B_X^{\gamma}[\bar{x}, r]$. If $I\!B_Y^{\varrho}[\bar{y}, r] \cap F(x) = \emptyset$, then there is nothing to prove. Suppose that this is not the case. Fix any $v \in I\!B_Y^{\varrho}[\bar{y}, r] \cap F(x)$. If $v = \bar{y}$, then, clearly, (5.8) holds. Suppose that $v \neq \bar{y}$. Let $t := \varrho(\bar{y}, v) > 0$, then $I\!B_Y^{\varrho}[\bar{y}, t] \cap F(x) \neq \emptyset$, $t \leq r$, and, by (5.7), there is $x' \in F^{-1}(\bar{y})$ such that

$$\operatorname{dist}_X^{\varphi}(x, F^{-1}(\bar{y})) \leq \varphi(x, x') \leq t = \varrho(\bar{y}, v).$$

Further, assume that $\varphi(x, \bar{x}) \leq \gamma(\bar{x}, x)$ for each $x \in I\!B_X^{\gamma}[\bar{x}, r]$. Fix any $x \in I\!B_X^{\gamma}[\bar{x}, r]$ and any $v \in F(x)$. If $\varrho(\bar{y}, v) \leq r$, then (5.8) holds. Suppose that $\varrho(\bar{y}, v) > r$, then

$$\operatorname{dist}_X^{\varphi}(x, F^{-1}(\bar{y})) \leq \varphi(x, \bar{x}) \leq \gamma(\bar{x}, x) \leq r < \varrho(\bar{y}, v).$$

To sum up, we get that for each $x \in I\!B_X^{\gamma}[\bar{x}, r]$ and each $v \in F(x)$ we have

$$\operatorname{dist}_X^{\varphi}(x, F^{-1}(\bar{y})) \leq \varrho(\bar{y}, v).$$

For such a fixed x, taking infimum over $v \in F(x)$ on the right hand side, we get (5.9). ■

Further, we are able to deduce from (5.7) some nonlinear and directional versions of metric subregularity and pseudo-openness of a mapping occurring in the literature.

Example 5.1.2 *Consider a set-valued mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$.*

(i) *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Suppose that there are positive numbers r, κ, and q such that* (5.7)*, with $\gamma(x, u) := \varphi(x, u) := \|x - u\|_X$ for $x, u \in X$ and $\varrho(y, v) := \kappa\|y - v\|_Y^{1/q}$ for $y, v \in Y$, holds. By Lemma 5.1.2 we have*

$$\operatorname{dist}(x, F^{-1}(\bar{y})) \leq \kappa \operatorname{dist}(\bar{y}, F(x))^{1/q} \quad \text{for each} \quad x \in I\!B_X[\bar{x}, r].$$

Such an F is said to be metrically q-subregular at $(\bar{x}, \bar{y})$ in [46, Definition 3.1 (i)].

(ii) *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, $d \in X$, and $\delta > 0$ be given. Let $L := \mathbb{S}_X \cap$ cone $I\!B_X[d, \delta]$, then* cone $L =$ cone $I\!B_X[d, \delta]$*. Suppose that there are $r > 0$, $\kappa > 0$, and $q \in (0, 1]$ such that* (5.7)*, with $\varphi(x, u) := \|x - u\|_X$ and $\gamma(x, u) := T_L(x, u)$ for $x, u \in X$ and $\varrho(y, v) := \kappa\|y - v\|_Y^q$ for $y, v \in Y$, holds. By Lemma 5.1.2 we have*

$$\operatorname{dist}(x, F^{-1}(\bar{y})) \leq \kappa \operatorname{dist}(\bar{y}, F(x))^q \quad \text{for each} \quad x \in I\!B_X[\bar{x}, r] \cap \big(\bar{x} + \operatorname{cone} I\!B_X[d, \delta]\big).$$

Such an F is said to be directionally Hölder metrically subregular at $(\bar{x}, \bar{y})$ in [47, p. 4].

(iii) *Let (X, d) and (Y, ϱ) be metric spaces. Consider a strictly increasing function $\phi : [0, \infty] \longrightarrow [0, \infty]$ with $\phi(0) = 0$. Suppose that there is $r > 0$ such that* (5.7)*, with $\varphi := \gamma := d$ and $\varrho := \phi \circ \varrho$, holds. Then, by* (5.6)*, we have a nonlinear version of pseudo-openness in the form: for any $x \in I\!B_X[\bar{x}, r]$ and any $t \in (0, \min\{r, \phi(r)\}]$, with $I\!B_Y\big[\bar{y}, \phi^{-1}(t)\big] \cap F(x) \neq \emptyset$, we have*

$$F\left(I\!B_X[x, t]\right) \ni \bar{y}.$$

Now, we focus on a generalized version of the openness with a linear rate at the reference point (cf. Definition 1.2.2).

Definition 5.1.3 *Let a set-valued mapping $F : X \rightrightarrows Y$, with $\bar{y} \in F(\bar{x})$, be given. The mapping F is said to be (φ, ψ)-open at $(\bar{x}, \bar{y})$, if there is $r > 0$ such that*

$$F\left(I\!B_X^{\varphi}[\bar{x}, t]\right) \supset I\!B_Y^{\psi}[\bar{y}, t] \quad \text{for each} \quad t \in (0, r]. \tag{5.10}$$

From this property, we derive some nonlinear and directional versions of the openness, however, to the best our knowledge, these properties are not studied in the literature so far.

Example 5.1.3 *Consider a set-valued mapping $F : X \rightrightarrows Y$, with $\bar{y} \in F(\bar{x})$, be given.*

(i) *Let (X, d) and (Y, ϱ) be metric spaces. Consider a strictly increasing function $\phi : [0, \infty] \longrightarrow [0, \infty]$ with $\phi(0) = 0$. Suppose that there is $r > 0$ such that* (5.10)*, with $\varphi := d$ and $\psi := \phi \circ \varrho$, holds.*

Therefore, by (5.6)*, we get the nonlinear version of the openness at the reference point in the form:*

$$F(\mathbb{B}_X[\bar{x}, t]) \supset \mathbb{B}_Y[\bar{y}, \phi^{-1}(t)] \quad \text{for each} \quad t \in (0, \min\{r, \phi(r)\}].$$

(ii) *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, M be a nonempty subset of $\mathbb{S}_Y$, and L be a nonempty subset of $\mathbb{S}_X$. Suppose that there are $r > 0$ and $c > 0$ such that* (5.10)*, with $\varphi(x, u) := T_L(x, u)$ for $x, u \in X$ and $\psi(y, v) := \frac{1}{c} T_M(y, v)$ for $y, v \in Y$, holds.*

Therefore we get the directional version of the openness at the reference point in the form:

$$F(\mathbb{B}_X[\bar{x}, t] \cap (\bar{x} + \operatorname{cone} L)) \supset \mathbb{B}_Y[\bar{y}, ct] \cap (\bar{y} + \operatorname{cone} M) \quad \text{for each} \quad t \in (0, r].$$

5.2 Semiregularity criteria

We start with the extension of Theorem 2.2.5, that gives us sufficient conditions for the property from Definition 5.1.3 for a single-valued mapping.

Proposition 5.2.1 *Let a function φ have the properties $(\mathcal{A}_2)-(\mathcal{A}_4)$ and satisfy $\gamma(x, \bar{x}) \leq \varphi(x, \bar{x})$ for each $x \in X$, and a function ψ have the property $(\mathcal{A}_3)$. Consider a single-valued mapping $g : X \longrightarrow Y$, defined on whole X, such that for each $y \in Y$ the mapping $X \ni x \longmapsto \psi(g(x), y)$ is τ_φ-lower semicontinuous. Suppose that there is $r > 0$ such that for any $x \in \mathbb{B}_X^{\overline{\gamma}}[\bar{x}, r]$ and any $y \in \mathbb{B}_Y^{\psi}[g(\bar{x}), r]$ satisfying*

$$0 < \psi(g(x), y) \leq \psi(g(\bar{x}), y) - \varphi(x, \bar{x}), \tag{5.11}$$

there is a point $x' \in X$ satisfying

$$\varphi(x', x) < \psi(g(x), y) - \psi(g(x'), y). \tag{5.12}$$

Then

$$g\big(\mathbb{B}_X^{\overline{\varphi}}[\bar{x}, t]\big) \supset \mathbb{B}_Y^{\psi}[g(\bar{x}), t] \quad \text{for each} \quad t \in (0, r]. \tag{5.13}$$

Proof. Fix any $t \in (0, r]$. Pick an arbitrary $y \in \mathbb{B}_Y^{\psi}[g(\bar{x}), t]$. If $y = g(\bar{x})$, then (5.13) holds trivially. Suppose that $y \neq g(\bar{x})$. Since the mapping $X \ni x \longmapsto \psi(g(x), y)$ is τ_φ-lower semicontinuous, applying Theorem 4.3.1, with $f := \psi(g(\cdot), y)$, we find $u \in X$ such that

$$\psi(g(u), y) + \varphi(u, \bar{x}) \leq \psi(g(\bar{x}), y) \tag{5.14}$$

and

$$\psi(g(u), y) < \psi(g(v), y) + \varphi(v, u) \quad \text{whenever} \quad v \in X \setminus \{u\}. \tag{5.15}$$

By (5.14), we have

$$\overline{\gamma}(\bar{x}, u) \leq \overline{\varphi}(\bar{x}, u) = \varphi(u, \bar{x}) \leq \psi(g(\bar{x}), y) - \psi(g(u), y) \leq \psi(g(\bar{x}), y) \leq t.$$

Therefore $u \in \mathbb{B}_X^{\overline{\varphi}}[\bar{x}, t] \subset \mathbb{B}_X^{\overline{\gamma}}[\bar{x}, r]$ and $\psi(g(u), y) \le t$. As $y \in \mathbb{B}_Y^{\psi}[g(\bar{x}), t]$ is arbitrary, (5.13) will follow once we show that $y = g(u)$.

Suppose on the contrary that $y \neq g(u)$. Therefore (5.11), with $x := u$, holds. Thus find $x' \in X$ such that (5.12), with $x := u$, holds. Clearly, $\psi(g(x'), y) < \infty$ and $x' \neq u$. Combining (5.15), with $v := x'$, and (5.12), with $x := u$, we get

$$\varphi(x', u) < \psi(g(u), y) - \psi(g(x'), y) < \varphi(x', u),$$

a contradiction. Hence $y = g(u)$. ■

Unlike Definition 5.1.3, the previous statement contains the function γ, but it can be useful in applications (see Proposition 5.4.1). In the simplest case, we can set $\gamma := \varphi$.

