S&P 500 AND CRYPTOCURRENCY MARKET RISK MEASURED BY STANDARD DEVIATION, VAR AND CVAR: A COMPARISON STUDY

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Abstract: VaR and CVaR are effective quantitative measurements of market risk. These measures can quantify the risk of unexpected changes within a given period. In this paper, we examine the market risk of the US stock market index S&P 500 and cryptocurrencies bitcoin and ripple. The returns of these three instruments are approximated using normal inverse Gaussian distribution and alpha stable distribution. For comparison, the normal distribution is also included. Subsequently, the VaR99 and CVaR97.5 values corresponding to four candidate distributions are calculated for these instruments. We also analyze the ability of theoretical distributions to approximate the left tail behavior of stock market index returns. It turns out that the normal distribution is not suitable for this purpose. Furthermore, it appears that CVaR97.5 is higher (in absolute value) for all indices than the corresponding VaR99 which may require higher need for economic capital which banks should allocate.

Key words: US index S&P 500, cryptocurrencies, normal inverse Gaussian distribution, alpha stable distribution, VaR and CVaR

JEL Classification: C58, G14

INTRODUCTION

Volatility is an integral part of stock market dynamics. It provides opportunities to make a lot of money as well as to incur huge losses. Therefore, any market participant has to take adequate risk management measure to counter its negative exposure. In order to do so, risk has first to be quantified. The same holds for regulatory purposes. Value at Risk (VaR) and Conditional Value at Risk (CVaR) are two of the main measurements of market risk. These measures are very simple and popular quantificators of market risk and they are widely used in practice. Moreover, they are applicable in measuring other types of risk as well.

Computing VaR and CVaR heavily depends on the specification of distribution used for modeling price or return dynamics. It has been known for quite a long time that the distribution of financial asset returns in general as well as stock price returns have heavier tails and sharper peak than the corresponding normal distribution. Not only can the correct choice for their distribution help to find the answer to our problem, but it is also of great importance for VaR and CVaR evaluation as well as for asset. So far, many efforts of researchers as well as practitioners have been devoted to this task. There are two ways how to deal with it.

The first one, which is less inconvenient but may not yield the needed accuracy, is to replace the normal distribution by an alternative distribution with the same number of parameters as the normal one which exhibits the leptokurtic property. In general, the probability distribution with heavy ends (alpha stable distribution) or the distribution with so-called semi-heavy tails (generalized hyperbolic distribution and its special cases) are considered. For the generalized hyperbolic distribution family, (see Prause, 1999), and (Eberlein & Keller, 1995), for the skewed generalized t-distribution family, (see Theodossiou, 1998), Zhu &Galbraith, 2012), (Platen & R. Rendek, 2008) and (Guo, 2017). The second way how to solve this problem is to use a candidate distribution with more than two parameters. In this case, the additional parameter(s) will capture the tail and peak behavior of the distribution of financial asset returns. However, additional parameters also make estimation procedure more complicated. In the literature, two distributions with semi-heavy and heavy tails are often chosen for this task: normal inverse Gaussian and alpha stable distribution. Hence, VaR and CVaR as measures of stock market risk can be calculated with these two distributions. The objective of this research is to find how market risk can be adequately quantified by these two measures and which distribution is a good one for approximation of returns of stock market indices.

In this paper, we compare the daily returns of the well-known S&P 500 index and the two cryptocurrencies Bitcoin and Ripple. Bitcoin (BTC) was proposed by an unknown person or persons, under the name Satoshi Nakamoto in October 2008, as a combination of a digital asset and a peer-to-peer payment system in his study: Bitcoin: A Peer-to-Peer Electronic Cash System. The first bitcoin was minted on January 4, 2009, and the first payment was January 11, 2009. The software was released as an open source on January 15, 2009, allowing anyone with sufficient technical skills and computer equipment to engage in development. For a long time, Bitcoin was of little interest. However, from the second quarter of 2012, transaction volumes began to grow dramatically. The current daily average volume of Bitcoin transactions during their lifetime (from January 4, 2010 to January 1, 2017) is 19,301,677 USD.

Bitcoin “coins” are created by a network of computers with specialized software programmed to release new coins at a steady but still declining pace. The number of coins in circulation should reach 21 million in year 2140 when the coinage should be terminated.

Bitcoin weekly volatility reached up to 60% per annum during the 2016, but other cryptocurrencies shows even bigger swings. Bitcoin is, however, still more volatile than any fiat currency pairs.

