R E V I E W

of the doctoral thesis entitled "Openness of Mappings"

for obtaining PhD degree by RNDr. Tomáš Roubal

The review has been prepared according to Study and Examination Regulations of the University of West Bohemia (UWB), Section 107.

PhD thesis of T. Roubal consists of 80 pages. It includes Abstract, Preface, five Chapters, Conclusion, Bibliography counting 69 titles, and Appendix.

In the work author investigates various notions of regularity of single-valued and set-valued mappings. Metric regularity (equivalent to openness with linear rate and to the Aubin property of the inverse) is well-known property which is widely used in the literature while studying mappings. Metric regularity can be relaxed in two ways: to metric subregularity (equivalent to pseudo-openness and calmness of the inverse) and to metric semiregularity (equivalent to openness with linear rate at the reference point and recessiveness of the inverse).

In the first two chapters T. Roubal introduces these properties and gives a good historical overview of the known results.

In Chapter 3 are studied conditions for constrained and directional semiregularity of single-valued mappings in finite dimensional setting. The conditions are related to approximation of the single-valued mapping by a linear one in Section3.2, or by a family of linear mappings in Section 3.3, respectively. The conditions for semiregularity in question are given in Theorems 3.2.1 and 3.2.3, and in Proposition 3.3.1 and Corollary 3.3.1, as well. A modulus of constrained (semi)regularity of a linear mapping, where the constraints are given subspaces or cones, is studied in Section 3.4.

The considerations in Chapter 4 are in so-called quasi-metric space – a metric space but without the symmetry requirement for the metric – notion introduced by W. A. Wilson in 1930's. An extension of Ekeland Variational Principle in this setting is proved in Theorem 4.3.1. Some applications of the result are given in Section 4.3.

In Chapter 5 A. D. Ioffe's criterion for metric regularity is extended to quasi-metric spaces and for nonlinear and directional versions of regularity, subregularity, and semiregularity. Proofs are based on Ekeland Variational Principle in quasi-metric spaces, more precisely on Theorem 4.3.1.

Significance of the doctoral thesis for the field

Studying openness of mappings, single-valued as well as multi-valued, is challenging both from theoretical point of view, and from the point of view of applications

– to optimization problems, for example. In the thesis are considered interesting topics that permit extensions linked to significant applications.

Approach to solving the problem, methods used, and fulfilment of the given objective

The approach of the author is very fresh and full of enthusiasm. He skillfully handles topological and non-smooth optimization tools. Combining them T. Roubal succeed in achieving the goal of his work – extending the important notion of map regularity in different directions. The proofs are correct and carefully written. In general, the dissertation leaves very good impression.

Contributions to the given area of knowledge

Results presented in the thesis are new and original. I'm convinced they will have future developments.

Systematic approach, clarity, appropriateness of form and language

Thesis is very well written. The level of English language is good. There are only a little number of typos and grammar errors that do not reflect the understanding.

Some minor remarks

In the statement of Proposition 3.4.2. at p. 39 is used Moore-Penrose inverse matrix $A^\dagger$ without being properly introduced before with a definition. It is done later – at p. 77 in the Appendix.

In Example 4.2.1 at p. 49 in a normed space $(X, \|\cdot\|)$ is considered the set $\mathbb{B}_X^{T_L}(x,r) := \mathbb{B}_X(x,r) \cap (x + \operatorname{cone} L)$, where $\mathbb{B}_X(x,r)$ is the open ball centered at x with radius r and L is a non-empty set of the unit sphere. This set is called "open ball" without being neither open set nor a ball. I agree that this notion comes from the works of S. Cobzaş cited in the bibliography, but keeping the consistency of the notions should be the primal goal of any mathematician.

At p. 50 as Corollary 4.3.1 is stated Ekeland's Variational Principle. This is a fundamental result and as such it looks strange for me to be stated as a corollary even if it can follows from, saying, more general considerations.

In Example 5.1.2 in (ii) letter d denotes an element of a normed space X, while in (iii) letter d denotes a metric in a metric space X. While used one and the same letter for denoting different things in one and the same example this is a little bit confusing.

Does in the beginning of the statement of Proposition 5.2.1 "Let a function φ have the properties $(\mathcal{A}2) - (\mathcal{A}4)$ and satisfy $\gamma(x,\bar{x}) \leq \varphi(x,\bar{x})...$." mean "Let a function φ have the properties $(\mathcal{A}2) - (\mathcal{A}4)$ and for some function γ satisfy $\gamma(x,\bar{x}) \leq \varphi(x,\bar{x})...$."?. The same is the situation in Corollary 5.2.2, Proposition 5.2.3, and Corollary 5.2.4. However, similar statements in § 5.3 and § 5.4 are formulated precisely with regard to the function γ.