The above result contains Theorem 2.2.5 as well as its directional version which seems to be new. The statement is in the spirit of [12, Proposition 11] and contains sufficient conditions guaranteeing the property in Example 5.1.3 (ii).

Corollary 5.2.1 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and nonempty closed sets $L \subset \mathbb{S}_X$ and $M \subset \mathbb{S}_Y$ be such that* cone L *is convex. Consider a continuous single-valued mapping $g : X \longrightarrow Y$, defined on whole X, for which there are positive constants c and r such that for any $x \in \mathbb{B}_X[\bar{x}, r]$ and any $y \in \mathbb{B}_Y[g(\bar{x}), cr] \cap (g(\bar{x}) + \operatorname{cone} M)$ satisfying*

$$0 < T_M(g(x), y) \le T_M(g(\bar{x}), y) - c\,T_L(\bar{x}, x), \tag{5.16}$$

there is a point $x' \in X$ satisfying

$$c\,T_L(x, x') < T_M(g(x), y) - T_M(g(x'), y). \tag{5.17}$$

Then

$$g\big(\mathbb{B}_X[\bar{x}, t] \cap (\bar{x} + \operatorname{cone} L)\big) \supset \mathbb{B}_Y[g(\bar{x}), ct] \cap (g(\bar{x}) + \operatorname{cone} M) \quad \textit{for each} \quad t \in (0, r].$$

Proof. Fix any $x \in \mathbb{B}_X[\bar{x}, r]$ and any $y \in \mathbb{B}_Y[g(\bar{x}), cr] \cap (g(\bar{x}) + \operatorname{cone} M)$ such that (5.16) holds. Then $x \in \mathbb{B}_X[\bar{x}, r] \cap (\bar{x} + \operatorname{cone} L)$ and there is $x' \in X$ such that (5.17) holds. Hence it suffices to apply Proposition 5.2.1, with $\varphi(x, u) := T_{-L}(x, u)$ and $\gamma(x, u) := \|x - u\|_X$ for x, $u \in X$, and $\psi(y, v) := \frac{1}{c} T_M(y, v)$ for y, $v \in Y$. Indeed, in view of Example 4.2.1, it is enough to observe that the mapping $X \ni x \longmapsto T_M(g(x), y)$ is lower semicontinuous on X, since g is continuous on X and $T_M(\cdot, y)$ is lower semicontinuous because M is a closed subset of $\mathbb{S}_Y$. ■

Proposition 5.2.1 also contains a criterion for a nonlinear version of semiregularity in Example 5.1.3(i).

Corollary 5.2.2 *Let a function φ have the properties $(\mathcal{A}_2) - (\mathcal{A}_4)$ and satisfy $\gamma(x, \bar{x}) \le \varphi(x, \bar{x})$ for each $x \in X$, and a function ϱ have the property $(\mathcal{A}_3)$. Consider a single-valued mapping $g : X \longrightarrow Y$, defined on whole X, such that for each $y \in Y$ the mapping $X \ni x \longmapsto \varrho(g(x), y)$ is τ_φ-lower semicontinuous, along with a continuous strictly increasing function $\phi : [0, \infty] \longrightarrow [0, \infty]$ such that $\phi(0) = 0$. Suppose that there is $r > 0$ such that for any $x \in \mathbb{B}_X^{\overline{\gamma}}[\bar{x}, r]$ and any $y \in \mathbb{B}_Y^{\varrho}[g(\bar{x}), r]$ satisfying*

$$0 < \phi(\varrho(g(x), y)) \le \phi(\varrho(g(\bar{x}), y)) - \varphi(x, \bar{x}), \tag{5.18}$$

there is a point $x' \in X$ satisfying

$$\varphi(x', x) < \phi(\varrho(g(x), y)) - \phi(\varrho(g(x'), y)). \tag{5.19}$$

Then

$$g\big(\mathbb{B}_X^{\overline{\varphi}}[\bar{x}, t]\big) \supset \mathbb{B}_Y^{\varrho}\big[g(\bar{x}), \phi^{-1}(t)\big] \quad \textit{for each} \quad t \in (0, \min\{r, \phi(r)\}].$$

Proof. Let $\psi := \phi \circ \varrho$. Then ψ has the properties $(\mathcal{A}_1)$ and $(\mathcal{A}_3)$. Since ϕ is continuous, we have that for each $y \in Y$ the mapping $X \ni x \longmapsto \psi(g(x), y)$ is τ_φ-lower semicontinuous. By assumptions, for any $x \in I\!B_X^{\overline{\gamma}}[\bar{x}, r]$ and any $y \in I\!B_Y^{\psi}[g(\bar{x}), \phi(r)]$ satisfying (5.18) there is $x' \in X$ such that (5.19) holds.

Proposition 5.2.1, with $r := \min\{r, \phi(r)\}$, implies that

$$g\big(I\!B_X^{\overline{\varphi}}[\bar{x}, t]\big) \supset I\!B_Y^{\psi}\big[g(\bar{x}), t\big] \quad \text{for each} \quad t \in (0, \min\{r, \phi(r)\}].$$

By (5.6), we get $g\big(I\!B_X^{\overline{\varphi}}[\bar{x}, t]\big) \supset I\!B_Y^{\varrho}\big[g(\bar{x}), \phi^{-1}(t)\big]$ for each $t \in (0, \min\{r, \phi(r)\}]$. ■

Of course, we are also able to derive a necessary condition in the spirit of Theorem 2.2.1.

Proposition 5.2.2 *Let a function φ have the properties $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$ and a function ψ have the property $(\mathcal{A}_3)$. Consider a single-valued mapping $g : X \longrightarrow Y$, which is τ_φ-τ_ψ-continuous on X. Assume that there is a positive constant r such that*

$$g\big(I\!B_X^{\varphi}[\bar{x}, t]\big) \supset I\!B_Y^{\psi}[g(\bar{x}), t] \quad \textit{for each} \quad t \in (0, r].$$

Then for any $x \in I\!B_X^{\varphi}[\bar{x}, r]$ and any $y \in I\!B_Y^{\psi}[g(\bar{x}), r]$ satisfying

$$0 < \psi(g(\bar{x}), y) \leq \psi(g(x), y) - \varphi(x, \bar{x}) \tag{5.20}$$

there is a point $x' \in X$ such that

$$\psi(g(x'), y) \leq \psi(g(x), y) - \varphi(x, x').$$

Proof. Pick any $(x, y) \in I\!B_X^{\varphi}[\bar{x}, r] \times I\!B_Y^{\psi}[g(\bar{x}), r]$ satisfying (5.20). Let $t := \psi(g(\bar{x}), y)$. The choice of y implies that $0 < t \leq r$. As $y \in I\!B_Y^{\psi}[g(\bar{x}), t]$ there is $x' \in I\!B_X^{\varphi}[\bar{x}, t]$ such that $g(x') = y$. Then

$$\varphi(x, x') \leq \varphi(x, \bar{x}) + t \overset{(5.20)}{\leq} \psi(g(x), y) - \psi(g(\bar{x}), y) + t = \psi(g(x), y) = \psi(g(x), y) - \psi(g(x'), y).$$

■

Further, we are able to derive a criterion for semiregularity of a set-valued mapping using the restriction of the canonical projection to the graph of this mapping.

Proposition 5.2.3 *Let a function φ have the properties $(\mathcal{A}_2) - (\mathcal{A}_4)$ and satisfy $\gamma(x, \bar{x}) \leq \varphi(x, \bar{x})$ for each $x \in X$, a function ϱ have the properties $(\mathcal{A}_2) - (\mathcal{A}_4)$, a function ψ have the property $(\mathcal{A}_3)$, and a constant $\alpha \in (0, 1)$ be given. Assume that $\omega : (X \times Y)^2 \longrightarrow [0, \infty]$ is defined by (4.2), and that for each $y \in Y$ the mapping $Y \ni v \longmapsto \psi(v, y)$ is τ_ϱ-lower semicontinuous. Consider a set-valued mapping $F : X \rightrightarrows Y$, with $\bar{y} \in F(\bar{x})$, such that the set $\operatorname{gph} F$ is τ_ω-closed. Suppose that there is $r > 0$ such that for any $x \in I\!B_X^{\overline{\gamma}}[\bar{x}, r]$, any $v \in I\!B_Y^{\overline{\varrho}}[\bar{y}, r/\alpha] \cap F(x)$, and any $y \in I\!B_Y^{\psi}[\bar{y}, r]$ satisfying*

$$0 < \psi(v, y) \leq \psi(\bar{y}, y) - \max\{\varphi(x, \bar{x}), \alpha\varrho(v, \bar{y})\}, \tag{5.21}$$

there is a pair $(x', v') \in \operatorname{gph} F$ such that

$$\max\{\varphi(x', x), \alpha\varrho(v', v)\} < \psi(v, y) - \psi(v', y). \tag{5.22}$$

Then $F\big(I\!B_X^{\overline{\varphi}}[\bar{x}, t]\big) \supset I\!B_Y^{\psi}[\bar{y}, t]$ for each $t \in (0, r]$.

Proof. Let $\widetilde{X} := \operatorname{gph} F$. We already know that ω has the properties $(\mathcal{A}_1) - (\mathcal{A}_4)$. Define a function $(X \times Y)^2 \ni \big((x,y),(u,v)\big) \longmapsto \chi\big((x,y),(u,v)\big)$ by $\chi((x,y),(u,v)) := \max\{\gamma(x,u), \alpha\varrho(y,v)\}$ for $(x,y),(u,v) \in X \times Y$, then $\overline{\chi}\big((\bar{x},\bar{y}),(x,y)\big) \leq \overline{\omega}\big((\bar{x},\bar{y}),(x,y)\big)$ for each $(x,y) \in X \times Y$ and χ has the property $(\mathcal{A}_1)$.

Define a single-valued mapping $g : \widetilde{X} \longrightarrow Y$ by $g(x,y) := y$ for $(x,y) \in \widetilde{X}$. The mapping g is τ_ω-to-τ_ϱ-continuous on $\widetilde{X}$, hence for each $y \in Y$ the mapping $\widetilde{X} \ni (x,v) \longmapsto \psi(g(x,v),y)$ is τ_ω-lower semicontinuous.

