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Illustrating Geometric Algebra and Differential Geometry in 5D Color Space

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Vectors in three-dimensional Euclidean space are a fundamental concept in computer graphics and physics. Linear Algebra provides the well-known operations of adding vectors or multiplying vectors with scalars. Together with matrix algebra this framework allows for pretty much all operations that are needed for practical work. However, this set of operations is inherently incomplete such that not all operations known for scalar numbers can be applied to vectors. Particularly we can divide by numbers, but what does it mean to divide by a vector? Such an operation is not defined in Linear Algebra as there is no invertible product of vectors: There is the inner (dot) product $v \cdot u$ and the exterior (cross) product $v \wedge u$, but neither of them is invertible. It was the idea of William Kingdon Clifford to combine both products, defining the "geometric product" thereby as

$$uv := v \cdot u + v \wedge u \quad ,$$

which turns out to be invertible, though at the cost of introducing a higher dimensional space of so-called "multivectors". This extension of Linear Algebra is known as Clifford Algebra or Geometric Algebra (GA)[Hes03, Hil13, DL03]. This formalism allows for a complete algebra on vectors same as for scalar or complex numbers. It is particularly suitable for rotations in arbitrary dimensions. In Euclidean 3D space quaternions are known to be numerically superior to rotation matrices and already widely used in computer graphics. However, their meaning beyond its numerical formalism often remains mysterious. GA allows for an intuitive interpretation in terms of planes of rotations. This algebraic framework extends easily to arbitrary dimensions and is not limited to 3D, like quaternions. However, our intuition of more than three spatial dimensions is deficient. The space of colors forms a vector space as well, though one of non-spatial nature, but spun by the primary colors red, green, blue. The GA formalism can be applied here as well, amalgamating surprisingly well with the notion of vectors and co-vectors known from differential geometry: tangential vectors on a manifold correspond to additive colors red/green/blue, whereas co-vectors from the co-tangential space correspond to subtractive primary colors magenta, yellow, cyan. GA in turn considers vectors, bi-vectors and anti-vectors as part of its generalized multi-vector zoo of algebraic objects. In 3D space vectors, anti-vectors, bi-vectors and co-vectors are all three-dimensional objects that can be identified with each other, so their distinction is concealed. In particular, in 3D all three basis vectors are given by the three primary colors red, green, blue. A bi-vector is the outer product of vectors. The bi-vector given by the $\vec{x}$ and $\vec{y}$ axis in Euclidean space is therefore the plane spun by the xy plane. Three such planes exist in three dimensions: xy, xz and zx (in cyclic notation). In the color space those combinations of two basis color vectors are then yellow=red∧green, cyan=green∧blue, and =blue∧red. Same as in Euclidean space, where we can identify a vector with a plane via the notion of a "normal vector", we can identify a color with a mixed color via its complementary color. This identification may ease some usage, but also leads to confusions, because the underlying objects - a vector versus a plane, or a pure color versus a mixed color - are inherently different.

Higher dimensional spaces exhibit the differences more clearly. In four dimensions there exist four vectors but six bi-vectors. Using space and time as the four dimension space, the four basis vectors $\vec{x}$,$\vec{y}$,$\vec{z}$ and $\vec{t}$ result in the six possible combinations xy, yz, zx (three "spatial" bi-vectors) and xt, yt, zt (three "temporal" bivectors). Evidently, identifying every vector with every bi-vector is no longer possible in 4D as it was in 3D. The distinction between direction vectors and planes becomes unavoidable. Using colors instead of spatial dimensions we can expand our intuition by considering "transparency" as an independent, four-dimensional property of a color vector. We can thereby explore 4D GA alternatively to spacetime in special/general relativity. Here, we start with red, green, blue and transparent as the basis vectors and construct three non-transparent mixed colors yellow, cyan, magenta and three transparent pure colors transparent red, transparent green, transparent blue. Clearly, there is no way to identify those six bi-vectors with the six vectors in 4D space, not even via some complement.