Bitcoin has volatility seven times greater than gold, eight times greater than the S&P 500, and 18 times greater than the U.S. dollar.

Currently, there are also a few places where you can trade with options on Bitcoin (e.g. Deribit). In general, the more liquid the derivatives markets are, the more deeply liquid the underlying product is. The appearance of options of Bitcoin shows that liquidity of Bitcoin increased.

Ripple (XRP) is a global currency exchange and remittance network that aims to lower the cost and improve the speed of international bank transfers relative to legacy financial infrastructure. Also called the Ripple Transaction Protocol (RTXP) or Ripple protocol, it is built upon a distributed open-source Internet protocol, consensus ledger and native currency called Ripples (XRP). Ripples digital currency acts as a bridge currency to other currencies and does not discriminate between fiat or cryptocurrency, making it easier for currency to be exchanged. Each currency on the ecosystem has its own gateway, allowing users to send payments in one currency, and the recipient of the payment to receive the payment in their preferred currency. XRP Ledger is publicly available code, so-called open source, and as with Bitcoin it is decentralized, it consists of individual users' computers. There is no server that checks and records individual transactions. The so-called nodes and validators are used for this. Anyone can become a node and validator in the XRP Ledger network, just download the appropriate software and run it on your computer. In addition to individuals, validators are run by companies such as Microsoft or universities such as the Massachusetts Institute of Technology. The main function of the XRP cryptocurrency is not the value itself, as in the case of bitcoin, for example, but as a functional element of the payment network. XRP’s price chart indeed reveals that daily swings are wide. Furthermore, the direction of the next move usually surprises investors. Therefore, short-term Ripple traders should be cautious. If you are an investor with a two-year to three-year horizon, you could consider having a small exposure below $1. If RippleNet increases its partnership with global financial institutions, it is likely to continue creating value for XRP investors.

XRP are subject to very high volatility. An example of this was in 2017 when XRP jumped 54,900% from $0.006 in March 2017 to $3.30 in December of that same year. Then in April 2018, the coin plummeted 86% to $0.44.

Finally, those market participants who do not want to experience the daily choppiness in Ripple could consider investing in exchange-traded funds (ETFs) that give exposure to the cryptocurrency and blockchain, the technology behind these digital assets.
Paper is organized as follows
In section 1 we first present both considered distributions (alpha stable and normal inverse Gaussian). Since we use the MLE method for estimation and the stable distribution does not have density in the analytical form, the method of Borak, Hardle, Weron (2005) using inverse Fourier transform of the characteristic function is briefly explained. Then, a definition of VaR and CVaR is followed and a brief discussion of the advantages and disadvantages of these risk measures: VaR values are calculated at 99% and CVaR values at 97.5%, according to the BIS (Bank for International Settlements) recommendation, which is based on the fact that for normal distributions these values are the same. The following section after the first section presents the results of parameter estimation for both distributions and the corresponding VaR and CVaR values. It turns out that the heavier the ends of the considered approximate distribution, the more the CVaR value exceeds the VaR value.

1. METODOLOGY

In this section a brief description of heavy and semi-heavy tailed distributions of our interest is provided. We will consider alpha stable distribution, and normal-inverse Gaussian (NIG) and for parameter estimation maximal likelihood method is used.

1.1 Stable distributions

This probabilistic distribution is formulated by its characteristic function because density function does not exist in explicit form (unlike NIG)

$$\Phi(t) = \exp\left\{-\sigma^{\alpha}|t|\left(1 - i\beta sign\ (t)\tan\frac{\pi\alpha}{2}\right) + i\mu t\right\} \quad \text{for } \alpha \neq 1 \tag{1}$$

$$\Phi(t) = \exp\left\{-\sigma|t|\left(1 - i\beta\frac{2}{\pi} sign(t)\log|t|\right) + i\mu t\right\} \quad \text{for } \alpha = 1 \tag{2}$$

where parameters $\alpha \in (0, 2], \beta \in [-1,1], \mu \in \mathbb{R}$ and $\sigma \in [0, \infty)$

So the α - stable distribution has four parameters. These parameters may be interpreted as:

α… tail power (tail index), as α decreses tail thicknes increases
β… skewness parameter, determines asymmetry, a positive β indicates that right tail is fatter then left tail and vice versa, $\beta = 0$ corresponding to a symmetric distribution
μ … location parameter (corresponding to a mean for $\alpha > 1$)
σ… scale parameter, generalized standard deviation, for $\alpha = 2$ corresponding to a standard deviation of normal distribution

Power of the tails

The power of the tail is the index α which approximately means that $P(X < x) \approx c_{\alpha}|x|^{-\alpha}$ as $x \to -\infty$. The exact formula for c_{α} can be found in (Nolan, 2020).