Fix any $(x,v) \in I\!B^{\overline{\chi}}_{\widetilde{X}}\big[(\bar{x},\bar{y}),r\big]$, that is, $x \in I\!B^{\overline{\gamma}}_X[\bar{x},r]$ and $v \in I\!B^{\overline{\varrho}}_Y[\bar{y},r/\alpha] \cap F(x)$, and any $y \in I\!B^{\psi}_Y[\bar{y},r]$, such that (5.21) holds. Find a pair $(x',v') \in \operatorname{gph} F = \widetilde{X}$ satisfying (5.22). Proposition 5.2.1, with $X := \widetilde{X}, \varphi := \omega$, and $\gamma := \chi$, implies that for each $t \in (0,r]$ we have

$$I\!B^{\psi}_Y[\bar{y},t] \quad \subset \quad g\big(I\!B^{\overline{\omega}}_{\widetilde{X}}\big[(\bar{x},\bar{y}),t\big]\big) = g\left(\big(I\!B^{\overline{\varphi}}_X[\bar{x},t] \times I\!B^{\overline{\varrho}}_Y[\bar{y},t/\alpha]\big) \cap \operatorname{gph} F\right).$$

Fix any $t \in (0,r]$ and any $y \in I\!B^{\psi}_Y[\bar{y},t]$. Then there are $x \in I\!B^{\overline{\varphi}}_X[\bar{x},t] \cap \operatorname{dom} F$ and $y' \in I\!B^{\overline{\varrho}}_Y[\bar{y},t/\alpha] \cap F(x)$ such that $g(x,y') = y$. Hence $y' = y$. So $y \in F(x) \subset F\big(I\!B^{\overline{\varphi}}_X[\bar{x},t]\big)$. ■
We can also derive a set-valued version of Corollary 5.2.1.

Corollary 5.2.3 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and nonempty closed sets $L \subset \mathbb{S}_X$ and $M \subset \mathbb{S}_Y$ be such that* $\operatorname{cone} L$ *is convex. Consider a set-valued mapping $F : X \rightrightarrows Y$, with $\bar{y} \in F(\bar{x})$, such that the set* $\operatorname{gph} F$ *is closed and for which there are positive constants c, r, and α, with $\alpha c < 1$, such that for any $x \in I\!B_X[\bar{x},r]$, any $v \in I\!B_Y[\bar{y},r/\alpha] \cap F(x)$, and any $y \in I\!B_Y[\bar{y},cr] \cap \big(\bar{y} + \operatorname{cone} M\big)$ satisfying*

$$0 < T_M(v,y) \leq T_M(\bar{y},y) - c\max\{T_L(x,\bar{x}), \alpha\|\bar{y} - v\|_Y\}, \tag{5.23}$$

there is a pair $(x',v') \in \operatorname{gph} F$ such that

$$c\max\{T_L(x,x'), \alpha\|v - v'\|_Y\} < T_M(v,y) - T_M(v',y). \tag{5.24}$$

Then

$$F\big(I\!B_X[\bar{x},t] \cap (\bar{x} + \operatorname{cone} L)\big) \supset I\!B_Y[\bar{y},ct] \cap (\bar{y} + \operatorname{cone} M) \quad \textit{for each} \quad t \in (0,r]. \tag{5.25}$$

Proof. By assumptions, for any $(x,v) \in (I\!B_X[\bar{x},r] \times I\!B_Y[\bar{y},r/\alpha]) \cap \operatorname{gph} F$ and any $y \in I\!B_Y[\bar{y},cr] \cap (\bar{y} + \operatorname{cone} M)$ satisfying (5.23), there is a pair $(x',v') \in \operatorname{gph} F$ such that (5.24) holds. Therefore, in the view of Example 4.2.1, Proposition 5.2.3, with $\varphi(x,u) := T_{-L}(x,u)$ and $\gamma(x,u) := \|x-u\|_X$ for $x,u \in X$, $\varrho(y,v) := \|y-v\|_Y$ and $\psi(y,v) := \frac{1}{c}T_M(v,y)$ for $y,v \in Y$, and $\alpha := \alpha c$, implies that (5.25) holds. ■

Similarly, we get a set-valued version of Corollary 5.2.2.

Corollary 5.2.4 *Let a function φ have the properties $(\mathcal{A}_2) - (\mathcal{A}_4)$ and satisfy $\gamma(x,\bar{x}) \leq \varphi(x,\bar{x})$ for each $x \in X$, a function ϱ have the properties $(\mathcal{A}_2) - (\mathcal{A}_4)$, and a constant $\alpha \in (0,1)$ be given. Assume that $\omega : (X \times Y)^2 \longrightarrow [0,\infty]$ is defined by (4.2). Consider a set-valued mapping $F : X \rightrightarrows Y$, with $\bar{y} \in F(\bar{x})$, such that the set* $\operatorname{gph} F$ *is τ_ω-closed, along with a continuous strictly increasing function $\phi : [0,\infty] \longrightarrow [0,\infty]$ such that $\phi(0) = 0$. Suppose that there is $r > 0$ such that for any $x \in I\!B^{\overline{\gamma}}_X[\bar{x},r]$, any $v \in I\!B^{\overline{\varrho}}_Y[\bar{y},r/\alpha] \cap F(x)$, and any $y \in I\!B^{\varrho}_Y[\bar{y},r]$ satisfying*

$$0 < \phi(\varrho(v,y)) \leq \phi(\varrho(\bar{y},v)) - \max\{\varphi(x,\bar{x}), \alpha\varrho(v,\bar{y})\}, \tag{5.26}$$

there is a pair $(x',v') \in \operatorname{gph} F$ such that

$$\max\{\varphi(x',x), \alpha\varrho(v',v)\} < \phi(\varrho(v,y)) - \phi(\varrho(v',y)). \tag{5.27}$$

Then

$$F\big(I\!B^{\overline{\varphi}}_X[\bar{x},t]\big) \supset I\!B^{\varrho}_Y\big[\bar{y},\phi^{-1}(t)\big] \quad \textit{for each} \quad t \in (0,\min\{r,\phi(r)\}]. \tag{5.28}$$

Proof. Let $\psi := \phi \circ \varrho$, then ψ has the properties $(\mathcal{A}_1)$ and $(\mathcal{A}_3)$. Since ϕ is continuous, we have that for each $y \in Y$ the mapping $Y \ni v \longmapsto \psi(v, y)$ is τ_ϱ-lower semicontinuous. By assumptions for any $x \in I\!B^{\overline{\gamma}}_X[\bar{x}, r]$, any $v \in I\!B^{\overline{\varrho}}_Y[\bar{y}, r/\alpha] \cap F(x)$, and any $y \in I\!B^{\psi}_Y[\bar{y}, \phi(r)]$ satisfying (5.26) there is $(x', v') \in \operatorname{gph} F$ such that (5.27) holds.

Proposition 5.2.3, with $r := \min\{r, \phi(r)\}$, implies that

$$F\big(I\!B^{\overline{\varphi}}_X[\bar{x}, t]\big) \supset I\!B^{\psi}_Y[\bar{y}, t] \quad \text{for each} \quad t \in (0, \min\{r, \phi(r)\}].$$

By (5.6), we get (5.28). ■

5.3 Subregularity criteria

Similarly, as in the criteria in the previous section we can easily derive criteria for the pseudo-openness from Definition 5.1.2 in the spirit of Theorem 2.2.3 and Theorem 2.2.4.

We start with a sufficient condition for the subregularity of a single-valued mapping.

Proposition 5.3.1 *Let a function φ have the properties $(\mathcal{A}_2) - (\mathcal{A}_4)$, a function γ have the property $(\mathcal{A}_2)$ and satisfy $\gamma(x, u) \le \varphi(x, u)$ for each $x, u \in X$, and a function ϱ have the property $(\mathcal{A}_3)$. Consider a single-valued mapping $g : X \longrightarrow Y$, defined on whole X, such that the mapping $X \ni x \longmapsto \varrho(g(x), g(\bar{x}))$ is τ_φ-lower semicontinuous. Suppose that there is $r > 0$ such that for any $u \in I\!B^{\overline{\gamma}}_X[\bar{x}, 2r]$, with $0 < \varrho(g(u), g(\bar{x})) < \infty$, there is a point $x' \in X$ satisfying*

$$\varphi(x', u) < \varrho(g(u), g(\bar{x})) - \varrho(g(x'), g(\bar{x})). \tag{5.29}$$

Then

$$g\big(I\!B^{\overline{\varphi}}_X[x, t]\big) \ni g(\bar{x}) \quad \textit{whenever} \quad x \in I\!B^{\overline{\gamma}}_X[\bar{x}, r] \quad \textit{and} \quad t \in (0, r], \ \textit{with } g(x) \in I\!B^{\overline{\varrho}}_Y[g(\bar{x}), t].$$

Proof. Fix any $t \in (0, r]$ and any $x \in I\!B^{\overline{\gamma}}_X[\bar{x}, r]$ with $g(x) \in I\!B^{\overline{\varrho}}_Y[g(\bar{x}), t]$. Then $\varrho(g(x), g(\bar{x})) \le t < \infty$. We are showing that $g(\bar{x}) \in g(I\!B^{\overline{\varphi}}_X[x, t])$. To show this, we are finding $u \in I\!B^{\overline{\varphi}}_X[x, t]$ such that $g(u) = g(\bar{x})$.

Since the mapping $X \ni x \longmapsto \varrho(g(x), g(\bar{x}))$ is τ_φ-lower semicontinuous, applying Theorem 4.3.1, with $f := \varrho(g(\cdot), g(\bar{x}))$ and $\bar{x} := x$, we find $u \in X$ such that

$$\varrho(g(u), g(\bar{x})) + \varphi(u, x) \le \varrho(g(x), g(\bar{x}))$$

and

$$\varrho(g(u), g(\bar{x})) < \varrho(g(v), g(\bar{x})) + \varphi(v, u) \quad \text{whenever} \quad v \in X \setminus \{u\}. \tag{5.30}$$

Then

$$\overline{\varphi}(x, u) = \varphi(u, x) \le \varrho(g(x), g(\bar{x})) - \varrho(g(u), g(\bar{x})) \le \varrho(g(x), g(\bar{x})) \le t$$

and

$$\overline{\gamma}(\bar{x}, u) \le \overline{\gamma}(\bar{x}, x) + \overline{\gamma}(x, u) \le r + \overline{\varphi}(x, u) \le r + t \le 2r.$$

Therefore $u \in I\!B^{\overline{\varphi}}_X[x, t] \cap I\!B^{\overline{\gamma}}_X[\bar{x}, 2r]$. We claim that $g(u) = g(\bar{x})$. Suppose on the contrary that $g(u) \neq g(\bar{x})$; hence $0 < \varrho(g(u), g(\bar{x}))$ by $(\mathcal{A}_3)$. By the assumptions, we find $x' \in X$ such that (5.29) holds. Clearly, $\varrho(g(x'), g(\bar{x})) < \infty$ and $x' \neq u$. Using (5.30), with $v := x'$, and (5.29) we get

$$\varphi(x', u) < \varrho(g(u), g(\bar{x})) - \varrho(g(x'), g(\bar{x})) < \varphi(x', u),$$

a contradiction. Therefore $g(u) = g(\bar{x})$. ■

From the previous statement we are able to derive a criterion for directional subregularity in the spirit of [12, Proposition 13].