However, even in 4D possibly confusing ambiguities remain between vectors, co-vectors, bi-vectors and bi-co-vectors: bi-vectors and bi-co-vectors - both six-dimensional objects - are visually equivalent. A co-vector in differential geometry is a linear, scalar valued function on vectors. These functions form their own vector space and can be seen as dual vectors. Visually we may interpret co-vectors as the complement of a vector to form the full space: In 3D a plane complements a vector to form the full volume. Therefore a co-vector is equivalent to a bi-vector. Both have three components in 3D. In 4D a vector is complemented with a tri-vector to form a four-volume, thus in 4D co-vectors and tri-vectors are equivalent. Within the concept of color-spaces the co-vectors play the role of subtractive colors. Here, the basis co-vectors are built by the CMY system yellow, cyan and magenta. Their combination via light-subtractive filtering (expressed as the $\wedge$-product) forms the bi-co-vectors green = yellow$\wedge$cyan , blue = cyan$\wedge$magenta and red = magenta$\wedge$yellow. The equivalence of bi-co-vectors with vectors in 3D space is obvious: they are the same colors. The basis co-vectors of 4D color space are constructed by "cutting off" a basis vector from the full 4D "color-hypervolume" $\Omega :=$ red$\wedge$green$\wedge$blue$\wedge$transparent. transparent magenta, transparent cyan and transparent yellow. For instance, "cutting off" red from Ω yields green$\wedge$blue$\wedge$transparent, a 3D "color volume", which is equivalent to a transparent cyan co-vector. Four such co-vectors exist in 4D: transparent cyan, transparent magenta, transparent yellow and non-transparent white. They are the set of all combinations with three properties. A bi-co-vector is constructed by cutting off two properties from the color hypervolume Ω, for instance cutting off the bi-vector red$\wedge$transparent yields the bi-co-vector (non-transparent) cyan. It is visually identical to the bi-vector (non-transparent) cyan because cyan is "blue and green", but can equivalently be described in 4D color space as "not transparent and not red". Both bi-vectors and bi-co-vectors provided two color properties and are thus visually indistinguishable in 4D. Higher dimensions are needed for such an unequivocal distinctions.

Envisioning five-dimensional geometry is even more challenging to the human mind than four-dimensional geometry, which we can at least associate with space-time. In color space we can add another property to the three primary colors and transparency. For instance, we can add "texture" or ~~strikethrough text~~ to constitute a five-dimensional vector space. The five-dimensional hypervolume is then

$$\Omega_{5D} := \text{red}\wedge\text{green}\wedge\text{blue}\wedge\text{transparent}\wedge\cancel{\text{strikethrough}}$$

as constructed from the five base color/texture vectors. In 5D we have ten bi-vectors and ten bi-co-vectors. The bi-vectors are built from the $\wedge$-product of all basis vectors, there are ten possibilities to combine two properties in 5D: three mixed colors cyan, magenta, yellow, three transparent pure colors transparent red, transparent green, transparent blue, three textured pure colors ~~red~~, ~~green~~, ~~blue~~, and one ~~textured transparent~~ element. In contrast, the bi-co-vectors are built from all color elements that combine three properties in 5D, which are also 10 color space elements: one non-transparent, non-textured element built from all three colors, i.e. "white", three transparent, textured colors ~~textured transparent red~~, ~~textured transparent green~~, ~~textured transparent blue~~, three transparent mixed colors transparent magenta, transparent cyan, transparent yellow and three textured mixed colors ~~cyan~~, ~~magenta~~, ~~yellow~~. None of these bi-co-vectors is visually equivalent to any of the bi-vectors in 5D. The three-property color elements are distinct from the two-property color elements. Thus, in this five-dimensional color space we can see immediately that bi-co-vectors are distinct from bi-vectors, a distinction that is not obvious in 4D or 3D. While envisioning the same geometrically via five spatial dimensions is hard, but using color space it is easy to comprehend. This impression serves to demonstrate that vectors, bi-vectors, co-vectors and bi-co-vectors are actually different kinds of vectors, and they should be treated as objects with different properties before identifying them in special situations. An explicit distinction clarifies the meanings of algebraic objects in 3D Euclidean space such as "tangential vectors", "axial vectors" or "normal vectors", which are just 3D names of these vector quantities: a "tangential vector" is basis vector in 3D; an "axial vector" is a bi-vector in 3D; a "normal vector" is a co-vector in 3D. Confusing these different algebraic objects in 3D unavoidably leads to programming errors, such as applying a wrong coordinate transformation (normal vectors transform inversely to tangential vectors). A type-safe implementation that honors the mathematical differences therefor allows for better, clearer formulations of algorithms in 3D that are less prone to implementation errors.

The ideas presented here are meant to inspire using colors and beyond as alternative to spatial geometry. We did not make use of the inner product which may find its use in vision research to describe perceptual intensity, for instance. Also, we did not make use of the anti-symmetric property of the $\wedge$ product such that $x \wedge y = -y \wedge x$ which introduces an orientation (this is why the highest dimensional $\wedge$-product was called "Ω" in this text) to multivectors: with red$\wedge$green being a "left-polarized" yellow versus green$\wedge$red yielding a "right-polarized" yellow may open an approach to include more properties of light into a mathematical framework. This is left for future work and / or an inspired audience.

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