Estimation of Parameters of stable distribution using Maximal Likelihood Estimation (MLE)

Because closed-form formula for the probability density function of NIG distribution exists using MLE is straightforward. In the case of stable distributions there is no explicit for of density function, so we have to find it by using inverse Fourier transformation.
According (Borak, et al.,2005), after substitution $\zeta = -\beta\tan\frac{\pi\alpha}{2}$ the density of standard $\alpha -$ stable random variable (μ=0, σ =1) for $\alpha \neq 1$ can be expressed as:
for $x > \zeta$:

$$f(x;\alpha,\beta) = \frac{\alpha(x-\zeta)^{\frac{1}{\alpha-1}}}{\pi|\alpha-1|}\, W(x,\alpha,\beta,\zeta) \tag{3}$$

where

$$W(x,\alpha,\beta,\zeta) = \int_{-\xi}^{\frac{\pi}{2}} V(\theta;\alpha,\beta)\, exp\left(-(x-\ \zeta)^{\alpha/\alpha-1}\, V(\theta;\alpha,\beta)\right) d\theta,$$

for $x = \zeta$:

$$f(x;\alpha,\beta) = \frac{\Gamma\left(1+\frac{1}{\alpha}\right)\cos\xi}{\pi(1+\zeta^2)^{\frac{1}{2\alpha}}}$$

and for $x < \zeta$:

$$f(x;\alpha,\beta) = f(-x;\alpha,-\beta)$$

where

$$V(\theta;\alpha,\beta) = (\cos\alpha\xi)^{\frac{1}{\alpha-1}}\left(\frac{\cos\theta}{\sin\alpha(\xi+\theta)}\right)^{\alpha/\alpha-1}\frac{\cos[\alpha\xi(\alpha-1)\theta]}{\cos\theta}$$

$$\xi = \frac{1}{\alpha}\arctan(-\zeta)$$

In MLE we have to find from observation data x_i a maximum of the likelihood function

$$\sum_{i=1}^{n} \log f(z_i;\alpha,\beta,\delta,\mu)$$

with respect to parameters α,β,δ,μ , where $z_i = \frac{x_i-\mu}{\delta}$.

1.2. Normal inverse Gaussian distribution as a special case of generalized hyperbolic distributions

This generalized hyperbolical distributions was introduced by (Barndorff-Nielsen 1977) and at first applied them to model grain size distributions of wind-blown sands. (Eberlein and Keller 1995) were the first to apply these distributions to finance. The probability density function is as follows:

$$f(x) = \frac{(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\left(\lambda-\frac{1}{2}\right)}\delta^{\lambda}K_{\lambda}\left(\delta\sqrt{(\alpha^2-\beta^2)}\right)}\, P(x) \tag{4}$$

where

$$P(x) = (\delta^2+(x-\mu)^2)^{(\lambda-1/2)/2}K_{\lambda-1/2}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)\exp\left(\beta(-\mu)\right),$$

where $K_\lambda(x)$ is the modified Bessel function of the third (second) kind with index $\lambda \in \mathbb{R}$. It can be defined as

$$K_\lambda(x) = \frac{1}{2}\int_0^\infty s^{\lambda-1}\exp\frac{x(s+s^{-1})}{2}\, ds \tag{5}$$

For $\lambda = 1/2$, we get the normal-inverse Gaussian distribution (NIG). So, the probability density function of NIG distribution is (using some properties of Bessel functions):

$$f(x) = \frac{\alpha\delta\, K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\pi\sqrt{\delta^2+(x-\mu)^2}}\exp\left(\delta+\beta(x-\mu)\right) \tag{6}$$

The semi-fat tail property of generalized hyperbolic distribution family coming from the following asymptotic property of Bessel function:

$$P(X \leq x) \approx |x|^{\lambda-1} \exp[(\alpha+\beta)x] \quad \text{as} \quad x \to -\infty$$
(7)

So we have to find from observation data x_i a maximum of the likelihood function $\sum_{i=1}^{n} \log f(x_i; \alpha, \beta, \delta, \mu)$
with respect to parameters $\alpha, \beta, \delta, \mu$

1.3 Value at Risk and Conditional Value at Risk

Value at risk (VaR) at the level $\alpha \in (0,1)$ is defined by

$$VaR_\alpha(Y) = inf\{x \in \mathbb{R} | F_Y(x) \geq \alpha\}, \quad (8)$$

where *Y* is loss random variable (losses are positive, gains negative) with cumulative distribution function $F_Y(x)$.