Corollary 5.3.1 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and nonempty closed sets $L \subset \mathbb{S}_X$ and $M \subset \mathbb{S}_Y$ be such that* cone L *is convex. Consider a continuous single-valued mapping $g : X \longrightarrow Y$, defined on whole X, for which there are positive constants $c > 0$ and $r > 0$ such that for any $u \in I\!B_X[\bar{x}, 2r]$, with $0 < T_M\big(g(\bar{x}), g(u)\big) < \infty$, there is a point $x' \in X$ satisfying*

$$c\,T_L(u, x') < T_M(g(\bar{x}), g(u)) - T_M(g(\bar{x}), g(x')).$$

Then for any $x \in I\!B_X[\bar{x}, r]$ and any $t \in (0, r]$, with $g(x) \in I\!B_Y[g(\bar{x}), ct] \cap (g(\bar{x}) + \operatorname{cone} M)$, we have

$$g\big(I\!B_X[x, t] \cap (x + \operatorname{cone} L)\big) \ni g(\bar{x}).$$

Proof. It suffices to apply Proposition 5.3.1, with $\varphi(x, u) := T_{-L}(x, u)$ and $\gamma(x, u) := \|x - u\|_X$ for x, $u \in X$, and $\varrho(y, v) := c^{-1}\,T_{-M}(y, v)$ for y, $v \in Y$. Indeed, in view of Example 4.2.1, it is enough to observe that the mapping $X \ni x \longmapsto T_M(y, g(x))$ is lower semicontinuous on X, since g is continuous on X and $T_M(y, \cdot)$ is lower semicontinuous because M is a closed subset of $\mathbb{S}_Y$. ■

We mention necessary conditions for these properties.

Proposition 5.3.2 *Let a function ϱ have the property $(\mathcal{A}_3)$. Consider a mapping $g : X \longrightarrow Y$, which is τ_φ-to-τ_ϱ-continuous at $\bar{x}$ and for which there are positive constants c and r such that*

$$g\big(I\!B^{\varphi}_X[x, t]\big) \ni g(\bar{x}) \quad \textit{whenever} \quad x \in I\!B^{\gamma}_X[\bar{x}, r] \quad \textit{and} \quad t \in (0, r] \textit{ with } g(x) \in I\!B^{\varrho}_Y[g(\bar{x}), t].$$

Then there is $r' > 0$ such that for every $x \in I\!B^{\varphi}_X[\bar{x}, r']$, with $0 < \varrho(g(\bar{x}), g(x)) < \infty$, there is a point $x' \in X$ satisfying $\varphi(x, x') \le \varrho(g(\bar{x}), g(x)) - \varrho(g(\bar{x}), g(x'))$.

Proof. Since g is τ_φ-to-τ_ϱ-continuous at $\bar{x}$, we find $r' \in (0, r)$ such that $\varrho(g(\bar{x}), g(x)) < r$ for each $x \in I\!B^{\gamma}_X[\bar{x}, r']$. Pick an arbitrary $x \in I\!B^{\gamma}_X[\bar{x}, r']$ with $g(x) \neq g(\bar{x})$. Let $t := \varrho(g(\bar{x}), g(x))$, then $t \in (0, r)$. Consequently, there is a point $x' \in I\!B^{\varphi}_X(x, t)$ such that $g(x') = g(\bar{x})$. Then

$$\varphi(x, x') \le t = \varrho(g(\bar{x}), g(x)) = \varrho(g(\bar{x}), g(x)) - \varrho(g(\bar{x}), g(x')).$$

■

As in Section 5.2, Proposition 5.3.1 has the following set-valued counterpart.

Proposition 5.3.3 *Let a function φ have the properties $(\mathcal{A}_2) - (\mathcal{A}_4)$, a function γ have the property $(\mathcal{A}_2)$ and satisfy $\gamma(x, u) \le \varphi(x, u)$ for each $x, u \in X$, a function ϱ have the properties $(\mathcal{A}_2) - (\mathcal{A}_4)$, a function ψ satisfy $\varrho(y, v) \le \psi(y, v)$ for each $y, v \in Y$, and a constant $\alpha \in (0, 1)$ be given.*

Assume that $\omega : (X \times Y)^2 \longrightarrow [0, \infty]$ is defined by (4.2), and that for each $y \in Y$ the mapping $Y \ni v \longmapsto \psi(v, y)$ is τ_ϱ-lower semicontinuous. Consider a set-valued mapping $F : X \rightrightarrows Y$, with $\bar{y} \in F(\bar{x})$, such that the set gph F *is τ_ω-closed. Suppose that there is $r > 0$ such that for any $x \in I\!B^{\overline{\gamma}}_X[\bar{x}, 2r]$ and any $v \in I\!B^{\overline{\varrho}}_Y[\bar{y}, 2r/\alpha] \cap F(x)$, with $0 < \psi(v, \bar{y}) < \infty$, there is a point $(x', v') \in$ gph F satisfying*

$$\max\{\varphi(x', x), \alpha\varrho(v', v)\} < \psi(v, \bar{y}) - \psi(v', \bar{y}). \tag{5.31}$$

Then

$$F\big(I\!B^{\overline{\varphi}}_X[x, t]\big) \ni \bar{y} \quad \textit{whenever} \quad x \in I\!B^{\overline{\gamma}}_X[\bar{x}, r] \quad \textit{and} \quad t \in (0, r] \textit{ with } I\!B^{\overline{\psi}}_Y[\bar{y}, t] \cap F(x) \neq \emptyset.$$

Proof. Let $\widetilde{X} :=$ gph F and define a single-valued mapping $\widetilde{X} \ni (x, y) \longmapsto g(x, y)$ by $g(x, y) := y$ for $(x, y) \in \widetilde{X}$. We already know that ω has the properties $(\mathcal{A}_1) - (\mathcal{A}_4)$. Then g is τ_ω-to-τ_ϱ-continuous, hence for each $y \in Y$ the mapping $\widetilde{X} \ni (x, v) \longmapsto \varrho(g(x, v), y)$ is τ_ω-lower semicontinuous. Define a function $(X\times Y)^2 \ni \big((x,y),(u,v)\big) \longmapsto \chi\big((x,y),(u,v)\big)$ by $\chi((x,y),(u,v)) := \max\{\gamma(x,u), \alpha\varrho(y,v)\}$ for $(x,y),(u,v)\in X\times Y$, then $\chi\big((x,y),(u,v)\big) \le \omega\big((x,y),(u,v)\big)$ for each $x,u\in X$ and $y,v\in Y$ and χ has the properties $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$.

Fix any $(x,v)\in I\!B^{\overline{\chi}}_{\widetilde{X}}\big[(\bar{x},\bar{y}),2r\big] = \big(I\!B^{\overline{\gamma}}_X[\bar{x},2r]\times I\!B^{\overline{\varrho}}_Y[\bar{y},2r/\alpha]\big)\cap \operatorname{gph} F$, with $0<\psi(g(x,v),\bar{y})<\infty$, thus find a pair $(x',v')\in \operatorname{gph} F = \widetilde{X}$ satisfying (5.31).

Proposition 5.3.1, with $X := \widetilde{X}$, $\varrho := \psi$, $\varphi := \omega$, and $\gamma := \chi$, implies that for each $t\in(0,r]$ and each $(x,y)\in I\!B^{\overline{\chi}}_{\widetilde{X}}\big[(\bar{x},\bar{y}),r\big]$, with $g(x,y)\in I\!B^{\overline{\psi}}_Y\,[g(\bar{x},\bar{y}),t]$, we have

$$\bar{y} \;\in\; g\left(I\!B^{\overline{\omega}}_{\widetilde{X}}\,[(x,y),t]\right) = g\left(\big(I\!B^{\overline{\varphi}}_X[x,t]\times I\!B^{\overline{\varrho}}_Y[y,t/\alpha]\big)\cap \operatorname{gph} F\right).$$

Fix any $t\in(0,r]$ and any $x\in I\!B^{\overline{\gamma}}_X[\bar{x},r]$ with $I\!B^{\overline{\psi}}_Y[\bar{y},t]\cap F(x)\neq\emptyset$. Then there is $y\in I\!B^{\overline{\psi}}_Y[\bar{y},t]\cap F(x)\subset I\!B^{\overline{\varrho}}_Y[\bar{y},t]\cap F(x)$. Thus $g(x,y)=y\in I\!B^{\overline{\psi}}_Y[\bar{y},t]\cap F(x)$. Find $(u,v)\in \big(I\!B^{\overline{\varphi}}_X[x,t]\times I\!B^{\overline{\varrho}}_Y[y,t/\alpha]\big)\cap\operatorname{gph} F$ such that $\bar{y}=g(u,v)=v$. Then $\bar{y}=v\in F(u)\subset F(I\!B^{\overline{\varphi}}_X[x,t])$. ■

Our setting allows us easily prove a set-valued version of Proposition 5.3.3.

Corollary 5.3.2 *Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be Banach spaces and nonempty closed sets $L\subset \mathbb{S}_X$ and $M\subset\mathbb{S}_Y$ be such that* cone L *is convex. Consider a set-valued mapping $F: X\rightrightarrows Y$, with $\bar{y}\in F(\bar{x})$, for which there are positive constants c,r, and α such that $\alpha c<1$; that the set* gph F *is closed; and for any $x\in I\!B_X[\bar{x},2r]$ and any $v\in I\!B_Y[\bar{y},2r/\alpha]\cap F(x)$, with $0<T_M(\bar{y},v)<\infty$, there is a point $(x',v')\in\operatorname{gph}F$ satisfying*

$$c\max\{T_L(x,x'),\alpha\|v-v'\|_Y\} < T_M(\bar{y},v)-T_M(\bar{y},v').$$

Then for any $x\in I\!B_X[\bar{x},r]$ and any $t\in(0,r]$, with $F(x)\cap I\!B_Y[\bar{y},ct]\cap(\bar{y}+\operatorname{cone} M)\neq\emptyset$, we have

$$F\big(I\!B_X[x,t]\big)\cap(x+\operatorname{cone} L)\big)\ni\bar{y}.$$

Proof. It suffices to apply Proposition 5.3.3, with $\varphi(x,u):=T_{-L}(x,u)$ and $\gamma(x,u):=\|x-u\|_X$ for $x,u\in X$, $\varrho(y,v):=\frac{1}{c}\|v-y\|_Y$ and $\psi(y,v):=\frac{1}{c}T_{-M}(y,v)$ for $y,v\in Y$, and $\alpha:=\alpha c$. Indeed, in view of Example 4.2.1, it is enough to observe that $T_{-M}(\cdot,y)$ is lower semicontinuous because M is a closed subset of $\mathbb{S}_Y$. ■

5.4 Regularity criteria

In Example 5.1.1, we have showed that the openness contains several types of regularity. Now, we prove extensions of Theorem 2.2.1 and Theorem 2.2.2, which guarantee the property from Definition 5.1.1 and its particular versions.

