COMPUTATIONAL MECHANICS 2023
38th conference with international participation
Srní
October 23 - 25, 2023

Analyzing stochastic stability of a gyroscope through the stochastic Lyapunov function

J. Náprstek[a], C. Fischer[a]

[a]Institute of Theoretical and Applied Mechanics of the Czech Academy of Sciences, Prosecká 76, 190 00 Prague, Czech Republic

1. Introduction

Practical experience shows that the random excitation component can affect the system response and its dynamic stability not only negatively but also positively. For example, the presence of a certain artificially generated turbulence component can have a positive effect against the occurrence of resonance. Such mechanisms are usually developed heuristically and are often not sufficiently justified theoretically. On the other hand, the presence of random excitation can lead to dangerous interactions with deterministic processes and thus cause a reduction in the level of dynamic stability in conditions that do not seem serious at first sight (icing on cables or power lines, road roughness, etc.).

This contribution delves into the application of first integrals in the construction of Lyapunov functions (LF) for analyzing the stability of dynamic systems in stochastic domains. It emphasizes the distinct characteristics of first integrals that warrant the introduction of additional constraints to ensure the essential properties required for a Lyapunov function. These constraints possess physical interpretations associated with system stability. The general approach to testing stochastic stability is illustrated using the example of a 3-degrees-of-freedom system representing a gyroscope.

2. The stochastic Lyapunov function

In the sense presented by Bolotin [1], the deterministic LF, is replaced in the stochastic domain by the adjoint Fokker-Planck (FP) operator

$$\mathbf{L}\{\lambda(t,\mathbf{u})\} \;=\; \frac{\partial\lambda(t,\mathbf{u})}{\partial t} + \sum_{i=1}^{n}\frac{\partial\lambda(t,\mathbf{u})}{\partial u_i}\kappa_i + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\lambda(t,\mathbf{u})}{\partial u_i\partial u_j}\kappa_{ij}, \tag{1}$$

where κ_i, κ_{ij} are the drift and diffusion coefficients of the n-dimensional Markov process and m depends on the system structure

$$\kappa_i = \sum_{k=1}^{m} A_{ik}(t) f_{ik}(\mathbf{u}) + \frac{1}{2}\sum_{k,l=1}^{m}\sum_{p=1}^{n}\frac{\partial f_{ik}(\mathbf{u})}{\partial u_p} f_{ip}(\mathbf{u})\cdot s_{iklp}\,,\;\; \kappa_{ij} = \sum_{k,l=1}^{m} f_{ik}(\mathbf{u}) f_{jl}(\mathbf{u})\cdot s_{ikjl}\,. \tag{2}$$

Equations (1)–(2) relate to the original stochastic system, the stochastic stability of which is being assessed

$$\dot{u}_i \;=\; \sum_{k-1}^{m}(A_{ik}(t) + w_{ik}(t)) f_{ik}(\mathbf{u}), \quad \mathbf{u}(t_0) \;=\; \mathbf{u}_0, \tag{3}$$

where $\lambda(t, \mathbf{u})$ is the LF candidate, $A_{ik}(t)$ are the nominal values of the system coefficients, $w_{ik}(t)$ is the Gaussian white noise of cross-intensity s_{ikjl}, and $f_{ik}(\mathbf{u})$ are the continuous non-decreasing functions.

Function $\lambda(t, \mathbf{u})$ should be a continuous positive definite. Its derivatives $\partial_t \lambda(t, \mathbf{u})$ and $\partial_{\mathbf{u},\mathbf{u}} \lambda(t, \mathbf{u})$ should be continuous as well. Let $\psi(t, \mathbf{u}) = \mathbf{L}\{\lambda(t, \mathbf{u})\} < 0$ in $\mathbf{u} \in \mathbf{\Omega}$ and $\psi(t, 0) = 0$ or $\psi(t, 0)$ is not defined. Then $\lambda(t, \mathbf{u})$ can be considered a Lyapunov function. Thus, for any $||\mathbf{u}_0|| \neq 0$ function $\lambda(t, \mathbf{u})$ decreases for $t \to \infty$ and, consequently, the trivial solution of (3) is stable in terms of probability.