VaR has become a standard risk measure in finance. But it has a disadvantage of lacking subadditivity[1] which means that a (diversified) portfolio may have a higher risk (VaR) than the sum of its individual parts.

Conditional VaR (CVaR), sometimes called Expected shortfall (ES) [2], is defined by

$$ES_\alpha(Y) = \frac{1}{1-\alpha} \int_{1-\alpha}^{1} VaR_s(Y) ds.$$
(9)

CVaR can be interpreted as a conditional mean value of losses provided that the VaR has been exceeded. CVaR has been proposed as an alternative to VaR risk because its subaddition property. However, it is often criticized for computational difficulty, limited backtesting capabilities and high sensitivity to extreme data (lack of robustness).

From a statistical point of view, CVaR should not be preferred to VaR, but CVaR has one advantage because it is far more difficult to manipulate. Banks can manipulate risk measures by selecting a specific estimation method. However, there is no certainty that the estimation method that gives positive results for the bank today will do the same tomorrow.

Short discussion about VaR and CVaR

(Danielsson et al., 2005) report that for most practical applications VaR is sub-additive and there is no reason to choose a more complicated risk measure than VaR, solely for reasons of subadditivity. (Ibragimov and Walden, 2007) showed that for very heavy-tailed risks diversification does not necessarily decrease tail risk and sometimes can increase it which makes the subadditivity requirement unnatural. According these authors (and others) these objections to the subadditivity deserve to be considered and support the choice of robust risk measurement.

According to other authors, the main deficiency of VaR (in addition to lack of subaditivity) is that it does not cover tail risks beyond VaR although it makes VaRa risk measure more robust than the other risk measures. This deficiency can be particularly serious when one faces choices of various risks with different tails. ES makes good for the lack of subadditivity of VaR but it is not elicitable. This means that backtesting of ES is less straightforward than backtesting of VaR. But there are feasible approaches to the backtesting[3] of ES although to reach the same level of certainty more validation data is required for ES than for VaR.

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[1] The subaditivity property for the risk measure means that the sum of the risks of the two portfolios X and Y is greater or equal than the total risk of both merged portfolios X + Y (diversification effect).

[2] These two terms (ES and CVaR) are not exactly the same, but they are identical for continuous distributions.

[3] One of the possible backtests (also recommended by BIS) is based on a subsequent approximation: $ES_\alpha(X) \approx \frac{1}{4}\left[q_\alpha(X) + q_{0.75\alpha+0.25}(X) + q_{0.5\alpha+0.5}(X) + q_{0.)25\alpha+0.75}(X)\right]$

2. DATA AND RESULTS

For our empirical analysis two cryptocurrencies bitcoin and ripple and stock market index S&P500 are chosen. Data for both cryptocurrencies are daily series of close values from 9-11-2017 to 4-2-2022 and they are obtained from Yahoo database. Data for S&P 500 are also daily ones from 14-12-2015 to 4-2-2022 and they are obtained from FRED database. The original series then are transformed into log-return series. We display their basic descriptive statistics in two table 1

Table 1 Descriptive statistics of time series in the econometric analysis

	original series			logarithmic returns		
	BTC	XRP	SP 500	BTC	XRP	SP 500
mean	18673,3	0,531269	2966,604	0,001117	0,000694	0,00052
median	9630,664	0,37127	2801,31	0,001691	-0,00084	0,000768
minimum	3236,762	0,139635	1829,08	-0,46473	-0,5505	-0,12765
maximum	67566,83	3,37781	4796,56	0,225119	0,606885	0,089683
1st quartile	7218,135	0,263618	2429,375	-0,01601	-0,02368	-0,00303
3rd quartile	32078,41	0,67867	3317,475	0,018967	0,021478	0,005441
std	17631,41	0,392851	741,9039	0,041523	0,066877	0,011619
skewness	1,252392	2,231216	0,83991	-0,85853	0,854557	-1,0964
kurtosis	3,045015	11,00664	2,80673	15,41087	18,93303	24,73759
n. of obs.	**1549**	**1549**	**1549**	**1548**	**1548**	**1548**

Source: Own calculations

Table 2 Sharpe ratio (mean/std)

	mean	std	Sharpe ratio	rank
BTC	0,001117	0,041523	0,026906629	middle
XRP	0,000694	0,066877	0,010378463	worst
SP500	0,00052	0,011619	0,044750682	best

Source: Own calculations

We see that bitcoin has the highest average daily return and the S&P 500 the lowest. If we measure risk by standard deviation, the ripple is the most risky and the S&P 500 is the least risky. Sharpe ratio, which measures return per unit of risk, is highest for the S&P 500 stock index, lowest for ripples.