Proposition 5.4.1 *Let a function φ have the properties $(\mathcal{A}_2)-(\mathcal{A}_4)$, a function γ have the property $(\mathcal{A}_2)$ and satisfy $\gamma(x,u)\le\varphi(x,u)$ for each $x,u\in X$, a function ϱ have the properties $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$, and a function ψ satisfy $\varrho(y,v)\le\psi(y,v)$ for each $y,v\in Y$.*

Consider a single-valued mapping $g: X\longrightarrow Y$, defined on whole X, such that for each $y\in Y$ the mapping $X\ni x\longmapsto\psi(g(x),y)$ is τ_φ-lower semicontinuous and the mapping $X\ni x\longmapsto\varrho(y,g(x))$ is $\tau_{\overline{\varphi}}$-upper semicontinuous.

Suppose that there is $r>0$ such that for any $x\in I\!B^{\overline{\gamma}}_X[\bar{x},2r]$ and any $y\in I\!B^{\varrho}_Y[g(\bar{x}),2r]$, with $0<\psi(g(x),y)<\infty$, there is a point $x'\in X$ satisfying

$$\varphi(x',x)<\psi(g(x),y)-\psi(g(x'),y). \tag{5.32}$$

Then there is $r' > 0$ such that for any $x \in I\!B_X^{\overline{\gamma}}[\bar{x}, r']$ we have

$$g\big(I\!B_X^{\overline{\varphi}}[x,t]\big) \supset I\!B_Y^{\psi}[g(x),t] \quad \textit{for each} \quad t \in (0,r').$$

Proof. Find $r' \in (0,r)$ such that $\varrho(g(\bar{x}), g(x)) < r$ for each $x \in I\!B_X^{\overline{\varphi}}[\bar{x}, r']$. Pick an arbitrary $\tilde{x} \in I\!B_X^{\overline{\gamma}}[\bar{x}, r']$. We are applying Proposition 5.2.1, with $\bar{x} := \tilde{x}$ and $r := r'$. Fix any $x \in I\!B_X^{\overline{\gamma}}[\tilde{x}, r']$ and any $y \in I\!B_Y^{\psi}[g(\tilde{x}), r'] \subset I\!B_Y^{\varrho}[g(\tilde{x}), r']$ such that

$$0 < \psi(g(x), y) \le \psi(g(\tilde{x}), y) - \varphi(x, \tilde{x}).$$

Then $\psi(g(x), y) \le \psi(g(\tilde{x}), y) \le r' < \infty$ and $\overline{\varphi}(\tilde{x}, x) \le \psi(g(\tilde{x}), y) - \psi(g(x), y) < \psi(g(\tilde{x}), y) < r'$. Also $x \in I\!B_X^{\overline{\gamma}}[\bar{x}, 2r]$ and $y \in I\!B_Y^{\varrho}[g(\bar{x}), 2r]$ because

$$\overline{\gamma}(\bar{x}, x) \le \overline{\gamma}(\bar{x}, \tilde{x}) + \overline{\gamma}(\tilde{x}, x) \le r' + \overline{\varphi}(\tilde{x}, x) \le r' + r' < 2r.$$

and

$$\varrho(g(\bar{x}), y) \le \varrho(g(\bar{x}), g(\tilde{x})) + \varrho(g(\tilde{x}), y) \le \varrho(g(\bar{x}), g(\tilde{x})) + \psi(g(\tilde{x}), y) \le r + r' < 2r.$$

Find $x' \in X$ such that (5.32) holds. Proposition 5.2.1 implies that $g\big(I\!B_X^{\overline{\varphi}}[\tilde{x}, t]\big) \supset I\!B_Y^{\psi}[g(\tilde{x}), t]$ for each $t \in (0, r')$. ■

The previous proposition contains a sufficiency part of [12, Propostion 11].

Corollary 5.4.1 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and nonempty closed sets $L \subset \mathbb{S}_X$ and $M \subset \mathbb{S}_Y$ be such that* cone L *is convex. Consider a continuous mapping $g : X \longrightarrow Y$, defined on whole X, for which there are positive constants c and r such that for any $x \in I\!B_X[\bar{x}, 2r]$ and any $y \in I\!B_Y[g(\bar{x}), 2r]$, with $0 < T_M(g(x), y) < \infty$, there is a point $x' \in X$ satisfying*

$$c\, T_L(x, x') < T_M(g(x), y) - T_M(g(x'), y).$$

Then there is $r' > 0$ such that for any $x \in I\!B_X[\bar{x}, r']$ we have

$$g\big(I\!B_X[x,t] \cap (x + \operatorname{cone} L)\big) \supset I\!B_Y[g(x), ct] \cap (g(x) + \operatorname{cone} M) \quad \textit{for each} \quad t \in (0, r').$$

Proof. It suffices to apply Proposition 5.2.1, with $\varphi(x,u) := T_{-L}(x,u)$ and $\gamma(x,u) := \|x-u\|_X$ for x, $u \in X$, and $\psi(y,v) := c^{-1}\, T_M(y,v)$ and $\varrho(y,v) := \|y-v\|_Y$ for $y, v \in Y$. Indeed, in view of Example 4.2.1, it is enough to observe that the mapping $X \ni x \longmapsto T_M(g(x), y)$ is lower semicontinuous on X, since g is continuous on X. ■

Proposition 5.4.1 also contains a criterion for the nonlinear openness, but unlike the previous statements, we need the nonlinearity, which is concave.

Corollary 5.4.2 *Let a function φ have the properties $(\mathcal{A}_2) - (\mathcal{A}_4)$, a function γ have the property $(\mathcal{A}_2)$ and satisfy $\gamma(x,u) \le \varphi(x,u)$ for each $x, u \in X$, a function ϱ have the properties $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$, and a function ψ satisfy $\varrho(y,v) \le \psi(y,v)$ for each $y, v \in Y$. Consider a mapping $g : X \longrightarrow Y$, defined on whole X, such that for each $y \in Y$ the mapping $X \ni x \longmapsto \psi(g(x), y)$ is τ_φ-lower semicontinuous and the mapping $X \ni x \longmapsto \varrho(y, g(x))$ is $\tau_{\overline{\varphi}}$-upper semicontinuous, along with a continuous strictly increasing concave function $\phi : [0,\infty] \longrightarrow [0,\infty]$ such that $\phi(0) = 0$.*

Suppose that there is $r > 0$ such that for any $x \in I\!B_X^{\overline{\gamma}}[\bar{x}, 2r]$ and any $y \in I\!B_Y^{\varrho}[g(\bar{x}), 2r]$, with $0 < \psi(g(x), y) < \infty$, there is a point $x' \in X$ satisfying

$$\varphi(x', x) < \phi(\psi(g(x), y)) - \phi(\psi(g(x'), y)).$$

Then there is $r' > 0$ such that for any $x \in I\!B_X^{\overline{\gamma}}[\bar{x}, r']$ we have

$$g\big(I\!B_X^{\overline{\varphi}}[x,t]\big) \supset I\!B_Y^{\psi}\big[g(x), \phi^{-1}(t)\big] \quad \textit{for each} \quad t \in (0, r'). \tag{5.33}$$

Proof. Note that since ϕ is continuous then for each $y \in Y$ the mapping $X \ni x \longmapsto \phi(\psi(g(x), y))$ is τ_φ-lower semicontinuous and the mapping $X \ni x \longmapsto \phi(\varrho(y, g(x)))$ is $\tau_{\overline{\varphi}}$-upper semicontinuous. Clearly, the function $\phi \circ \varrho$ has the properties $(\mathcal{A}_1)$ and $(\mathcal{A}_3)$ and, by Lemma A.3.3, it has the property $(\mathcal{A}_2)$.

Proposition 5.4.1, with $\psi := \phi \circ \psi$, $\varrho := \phi \circ \varrho$, and $r := \frac{1}{2}\min\{2r, \phi(2r)\}$, implies that there is $\tilde{r} > 0$ such that

$$g\big(I\!B_X^{\overline{\varphi}}[x,t]\big) \supset I\!B_Y^{\phi\circ\psi}[g(x),t] \quad \text{for each} \quad x \in I\!B_X^{\overline{\gamma}}[\bar{x},\tilde{r}] \quad \text{and} \quad t \in (0,\tilde{r}].$$

By (5.6), we get (5.33) with $r' := \min\{\tilde{r}, \phi(\tilde{r})\}$. ■
Similarly, we get a set-valued version of Proposition 5.4.1 using the restriction of the canonical projection on the graph of a given set-valued mapping.

