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Dimensionless form of straight pipe pressure loss formula

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The design of the heat exchanger equipped with straight pipes solves the heat transfer and the pressure loss as a coupled problem. While so called heat transfer similitude formulas (Nusselt number criterion formulas) are commonly used in dimensionless form, the pressure loss phenomenon is traditionally analysed in the dimension form of the Darcy–Weisbach equation [4]

$$\frac{\Delta p}{L} = f_D \frac{\rho}{D_h} \frac{\overline{w}^2}{2}.$$

The Colebrook implicit equation [3]

$$1/f_C^{\frac{1}{2}} = -2 \ \log\left[2.51/\left(Ref_C^{\frac{1}{2}}\right) + \varepsilon/3.72\right]$$

is one of possible ways to express the value of the Darcy friction coefficient of turbulent flow as the function of two dimensionless quantities the Reynolds number and relative roughness of the pipe surface

$$f_D = f_C(\mathrm{Re}, \varepsilon) \ ,$$

where the Reynolds number is defined as

$$\mathrm{Re} = \frac{\overline{w} \ D_h \ \rho}{\mu}$$

and the relative roughness of the inner pipe surface is defined as

$$\varepsilon = k \ / \ D_h \ \ .$$

The formula of the mass flow rate of the fluid through the analysed pipe is added to the above equation system

$$m = \frac{\pi}{4} \ D^2 \ \overline{w} \ \rho \ .$$

The formula of magnitude of the pressure gradient along the straight pipe of constant cross section area is added to the above equation system

$$\left|\frac{\partial p}{\partial x}\right| = \frac{\Delta p}{L} \ .$$

The algebraic-only treatment of the above equation system does not allow the expression of the pressure drop as an explicit function due to the mathematical properties of the implicit Colebrook equation. Nevertheless, the explicit form of the pressure drop formula can be reached using the Lambert function from the so-called special mathematical functions category [2]. The following analysis shows the conversion of the above equation system to the explicit dimensionless formula representing straight pipe pressure loss hydrodynamic similitude. Although the Colebrook friction coefficient equation is empirical, this theoretical analysis uses it because it is known to be very precious, and several attempts were made to convert it to an explicit form, [1] for instance.

Let's substitute the definition of the Reynolds number, magnitude of the pressure gradient along the straight pipe and formal function of relative surface roughness to the Darcy–Weisbach equation to get an interim form of the dimensionless pressure loss formulation. For the sake of simplicity let's reduce our analysis to the circular pipe only

$$D_h = D\,.$$

After some formula treatment, we get the form

$$\frac{\rho\, D^3}{\mu^2}\left|\frac{\partial p}{\partial x}\right| = f_C[\mathrm{Re}, \varepsilon(k, D)]\ \frac{\mathrm{Re}^2}{2}.$$

The received interim formula is in dimensionless form already. It is one of the possible forms (the first form) of the pressure loss hydraulic similitude of the pressure loss of the straight pipe. The right-hand side of the formula consists of the dimensionless Reynolds number and relative surface roughness only; consequently, the left-hand side of the formula must be dimensionless, too.

Let's designate the left-hand side of the above formula as the first result of this study, as a new dimensionless parameter (similitude number of "hydrodynamic diameter of the pressure loss")

$$\Phi = \frac{\rho\, D^3}{\mu^2}\left|\frac{\partial p}{\partial x}\right|.$$

The next interim step of the analysis is eliminating the pipe diameter from of the pressure loss hydraulic similitude of the pressure loss of the straight pipe. Substituting the Reynolds number equation to the mass flow rate equation and after restructuring, we get the interim formula for geometric pipe diameter

$$D = \frac{4}{\pi}\frac{m}{\mu\,\mathrm{Re}}.$$

Substituting the interim pipe diameter formula to the pressure loss hydraulic similitude of the straight pipe and after restructuring, we get the form

$$\left(\frac{4}{\pi}\right)^3 \frac{m^3\,\rho}{\mu^5}\left|\frac{\partial p}{\partial x}\right| = f_C\left[\mathrm{Re}, \varepsilon\left(k, \frac{4}{\pi}\frac{m}{\mu\, Re}\right)\right]\ \frac{\mathrm{Re}^5}{2}\,.$$

The received interim formula is the second form of the pressure loss hydraulic similitude of the pressure loss of the straight pipe. Let's designate part of the left-hand side of the above formula as the second result of this study, as a new dimensionless parameter (similitude number of "hydrodynamic mass flow of the pressure loss")

$$\Psi = \frac{m^3\,\rho}{\mu^5}\left|\frac{\partial p}{\partial x}\right|.$$

Eliminating the friction factor out of both formulas, the following equation for both new dimensionless quantities can be obtained

$$\Psi = \left(\frac{\pi}{4}\right)^3 \Phi\ \mathrm{Re}^3\,.$$

The next interim step of the analysis is to express the Darcy friction factor defined by the Colebrook equation using an identity valid for the Lambert function and treat the surface roughness in the way of the new dimensionless quantity.