It can therefore be concluded that investing in a ripple (if we take into account the Sharpe ratio) is the least advantageous, because the average return is accompanied by high risk. From this point of view, it would be most advantageous to invest in a portfolio that replicates the S&P 500 index.

where $q_{\alpha}(X), q_{0.75\alpha+0.25}(X), q_{0.5\alpha+0.5}(X), q_{0.)25\alpha+0.75}(X)$ are corresponding quantiles. So, we have to backtest them just like we do at VaR.

Table 3 Estimation results of NIG parameter distribution

currency	parameter	Coefficient	SE	z-stat	p-value
BTC	alpha	13,81002	0,726915	18,99813	0
	beta	-0,54902	0,190318	-2,88475	0,003917
	delta	0,024529	0,005004	4,9018	9,50E-07
	mu	0,002093	0,06234	0,033576	0,973215
XRP	alpha	6,53982	1,197775	5,459973	4,76E-08
	beta	0,418575	0,246452	1,698404	0,089432
	delta	0,028381	0,001217	23,31747	0
	mu	-0,00113	0,001052	-1,07066	0,284321
SP 500	alpha	44,40342	5,304945	8,370194	0,00E+00
	beta	-5,39343	2,911659	-1,85236	0,063975
	delta	0,005456	0,000283	19,28385	0
	mu	0,001187	0,000213	5,587328	2,31E-08

Source: Own calculations

Table 4 Estimation results of parameters of stable distribution

currency	parameter	Coefficient	SE	z-stat	p-value
BTC	alpha	1,445392	0,038493	37,5497	0
	beta	-0,03446	0,071616	-0,4812	0,630373
	mu	0,019448	0,00055	35,343	0
	delta	0,001751	0,000856	2,045022	0,040853
XRP	alpha	1,357857	0,037248	36,4543	0
	beta	-0,01527	0,064215	-0,23784	0,812007
	mu	0,024837	0,000737	33,68731	0
	delta	-0,0008	0,001072	-0,74265	0,457695
SP 500	alpha	1,458221	0,038627	37,75123	0
	beta	-0,11804	0,072457	-1,62905	0,103303
	mu	0,004694	0,000132	35,64421	0
	delta	0,001257	0,000207	6,067191	1,30E-09

Source: Own calculations

Table 5 VaR 99

	VaR 99					
	Gaussian		NIG		stable	
BTC	-0,0955	middle	-0,1264	middle	-0,172	middle
XRP	-0,1549	worst	-0,1934	worst	-0,2689	worst
SP 500	-0,0265	best	-0,0357	best	-0,0418	best

Source: Own calculation

Table 6 CVaR 97.5

CVaR 97.5						
	Gaussian		nig		stable	
BTC	-0,096	middle	-0,1322	middle	-0,2669	middle
XRP	-0,1557	worst	-0,2045	worst	-0,4488	worst
SP 500	-0,0265	best	-0,0378	best	-0,0647	best

Source: Own calculation

Table 7 CVaR/VaR

	NIG		Stable	
BTC	1,045886	best	1,551744	middle
XRP	1,057394	middle	1,669022	worst
SP 500	1,058824	worst	1,547847	best

Source: Own calculations

CONCLUSIONS

We see that the VaR 99 and CVaR 97.5 values depend on the probability distributions that were chosen as an approximation for the empirical distribution. The results confirm our assumption that the fatter tail of a given approximate distribution, the higher the CVaR in absolute value than the corresponding VaR. The stable distribution, which has stronger ends than NIG, has higher values (in absolute value) of VaR and CVaR. The normal distribution that was used for comparison obviously has the same VaR and CVaR values, however, as an approximation, it turns out to be inappropriate. The ripple cryptocurrency has the fattest tail which is reflected in the magnitudes of VaR and CVaR, which are the highest (in absolute value). The S&P index, on the other hand, has by far the lowest VaR and CVaR values.
From Table 6 we see that the highest CVaR / VaR ratio is for ripple and the lowest for the S&P 500, which has the thinnest ends. Therefore, if a bank invests in a ripple, its need for economic capital to cover unexpected losses will be highest.

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