Proposition 5.4.2 *Let a function φ have the properties $(\mathcal{A}_2) - (\mathcal{A}_4)$, a function γ have the property $(\mathcal{A}_2)$ and satisfy $\gamma(x,u) \le \varphi(x,u)$ for each $x, u \in X$, a function ϱ have the properties $(\mathcal{A}_2) - (\mathcal{A}_4)$, a function ψ satisfy $\varrho(y,v) \le \psi(y,v)$ for each $y, v \in Y$, and a constant $\alpha \in (0,1)$ be given. Assume that $\omega : X \times Y \longrightarrow [0,\infty]$ is defined by (4.2). Consider a set-valued mapping $F : X \rightrightarrows Y$ such that $\bar{y} \in F(\bar{x})$ and that $\operatorname{gph} F$ is τ_ω-closed. Assume that for each $y \in Y$ the mapping $Y \ni v \longmapsto \psi(v,y)$ is τ_ϱ-lower semicontinuous and the mapping $Y \ni v \longmapsto \varrho(y,v)$ is $\tau_{\overline{\varrho}}$-upper semicontinuous.*

Suppose that there is $r > 0$ such that for any $x \in I\!B_X^{\overline{\gamma}}[\bar{x}, 2r]$, any $v \in I\!B_Y^{\overline{\varrho}}[\bar{y}, 2r/\alpha] \cap F(x)$, and any $y \in I\!B_Y^{\varrho}[\bar{y}, 2r]$, with $0 < \psi(v,y) < \infty$, there is a pair $(x', v') \in \operatorname{gph} F$ satisfying

$$\max\{\varphi(x',x), \alpha\varrho(v',v)\} < \psi(v,y) - \psi(v',y). \tag{5.34}$$

Then there is $r' > 0$ such that for any $x \in I\!B_X^{\overline{\gamma}}[\bar{x}, r']$ and any $v \in I\!B_Y^{\overline{\varrho}}[\bar{y}, r'] \cap F(x)$ we have

$$F\big(I\!B_X^{\overline{\varphi}}[x,t]\big) \supset I\!B_Y^{\psi}[v,t] \quad \text{for each} \quad t \in (0, r'].$$

Proof. Let $\widetilde{X} := \operatorname{gph} F$ and define a mapping $\widetilde{X} \ni (x,y) \longmapsto g(x,y)$ by $g(x,y) := y$ for $(x,y) \in \widetilde{X}$. Then the mapping g is τ_ω-to-τ_ϱ-continuous and $\tau_{\overline{\omega}}$-to-$\tau_{\overline{\varrho}}$-continuous, hence for each $y \in Y$ the mapping $\widetilde{X} \ni (x,v) \longmapsto \psi(g(x,v),y)$ is τ_ω-lower semicontinuous and the mapping $\widetilde{X} \ni (x,v) \longmapsto \varrho(y, g(x,v))$ is $\tau_{\overline{\omega}}$-upper semicontinuous. Define a function $(X \times Y)^2 \ni \big((x,y),(u,v)\big) \longmapsto \chi\big((x,y),(u,v)\big)$ by $\chi((x,y),(u,v)) := \max\{\gamma(x,u), \alpha\varrho(y,v)\}$ for $(x,y),(u,v) \in X \times Y$, then $\chi\big((x,y),(u,v)\big) \le \omega\big((x,y),(u,v)\big)$ for each $x, u \in X$ and each $y, v \in Y$ and χ has the properties $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$.

Fix any $(x,v) \in I\!B_{\widetilde{X}}^{\overline{\chi}}\big[(\bar{x},\bar{y}), 2r\big] = \big(I\!B_X^{\overline{\gamma}}[\bar{x}, 2r] \times I\!B_Y^{\overline{\varrho}}[\bar{y}, 2r/\alpha]\big) \cap \operatorname{gph} F$ and any $y \in I\!B_Y^{\varrho}[\bar{y}, 2r]$ with $0 < \psi(v,y) < \infty$. Find a pair $(x', v') \in \operatorname{gph} F = \widetilde{X}$ satisfying (5.34).

Proposition 5.4.1, with $X := \widetilde{X}, \varphi := \omega$, and $\gamma := \chi$, implies that there is $r' > 0$ such that for any $t \in (0, r']$ and any $(x,v) \in I\!B_{\widetilde{X}}^{\overline{\chi}}\big[(\bar{x},\bar{y}), r'\big]$ we have

$$I\!B_Y^{\psi}[v,t] \subset g\big(I\!B_{\widetilde{X}}^{\overline{\omega}}[(x,v),t]\big) = g\left(\big(I\!B_X^{\overline{\varphi}}[x,t] \times I\!B_Y^{\overline{\varrho}}[v,t/\alpha]\big) \cap \operatorname{gph} F\right).$$

Fix any $(x,v) \in I\!B_{\widetilde{X}}^{\overline{\chi}}\big[(\bar{x},\bar{y}), r'\big]$, any $t \in (0, r']$, and any $y \in I\!B_Y^{\psi}[v,t]$. Then there are $x' \in I\!B_X^{\overline{\varphi}}[x,t] \cap \operatorname{dom} F$ and $y' \in I\!B_Y^{\overline{\varrho}}[v, t/\alpha] \cap F(x')$ such that $g(x', y') = y$, hence $y' = y$ and consequently $y \in F(x')$. Thus $I\!B_Y^{\psi}[v,t] \subset F\big(I\!B_X^{\overline{\varphi}}[x,t]\big)$. ■
Proposition 5.4.2 contains a sufficiency part of [12, Proposition 13].

Corollary 5.4.3 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and nonempty closed sets $L \subset \mathbb{S}_X$ and $M \subset \mathbb{S}_Y$ be such that $\operatorname{cone} L$ is convex. Consider a set-valued mapping $F : X \rightrightarrows Y$, with $\bar{y} \in F(\bar{x})$, for which there are positive constants c, r, and α such that $\alpha c < 1$; that the set $\operatorname{gph} F$ is closed; and that for* *any* $x \in \mathbb{B}_X[\bar{x}, 2r]$, *any* $v \in \mathbb{B}_Y[\bar{y}, 2r/\alpha] \cap F(x)$, *and any* $y \in \mathbb{B}_Y[\bar{y}, cr]$, *with* $0 < T_M(v, y) < \infty$, *there is a pair* $(x', v') \in \operatorname{gph} F$ *such that*

$$c \max\{T_L(x, x'), \alpha \|v - v'\|_Y\} < T_M(v, y) - T_M(v', y).$$

Then there is $r' > 0$ *such that for any* $x \in \mathbb{B}_X[x, r']$ *and for any* $v \in \mathbb{B}_Y[\bar{y}, r'] \cap F(x)$ *we have*

$$F\big(\mathbb{B}_X[x, t] \cap (x + \operatorname{cone} L)\big) \supset \mathbb{B}_Y[v, ct] \cap (v + \operatorname{cone} M) \quad \textit{for each} \quad t \in (0, r']$$

Proof. It suffices to apply Proposition 5.4.2, with $\alpha := \alpha c$, $\varphi(x, u) := T_{-L}(x, u)$ and $\gamma(x, u) := \|x - u\|_X$ for x, $u \in X$, $\varrho(y, v) := \frac{1}{c}\|v - y\|_Y$ and $\psi(y, v) := \frac{1}{c} T_M(y, v)$ for $y, v \in Y$. Indeed, in view of Example 4.2.1, it is enough to observe that the mapping $Y \ni v \longmapsto T_M(v, y)$ is lower semicontinuous because M is a closed subset of $\mathbb{S}_Y$. ■

It follows the criterion for nonlinear openness of set-valued mappings.

Corollary 5.4.4 *Let a function* φ *have the properties* $(\mathcal{A}_2) - (\mathcal{A}_4)$, *a function* γ *have the property* $(\mathcal{A}_2)$ *and satisfy* $\gamma(x, u) \le \varphi(x, u)$ *for each* $x, u \in X$, *a function* ϱ *have the properties* $(\mathcal{A}_2) - (\mathcal{A}_4)$, *a function* ψ *satisfy* $\varrho(y, v) \le \psi(y, v)$ *for each* $y, v \in Y$, *and a constant* $\alpha \in (0, 1)$ *be given. Assume that* $\omega : X \times Y \longrightarrow [0, \infty]$ *is defined by* (4.2).

Consider a set-valued mapping $F : X \rightrightarrows Y$ *such that* $\bar{y} \in F(\bar{x})$ *and that* $\operatorname{gph} F$ *is* τ_ω*-closed, and a continuous strictly increasing concave function* $\phi : [0, \infty] \longrightarrow [0, \infty]$ *such that* $\phi(0) = 0$. *Assume that for each* $y \in Y$ *the mapping* $Y \ni v \longmapsto \psi(v, y)$ *is* τ_ϱ*-lower semicontinuous and the mapping* $Y \ni v \longmapsto \varrho(y, v)$ *is* $\tau_{\overline{\varrho}}$*-upper semicontinuous.*

Suppose that there is $r > 0$ *such that for any* $x \in \mathbb{B}_X^{\overline{\gamma}}[\bar{x}, 2r]$, *any* $v \in \mathbb{B}_Y^{\overline{\varrho}}[\bar{y}, 2r/\alpha] \cap F(x)$, *and any* $y \in \mathbb{B}_Y^{\varrho}[\bar{y}, 2r]$, *with* $0 < \psi(v, y) < \infty$, *there is a pair* $(x', v') \in \operatorname{gph} F$ *satisfying*

$$\max\{\varphi(x', x), \alpha \varrho(v', v)\} < \phi(\psi(v, y)) - \phi(\psi(v', y)).$$

Then there is $r' > 0$ *such that for any* $x \in \mathbb{B}_X^{\overline{\gamma}}[\bar{x}, r']$ *and any* $v \in \mathbb{B}_Y^{\overline{\varrho}}[\bar{y}, r'] \cap F(x)$ *we have*

$$F\big(\mathbb{B}_X^{\overline{\varphi}}[x, t]\big) \supset \mathbb{B}_Y^{\psi}\big[v, \phi^{-1}(t)\big] \quad \textit{for each} \quad t \in (0, r']. \tag{5.35}$$

Proof. Note that since ϕ is continuous then for each $y \in Y$ the mapping $Y \ni v \longmapsto \phi(\psi(v, y))$ is τ_φ-lower semicontinuous and the mapping $Y \ni v \longmapsto \phi(\varrho(y, v))$ is $\tau_{\overline{\varphi}}$-upper semicontinuous. Clearly, the function $\phi \circ \varrho$ has the properties $(\mathcal{A}_1)$, $(\mathcal{A}_3)$, and $(\mathcal{A}_4)$ and, by Lemma A.3.3, it has the property $(\mathcal{A}_2)$. Find $\tilde{r} \in (0, r]$ such that $\frac{2\tilde{r}}{\alpha} \le \phi\big(\frac{2r}{\alpha}\big)$.

Proposition 5.4.2, with $\psi := \phi \circ \psi$, $\varrho := \phi \circ \varrho$, and $r := \tilde{r}$, implies that there is $\tilde{r}' > 0$ such that

$$F\big(\mathbb{B}_X^{\overline{\varphi}}[x, t]\big) \supset \mathbb{B}_Y^{\phi \circ \psi}[v, t] \quad \text{for each} \quad x \in \mathbb{B}_X^{\overline{\gamma}}[\bar{x}, \tilde{r}'], v \in \mathbb{B}_Y^{\overline{\phi \circ \varrho}}[\bar{y}, \tilde{r}'] \cap F(x) \quad \text{and} \quad t \in (0, \tilde{r}'].$$

By (5.6), we get (5.35) with $r' := \min\{\tilde{r}', \phi(\tilde{r}'), \phi^{-1}(\tilde{r}')\}$. ■

Conclusion

In this thesis, we studied regularity, subregularity, and semiregularity and their generalized versions. We focused on sufficient/necessary conditions in the spirit of Graves theorem in the finite dimensional spaces and in the spirit of Ioffe criterion of regularity in quasi-metric spaces.

