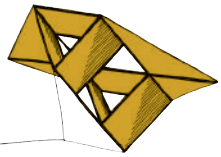

Minimization of energy functionals via FEM: implementation of hp-FEM

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1 Introduction

Nonlinear energy functionals appearing in the calculus of variations can be discretized by the finite element (FE) method and formulated as a sum of energy contributions from local elements. In Matonoha et al. (2022); Moskovka and Valdman (2022); Moskovka (2023) we demonstrated minimization of several functionals using vectorized implementation with the simplest linear nodal (P1) elements in MATLAB. An assembly of the discrete energy functional gradient required by the trust region method is crucial for the performance of minimization process. In this contribution, we extend our vectorization concept by incorporating two-dimensional rectangular hp-finite elements.

2 Models

Two models appearing in science and engineering are considered. The first, a p-Laplace equation, is given by

$$\begin{aligned} \Delta_\alpha u &= f \qquad \text{in } \Omega\,, \\ u &= g \qquad \text{on } \partial\Omega \end{aligned} \tag{1}$$

for some $\alpha > 1$. Solving (1) is equivalent to finding the minimum of the corresponding energy functional

$$J(u) = \min_{v \in V} J(v)\,, \qquad J(v) := \frac{1}{\alpha}\int_\Omega \|\nabla v\|^\alpha \,\mathrm{d}x - \int_\Omega f\, v \,\mathrm{d}x\,, \tag{2}$$

where

$$V = W_g^{1,\alpha}(\Omega) = \{v \in W^{1,\alpha}, v = g \ \text{ on } \partial\Omega\}.$$

One of the most well-known vector problems from mechanics is the nonlinear elasticity which describes the deformation of the material loaded by a force. The corresponding energy functional is given by

$$J(\mathbf{v}) = \int_\Omega W\big(\mathbf{F}(\mathbf{v}(\mathbf{x}))\big)\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{f}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,, \tag{3}$$

where $\mathbf{v} : \Omega \to \mathbb{R}^{dim}$ is a deformation mapping, $\mathbf{F} = \nabla(\mathbf{v})$, $\mathbf{f}$ is a loading, $\mathbf{g}$ is an external load on the Neumann part of the boundary and $W(\mathbf{F})$ is a density function which depends on the model choice.

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Fig. 1 depicts the solutions of the p-Laplace problem (2) (left) and hyperelasticity (3) (right). For p-Laplace we consider $\alpha = 3$ and a constant $f = -10$. For hyperelasticity a constant volumetric vector force $f = (-3.5 \cdot 10^7, -3.5 \cdot 10^7)$ is considered. Fig. 2 depicts the corresponding energy errors with respect to the number of dofs or evaluation times. For both p-Laplace and hyperelasticity the Q2 elements dominate over P1 and Q1.

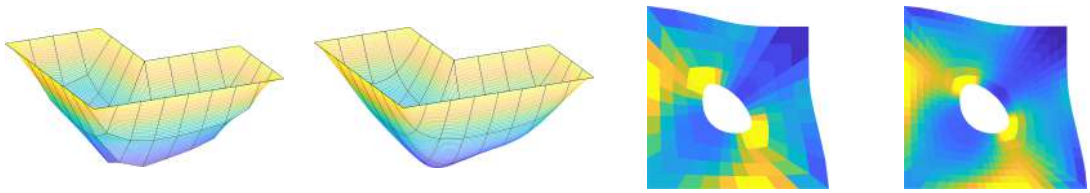

Figure 1: Solutions of p-Laplace (left) and hyperelasticity (right) problems. The same computational mesh for p-Laplace is considered using Q1 elements (the 1st image) and Q4 elements (the 2nd). Deformation of a body with the underlying NeoHookean density is depicted for a coarse mesh with Q4 elements (the 3rd image) and a fine mesh with Q1 elements (the 4th).

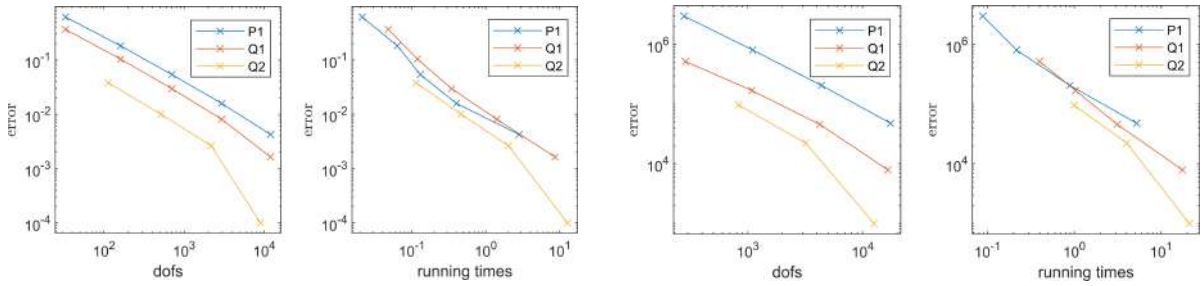

Figure 2: Two graphs on the left depict absolute energy errors of p-Laplace with respect to the number of dofs and evaluation times. Energy errors for hyperelasticity are displayed on two graphs on the right.

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