Publications

RNDr. Tomáš Roubal has presented five research co-authored papers, all indexed in Web of Science. Four articles are published in Q2 (for the year of publication) peer-reviewed mathematical journals in the field of dissertation – two in *Journal of Convex Analysis*, one in *Optimization*, and one in *Set-valued and Variational Analysis*. The high rank of the journals corresponds to the quality of the obtained results. The fifth paper is published in book series *Lecture notes in Economics and Mathematical Systems* and also is peer-reviewed.

Conclusion

Having become acquainted with the dissertation thesis and the accompanying scientific papers and on the basis of the analysis of their importance and the scientific and applied contributions contained therein, I confirm that the dissertation presented, as well as the quality and originality of the results and achievements presented therein, meet the requirements for acquisition of PhD degree by the candidate.

Based on the above, I strongly recommend the PhD thesis presented by RNDr. Tomáš Roubal for defence.

July 6, 2021 Signature: .....

Sofia, Bulgaria /Prof. DSci Nadia Zlateva/

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

Opponent Report on Dissertation Thesis

Openness of mappings

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The thesis is devoted to studying various criteria for regularity and its generalizations of single-valued and set-valued mappings, which is an actual and important topic in the fields of non-smooth and variational analysis, and mathematical optimization.

In the first chapter, we can find a motivation for the research. It also contains basic definitions and properties of metric regular, subregular and semiregular mappings.

The second chapter contains criteria for the above conditions which were published during last years. It reviews basic Graves, Robinson-Ursescu, and Robinson theorems and it ends with an overview of the Ioffe-type criteria.

Chapters 3, 4, and 5 bring then new findings. In particular, Chapter 3 is focused on constrained and directional semiregularity of single-valued mappings. Important new results can be found in Theorem 3.2.1, which establishes criteria for constrained semiregularity of a nonlinear mapping defined on locally convex closed set. These criteria are based on linear approximation of the mapping. This theorem is then employed to derive the conditions for the semiregularity for the sum of nonlinear and linear mapping, c.f. Theorem 3.2.3. Section 3.3 introduces new sufficient conditions for constrained semiregularity which are derived using an approximation by a bunch of linear mappings. Section 3.4 is then focused on the directional (semi)regularity of linear mappings with constraints given by subspaces and cones. Moduli of (semi)regularity are derived using the singular value decomposition and conical eigenvalues, see Propositions 3.4.1 – 3.4.5.

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Chapter 4 is devoted to topological and quasi-metric spaces, c.f. Sections 4.1 and 4.2. In Section 4.3, the original Ekeland variantional principle (Ekeland 1974) is extended from complete metric spaces to the quasi-metric space, see Theorem 4.3.1. Several applications are then presented.

In Chapter 5, new types of openness for the set-values mapping are introduced, in particular, openness around and at a reference point, and pseudo-openness at a reference point. Connection between these definitions and (sub)regularity is studied in Section 5.1. In Section 5.2, necessary and sufficient conditions for these properties are introduced using the Ekeland variational principle. Ioffe-type criteria are derived in quasi-metric spaces, see Propositions 5.2.1 and 5.2.3 for semiregularity criteria, Prepositions 5.3.1 and 5.3.3 for subregularity criteria, and finally Prepositions 5.4.1 and 5.4.2 for regularity criteria.

I would like to ask two questions:

1. If the approximating mapping g in Proposition 3.1.1 is an affine mapping, then this statement can be extended from finite dimensional spaces to Fréchet spaces, see [15, Theorem 5]. Is it possible to prove Proposition 3.1.1 within this more general framework? Moreover, can we generalize Proposition 3.1.3 in the similar way?

2. Is there any hope to obtain Ioffe-type necessary and/or sufficient conditions (e.g., Proposition 5.2.1, Proposition 5.3.1, Proposition 5.4.1) without the completeness assumption (A_4) on the domain space? Especially, in connection with a recent work: M. M. Ait, M. A. Bahraoui, and A. El Bekkali. *Metric Regularity and Lyusternik-Graves Theorem via Approximate Fixed Points of Set-Valued Maps in Noncomplete Metric Spaces*. Set-Valued Var. Anal.:1-24, 2020.

To summarize, the thesis brings original results which contribute significantly to the fields of variational analysis, non-smooth analysis, and mathematical optimization. The mathematical methods used to derive new results are original and in most cases highly non-trivial. I consider the goals of the work fulfilled. I would like to appreciate many examples that complement the theorems and make it easier to understand the complex results.

The thesis is written very precisely. I found almost no typos or ambiguities. The candidate works correctly with the sources, they are properly cited in the text and his contribution is clearly specified.

The candidate has already published 5 papers (as a coauthor), where 4 of the papers are published in leading international journals of the field of variational analysis with positive WOS Impact Factor, i.e. Journal of Convex Analysis (2 papers), Set-Values and Variational Analysis, and Optimization. My usual expectations for young Ph.D. students are two articles, which the applicant exceeds twice.

Based on the above facts, **I recommend the submitted thesis for defense.**

In Prague, May 28, 2021

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