At first, the definitions of the properties were presented in Section 1.2 as well as a brief survey of the corresponding criteria for them, which have been published during several last decades in Chapter 2.

Then we studied conditions guaranteeing a constrained and directional semiregularity of single-valued mappings in the finite dimensional spaces in Chapter 3. These conditions are based on an approximation of a (nonlinear) single-valued mapping by, in the first case, a linear mapping in Section 3.2 and, in the second case, by a bunch of linear mappings in Section 3.3. We introduced conditions guaranteeing semiregularity with constraints given by the closed convex set containing the origin in the domain space and a locally conical constraint in the image space, in particular in Theorem 3.2.1 and Proposition 3.3.1. We reached the same result for a single-valued mapping perturbed by a constant set-valued mapping determined by a closed convex set in Theorem 3.2.3 and Corollary 3.3.1. We used singular value decomposition and conical eigenvalues for finding moduli of (semi)regularity for linear mappings with the constraints given by subspaces and cones, respectively, in Section 3.4. We showed the uniformity of regularity moduli with respect to certain kinds of closed sets in Proposition 3.3.2 and Corollary 3.3.2.

Furthermore, the basics of topology were presented in Section 4.1 and the definition of quasi-metric space with its topological properties in Section 4.2. Moreover, an extension of Ekeland variational principle to a quasi-metric space (Theorem 4.3.1) was presented with several applications in Setion 4.3.

In Section 5.1, we introduced the new type of openness around the reference point, openness at the reference point, and pseudo-openness at the reference point for set-valued mappings in Definition 5.1.1, Definition 5.1.3, and Definition 5.1.2, respectively. We studied the connection between these properties and nonlinear and directional (sub)regularity appearing in the literature also in Section 5.1. Further, we focused on sufficient/necessary conditions for these properties based on Ekeland variational principle. In particular, we introduced the Ioffe-type criterion (Proposition 5.2.1, 5.2.3, 5.3.1, 5.3.3, 5.4.1, and 5.4.2) for each mentioned property in quasi-metric spaces. We also derived from this criterion the sufficient conditions for some generalized versions of the properties in metric and normed spaces.

List of publications

- Cibulka, R., Dontchev, A. L., Preininger, J., Veliov, V., and Roubal, T. Kantorovich-type theorems for generalized equations. *J. Convex Anal. 25*, 2 (2018), 459–486

- Cibulka, R., and Roubal, T. Solution stability and path-following for a class of generalized equations. In *Control systems and mathematical methods in economics*, vol. 687 of *Lecture Notes in Econom. and Math. Systems*. Springer, Cham, 2018, pp. 57–80

- Cibulka, R., Preininger, J., and Roubal, T. On uniform regularity and strong regularity.

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- Cibulka, R., Fabian, M., and Roubal, T. An inverse mapping theorem in Fréchet-Montel spaces. *Set-Valued Var. Anal. 28*, 1 (2020), 195–208
- Cibulka, R., and Roubal, T. Ioffe-type criteria in extended quasi-metric spaces. *J. Convex Anal. 27*, 1 (2020), 207–228

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Appendix

A.1 Singular value decomposition and Moore-Penrose inverse

The following lemma and text below it are a (brief) summary of basic results of the singular value decomposition and corresponding Moore–Penrose inverse. For more details, see [63, Chapter 5].

Lemma A.1.1 *Let a matrix $A \in \mathbb{R}^{m\times n}$ be given. Then there are orthogonal matrices $V := (v_1, v_2, \ldots, v_n) \in \mathbb{R}^{n\times n}$ and $U := (u_1, u_2, \ldots, u_m) \in \mathbb{R}^{m\times m}$ and numbers $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_j > 0$, where $j := \dim \operatorname{rge} A$, such that*

$$U^T AV = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{m\times n}, \quad \text{with} \quad \Sigma := \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_j\}. \tag{A.1}$$

In particular, we have

$$A^T Av_i = \sigma_i^2 v_i \quad \text{and} \quad Av_i = \sigma_i u_i \quad \text{for each} \quad i = 1, 2, \ldots, j,$$

and

$$Av_i = 0 \quad \text{for each} \quad i = j+1, j+2, \ldots, n.$$

Moreover, $\ker A = \operatorname{span}\{v_{j+1}, v_{j+2}, \ldots, v_n\}$, $\operatorname{rge} A^T = \operatorname{span}\{v_1, v_2, \ldots, v_j\}$, $\operatorname{rge} A = \operatorname{span}\{u_1, u_2, \ldots, u_j\}$, *and* $\ker A^T = \operatorname{span}\{u_{j+1}, u_{j+2}, \ldots, u_m\}$.

The problem to find orthogonal matrices U and V and positive numbers $\sigma_1, \sigma_2, \ldots, \sigma_j$, such that (A.1) holds, is called the *singular value decomposition* of the matrix A. The numbers $\sigma_1, \sigma_2, \ldots, \sigma_j$ are called the *singular values* of the matrix A, the vectors $v_1, v_2, \ldots, v_n$ are called the *right singular vectors* of the matrix A, and the vectors $u_1, u_2, \ldots, u_m$ are called the *left singular vectors* of the matrix A.

Remark A.1.1 *The manner how to find singular values and vectors of a matrix $A \in \mathbb{R}^{m\times n}$, is to find orthonormal eigenvectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ of the matrix $A^T A$ and the corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then the vectors $v_1, v_2, \ldots, v_n$ are the right singular vectors and the numbers $\sigma_1, \sigma_2, \ldots, \sigma_j$, where $j := \dim \operatorname{rge} A$ and $\sigma_i := \sqrt{\lambda_i}$ for $i = 1, 2, \ldots, j$, are the corresponding singular values of the matrix A. The left singular vectors are given by*

$$u_i = \frac{1}{\sigma_i} Av_i \quad \text{for} \quad i = 1, 2, \cdots, j.$$

The remaining $m - j$ left singular vectors $u_{j+1}, u_{j+2}, \ldots, u_m$ can be find as eigenvectors of the matrix AA^T corresponding to an eigenvalue 0, which are all orthogonal to the vectors $u_1, u_2, \ldots, u_j$ and to each other.

With the use of notation from Lemma A.1.1, *Moore–Penrose inverse matrix* $A^\dagger$ of the matrix A is defined by

$$A^\dagger := V\Sigma^\dagger U^T, \quad \text{where} \quad \Sigma^\dagger := \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times m}. \tag{A.2}$$

Then $AA^\dagger$ and $A^\dagger A$ are matrices of orthogonal projections onto $\operatorname{rge} A$ and $\operatorname{rge} A^T$, respectively. Also, $\ker A^\dagger = \ker A^T$, $\operatorname{rge}(A^\dagger)^T = \operatorname{rge} A$, and $\operatorname{rge} A^\dagger = \operatorname{rge} A^T$. Hence for each $y \in \operatorname{rge} A$ we have $A^\dagger y \in \operatorname{rge} A^T$.

The spectral norm of any matrix is equal to its largest singular value, therefore

$$\|A\| = \sigma_1 \quad \text{and} \quad \|A^\dagger\| = \tfrac{1}{\sigma_j}.$$

By (A.2), $V^T A^\dagger U = \Sigma^\dagger$ and the uniqueness of the singular values implies that the numbers $1/\sigma_j, 1/\sigma_{j-1}, \cdots, 1/\sigma_1$, are the singular values of $A^\dagger$.

If $m \le n$ and $\operatorname{rank} A = m$, then

$$A^\dagger = A^T(AA^T)^{-1}$$

and if $m > n$ and $\operatorname{rank} A = n$, then

$$A^\dagger = (A^TA)^{-1}A^T.$$

When $m = n$ and $\operatorname{rank} A = n$, then $A^\dagger$ and A^{-1} coincide.

Remark A.1.2 *It is a well-know fact that for a matrix $A \in \mathbb{R}^{m\times n}$, the linear mapping $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a bijection between* $\operatorname{rge} A^T$ *and* $\operatorname{rge} A$. *Then, since* $\operatorname{rge}(A^\dagger)^T = \operatorname{rge} A$ *and* $\operatorname{rge} A^\dagger = \operatorname{rge} A^T$, *the linear mapping $A^\dagger : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a bijection between* $\operatorname{rge} A$ *and* $\operatorname{rge} A^T$.

A.2 Conical eigenvalues

Let a matrix $A \in \mathbb{R}^{n\times n}$ and a closed convex cone $K \subset \mathbb{R}^n$ be given. We say that $\lambda \in \mathbb{R}$ is *K-eigenvalue of the matrix A* if there is nonzero vector $x \in \mathbb{R}^n$ such that

$$K^* \ni Ax - \lambda x \perp x \in K, \tag{A.3}$$

where $K^* := \{u \in \mathbb{R}^n : \langle u, x\rangle \ge 0 \quad \text{for each} \quad x \in K\}$.

Set $\sigma_K(A) := \{\lambda \in \mathbb{R} : \quad \text{(A.3)} \quad \text{holds for some nonzero } x \in \mathbb{R}^n\}$. When K is the whole space then K-eigenvalues and classical eigenvalues coincide. The existence of K-eigenvalues is considered in [52] and [61]. The numerical method for finding K-eigenvalues is also considered in [52].

Let us point out, if a nonzero $x \in \mathbb{R}^n$ satisfies (A.3) for some $\lambda \in \mathbb{R}$, then $x/\|x\|$ also satisfies (A.3) with the same λ. So without any loss of generality, we can assume that such x satisfies $\|x\| = 1$.