Logarithmic equation

$$\ln(A_W + B_W x) + C_W x = \ln D_W$$

using of the Lambert function W is identical to

$$x = \frac{1}{C_W} \mathrm{W}\left[\frac{C_W\ D_W}{B_W}\ e^{\left(\frac{A_W\, C_W}{B_W}\right)}\right] - \frac{A_W}{B_W}.$$

The friction factor defined by the Colebrook equation expressed explicitly using the above shown identity is

$$f_C = \frac{1}{\left\{\frac{2}{\ln(10)} \mathrm{W}\left[\frac{50 \ln(10)\,\mathrm{Re}\, e^{\left(\frac{500 \ln(10)\,\mathrm{Re}\,\varepsilon}{9287}\right)}}{251}\right] - \frac{1000}{9287}\,\mathrm{Re}\,\varepsilon\right\}^2}.$$

Relative surface roughness in the above statement can be modified by substituting the pipe diameter as a function of Reynolds number

$$\varepsilon = \frac{\pi\,\mathrm{Re}\,k\,\mu}{4\,m}.$$

Substituting friction coefficient and relative surface roughness formulas derived above to the function of the second interim form of the hydraulic similitude of the pressure loss of straight pipe, we receive the final formula

$$\Psi = \frac{C_1\,\mathrm{Re}^5}{\left\{C_5\ \mathrm{W}\left[C_2\,\mathrm{Re}\,e^{(C_3\,\mathrm{Re}^2\,\Omega)}\right] - C_4\,\mathrm{Re}^2\,\Omega\right\}^2},$$

where the new dimensionless quantity describing the dimensionless parameter of hydrodynamic roughness of the pressure loss of the straight pipe is introduced by the formula

$$\Omega = \frac{k\,\mu}{m}.$$

Nomenclature

Variables:

D_h	[m]	hydraulic pipe diameter,
D	[m]	geometric pipe diameter (of circular pipe),
f_D	[1]	Darcy friction coefficient,
f_C	[1]	friction coefficient according to the Colebrook definition,
k	[m]	height of the roughness of the pipe surface,
L	[m]	pipe length,
m	[kg/s]	mass flow rate,
Δp	[Pa]	pressure drop of the straight pipe of the given length,
$\left\|\frac{\partial p}{\partial x}\right\|$	[Pa/m]	pressure gradient magnitude along the straight pipe,
$\overline{w}$	[m/s]	mean fluid velocity,
$\varepsilon = \varepsilon(k, D_h)$	[1]	relative roughness of the pipe surface,
μ	[Pa*s]	dynamic viscosity of the fluid,
ρ	[kg/m^3]	fluid density.

Dimensionless hydrodynamic parameters of pressure loss in the straight pipe:

Φ	[1]	dimensionless hydrodynamic mass flow rate,
Ψ	[1]	dimensionless hydrodynamic diameter,
Ω	[1]	dimensionless surface roughness.

Logarithms, constants and symbols of identity using of the Lambert function:

ln	[-]	natural logarithm,
log	[-]	decimal logarithm,
W	[-]	main branch of the Lambert function,
A_W, B_W, C_W, D_W	[1]	constants of identity using the Lambert function.

Constants of the final formula of hydraulic similitude of the pressure loss in the straight pipe:

$C_1 = \frac{\pi^3}{128} \approx 0.2422 \ldots$	[1]	constant,
$C_2 = \frac{50 \ln(10)}{251} \approx 0.4586 \ldots$	[1]	constant,
$C_3 = \frac{125\pi \ln(10)}{9287} \approx 0.0973 \ldots$	[1]	constant,
$C_4 = \frac{250\pi}{9287} \approx 0.0845 \ldots$	[1]	constant,
$C_5 = \frac{2}{\ln(10)} \approx 0.8685 \ldots$	[1]	constant.

References

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[4] Howell, G., Aerospace fluid component designers' handbook. Vol. I., TRW Systems Group, Redondo Beach CA, 1964.