Let K be a closed convex cone and $A \in \mathbb{R}^{m\times n}$, with $m \le n$, be matrix with a full rank. We are going to show that the smallest K-eigenvalues of matrices A^TA and $-A^TA$ correspond to the solutions of the conditional minimization. To be specific, we show that

$$\min_{x\in K\cap\mathbb{S}_{\mathbb{R}^n}} \|Ax\|^2 = \min \sigma_K(A^TA) \quad \text{and} \quad \min_{x\in K\cap\mathbb{S}_{\mathbb{R}^n}} -\|Ax\|^2 = \min \sigma_K(-A^TA). \tag{A.4}$$

Lemma A.2.1 *Let $K \subset \mathbb{R}^n$ be a non-trivial closed convex cone. Consider a continuously differentiable mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and a problem*

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad x \in K \cap \mathbb{S}_{\mathbb{R}^n}. \tag{A.5}$$

Let $x \in \mathbb{R}^n$ be a solution of (A.5), *then there is $\lambda \in \mathbb{R}$ such that*

$$K^* \ni \nabla f(x) - \lambda x \perp x \in K. \tag{A.6}$$

Proof. We are checking the assumptions of Theorem A.3.4 with $g_0 := f, g := \frac{1}{2}(1 - \|\cdot\|^2), U := K$, and $V := 0$. Let $x \in \mathbb{R}^n$ be a solution of (A.5). To show that the constrained qualification holds, suppose, on the contrary, that there is a nonzero $\lambda \in \mathbb{R}$ such that

$$K^* \ni -\lambda x \perp x \in K,$$

but $-\langle \lambda x, x \rangle = 0$ if and only if $\lambda = 0$, a contradiction. Then, by Theorem A.3.4 and Lemma A.3.1, there is $\lambda \in \mathbb{R}$ such that (A.6) holds. ■

Thanks to the previous lemma, we are able to prove the main statement of this section.

Proposition A.2.1 *Let a matrix $A \in \mathbb{R}^{m\times n}$ and a non-trivial closed convex cone $K \subset \mathbb{R}^n$ be given. Then* (A.4) *holds.*

Proof. At first, we are showing that the first equality in (A.4) holds. Let $x' \in K \cap \mathbb{S}_{\mathbb{R}^n}$ be such that $\|Ax'\|^2 = \min_{x\in K\cap\mathbb{S}_{\mathbb{R}^n}} \|Ax\|^2$ (such an x' exists since the function $\|A(\cdot)\|$ is continuous and the set $K \cap \mathbb{S}_{\mathbb{R}^n}$ is compact).

Lemma A.2.1, with $f := \frac{1}{2}\|A(\cdot)\|^2$, implies that (A.3), with $A := A^TA$ and $x := x'$, holds for some $\lambda \in \mathbb{R}$. Thus $\lambda \in \sigma_K(A^TA)$ and

$$\|Ax'\|^2 = \left\langle A^TAx', x'\right\rangle = \lambda\|x'\|^2 = \lambda.$$

On the contrary, suppose that there is $\lambda' \in \sigma_K(A^TA)$ with $\lambda > \lambda'$. Find a corresponding $\widehat{x} \in K \cap \mathbb{S}_{\mathbb{R}^n}$ to λ', then

$$\|A\widehat{x}\|^2 = \left\langle A^TA\widehat{x}, \widehat{x}\right\rangle = \lambda'\|\widehat{x}\|^2 = \lambda'.$$

That is a contradiction because the function f has a minimum at x' over $K \cap \mathbb{S}_{\mathbb{R}^n}$.

Now, we are showing that the second equality in (A.4) holds. Let $x' \in K \cap \mathbb{S}_{\mathbb{R}^n}$ be such that $-\|Ax'\|^2 = \min_{x\in K\cap\mathbb{S}_{\mathbb{R}^n}} -\|Ax\|^2$. Lemma A.2.1, with $f := -\frac{1}{2}\|A(\cdot)\|^2$, implies that (A.3), with $A := -A^TA$ and $x := x'$, holds for some $\lambda \in \mathbb{R}$. Thus $\lambda \in \sigma_K(-A^TA)$ and

$$-\|Ax'\|^2 = \left\langle -A^TAx', x'\right\rangle = \lambda\|x'\|^2 = \lambda.$$

On the contrary, suppose that there is $\lambda' \in \sigma_K(-A^TA)$ with $\lambda > \lambda'$. Find a corresponding $\widehat{x} \in K \cap \mathbb{S}_{\mathbb{R}^n}$ to λ', then $-\|A\widehat{x}\|^2 = \lambda'$. That is a contradiction because the function $-\|A(\cdot)\|^2$ has a minimum at x' over $K \cap \mathbb{S}_{\mathbb{R}^n}$. Therefore (A.4) holds. ■

Let us point out, that the author of [8] called the number $\min_{x\in K\cap\mathbb{S}_{\mathbb{R}^n}} \|Ax\|$ by the minimal conic singular value, to the best our knowledge there are no other references, which use this term. So we rely on the term K-eigenvalues of the matrices A^TA and $-A^TA$, that occurs in the literature.

A.3 Statements

In this section, we list several statements used in the proofs of our main results contained in the previous chapters. All of them seem to be well-known and can be found in the literature.

The following statement comes from [4, Theorem 3.2.3].

Theorem A.3.1 (Kakutani fixed point theorem) *Let Ω be a nonempty convex compact subset of $\mathbb{R}^n$. Let $\Psi : \Omega \rightrightarrows \Omega$ be a set-valued mapping with a closed graph and such that $\Psi(u)$ is nonempty convex set for each $u \in \Omega$. Then Ψ has a fixed point.*

The following statement is from [45, Theorem 2.5].

Theorem A.3.2 (The separation theorem) *Let X, Y be nonempty convex subsets of $\mathbb{R}^n$. Suppose that X and Y are disjoint. Then there is a nonzero $\xi \in \mathbb{R}^n$ such that*

$$\langle \xi, x \rangle \le \langle \xi, y \rangle \quad \text{for each} \quad x \in X \quad \text{and} \quad y \in Y.$$

The following statement is from [45, Lemma 1.30].

Theorem A.3.3 (The line segment principle) *Let Ξ be an open convex subset of $\mathbb{R}^m$. Then for each $y \in \Xi$, $v \in \overline{\Xi}$, and $\lambda \in (0,1)$ we have*

$$\lambda y + (1-\lambda) v \in \Xi.$$

For the following theorem, we recall the *normal cone mapping* $N_K : X \rightrightarrows X^*$, associated with a closed convex subset K of X, is given by

$$N_K(x) := \begin{cases} \{x^* \in X^* : \langle x^*, u - x \rangle \le 0 \text{ for each } u \in K\} & \text{for} \quad x \in K \\ \emptyset & \text{otherwise.} \end{cases}$$

and the statement is from [23, Theorem 2A.9].

Theorem A.3.4 (Lagrange multiplier rule) *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be nonempty closed convex sets and consider a problem*

$$\text{minimize} \quad g_0(x) \quad \text{subject to} \quad x \in U \quad \text{and} \quad g(x) \in V \tag{A.7}$$

for $g(x) := (g_1(x), g_2(x), \ldots, g_m(x))^T$, where $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, for $i = 0, 1, \ldots, m$ are continuously differentiable. Let $x \in \mathbb{R}^n$ be a solution of (A.7). Suppose that there is no $y \in N_V(g(x))$, with $y \neq 0$, such that $-y^T \nabla g(x) \in N_U(x)$, then there is $y \in N_V(g(x))$ such that

$$-\nabla g_0(x) - y^T \nabla g(x) \in N_U(x).$$

The following lemma is from [9, Proposition 1.1.1].

Lemma A.3.1 *If K is a nonempty closed convex cone in $\mathbb{R}^n$, then so is K^*. Moreover, $(K^*)^* = K$ and*

$$K \ni u \perp p \in K^* \quad \Leftrightarrow \quad -p \in N_K(u) \quad \Leftrightarrow \quad -u \in N_{K^*}(p).$$

In particular, when $K = \mathbb{R}^n_+$, then

$$\mathbb{R}^n_+ \ni u \perp p \in \mathbb{R}^n_+ \quad \Leftrightarrow \quad -p \in N_{\mathbb{R}^n_+}(u) \quad \Leftrightarrow \quad -u \in N_{\mathbb{R}^n_+}(p).$$

We need a corollary of Michael selection theorem, see [28, Theorem 5.27].

Theorem A.3.5 *Let Ω be a nonempty subset of $\mathbb{R}^n$ and $\mathcal{T}$ be a closed subset of $\mathbb{R}^{m\times n}$. Let $\Phi : \Omega \rightrightarrows \mathcal{T}$ be such that $\Phi(u)$ is a nonempty closed convex set for each $u \in \Omega$. Assume that Φ is lower semicontinuous on Ω. Then Φ admits a continuous selection.*

Lemma A.3.2 *Let X and Y be subsets of a finite dimensional space, such that X is bounded. Then*

$$\overline{X+Y} = \overline{X} + \overline{Y}.$$

Proof. At first, we are proving that $\overline{X+Y} \subset \overline{X} + \overline{Y}$. Fix any $z \in \overline{X+Y}$, then there is a sequence (z_k) in $X+Y$ such that $z_k \in X+Y$ for each $k \in \mathbb{N}$ and $z_k \to z$ as $k \to \infty$. Further, find a sequence (x_k) in X and (y_k) in Y such that $z_k = x_k + y_k$ for each $k \in \mathbb{N}$. Since $\overline{X}$ is compact, we can assume that $x_k \to x$ as $k \to \infty$ for some $x \in \overline{X}$. Then $y_k = z_k - x_k \to z - x \in \overline{Y}$ as $k \to \infty$. Hence $z = x + (z-x) \in \overline{X} + \overline{Y}$.

Now, we are proving that $\overline{X} + \overline{Y} \subset \overline{X+Y}$. Fix any $z \in \overline{X} + \overline{Y}$, then find $x \in \overline{X}$ and $y \in \overline{Y}$, such that $z = x + y$. There are a sequence (x_k) in X and (y_k) in Y such that $x_k \to x$ and $y_k \to y$ as $k \to \infty$. Hence $x_k + y_k \in X + Y$ for each $k \in \mathbb{N}$. Therefore $z = x + y \in \overline{X+Y}$. ■

Lemma A.3.3 *Consider a concave function $f : [0,\infty] \longrightarrow [0,\infty]$ with $f(0) = 0$. Then for each $x, u \in [0,\infty]$ we have*

$$f(x+u) \le f(x) + f(u). \tag{A.8}$$

Proof. Fix any $x, u \in [0,\infty]$. If $x = 0$ or $u = 0$ or $x = \infty$ or $u = \infty$, then (A.8) holds. If not, let $\lambda := \frac{x}{x+u}$, then $\lambda \in (0,1)$ and $1 - \lambda = \frac{u}{x+u}$. Thus since f is concave and $f(0) = 0$, we have

$$f(x) + f(u) = f\big(\lambda(x+u)\big) + f\big((1-\lambda)(x+u)\big) \ge \lambda f(x+u) + (1-\lambda) f(x+u) = f(x+u).$$

■

The following lemma is from [60, Lemma 2.1].

Lemma A.3.4 *Let $A, B,$ and C be subsets of $\mathbb{R}^n$. Suppose that B is closed and convex, C is bounded, and $A + C \subset B + C$. Then $A \subset B$.*