3. Construction of the Lyapunov function

Let us denote $J_1, \ldots J_s$ the following first integrals which satisfy the equations of motion,

$$J_1(\mathbf{u}) = C_1, ..., J_s(\mathbf{u}) = C_s \, . \tag{4}$$

The Lyapunov function can be selected as a linear combination of $J_1, \ldots J_s$ and their functions. A convenient selection of the λ may be

$$\lambda(\mathbf{u}) = \sum_{i=1}^{s} a_i \left(J_i(\mathbf{u}) - J_i(0) \right) + b_i \left(J_i^2(\mathbf{u}) - J_i^2(0) \right) \, , \tag{5}$$

where a_i, b_i are constants of the linear combination that have to be selected so that the function (5) is positive definite.

4. Gyroscope

A rotationally symmetrical gyroscope rotates around its z axis along with its massless shaft, which is hinged at the origin of coordinates. The centroid of the gyroscope is positioned above the point where the shaft is fastened. The primary motion of the gyroscope can be affected by perturbations resulting from potential parasitic rotations around horizontal axes. The moving coordinate system x, y, z associated with the gyroscope deviates from the fixed coordinate system x_0, y_0, z_0 by the Euler angles α and β, as shown in Fig. 1.

The movement of the gyroscope, as described in [2], is characterized by five coordinates, whose values in the absence of perturbations are

$$\alpha = 0 \, , \quad \dot{\alpha} = 0 \, , \quad \beta = 0 \, , \quad \dot{\beta} = 0 \, , \quad \varphi = \varphi_0 \, , \quad \dot{\varphi}_0 = \omega \, . \tag{6}$$

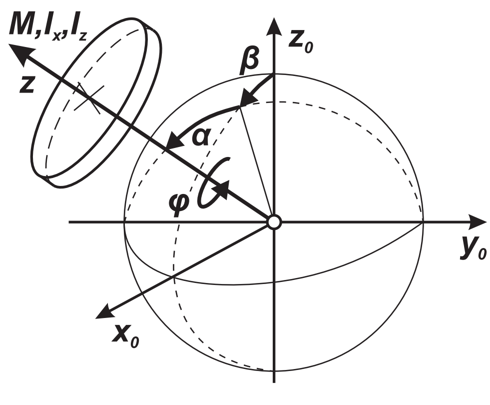

Fig. 1. Gyroscope outline with coordinates

The kinetic and potential energy of the unperturbed system can be determined as

$$\begin{aligned} T &= \frac{1}{2}I_x(\dot\alpha^2 + \dot\beta^2\cos^2\alpha) + \frac{1}{2}I_z(\dot\varphi - \dot\beta\sin\alpha)^2\,, \\ \Pi &= Mgl\cos\alpha\cos\beta\,, \end{aligned} \tag{7}$$

where $I_x = I_y, I_z$ represent the moments of inertia of the gyroscope, M denotes its mass, and l represents the distance between the centroid and the origin. The system possesses three first integrals. The first one is given by the simple sum of both energies, as expressed in (7),

$$T + \Pi = \frac{1}{2}I_x(\dot\alpha^2 + \dot\beta^2\cos^2\alpha) + \frac{1}{2}I_z(\dot\varphi - \dot\beta\sin\alpha)^2 + Mgl\cdot\cos\alpha\cos\beta = C_1\,. \tag{8}$$

Since the coordinate φ is cyclic, the corresponding Lagrange equation simplifies considerably so that the second first integral is

$$\frac{\partial T}{\partial\dot\varphi} = I_z(\dot\varphi - \dot\beta\sin\alpha) = I_z\cdot C_2\,. \tag{9}$$

The third first integral can be introduced, for example, as the integral of the angular momentum of the gyroscope with respect to the fixed axis z_o. It can be written in the following form:

$$I_x(-\dot\alpha\sin\beta + \dot\beta\cos\alpha\sin\alpha\cos\beta) + I_z(\dot\varphi - \dot\beta\sin\alpha)\cos\alpha\cos\beta = C_3\,. \tag{10}$$

Random perturbations of the individual components will be introduced as the coordinates $\mathbf{u} = [u_1, u_2, u_3, u_4, u_5]$ in the following form:

$$\alpha = u_1\,,\quad \dot\alpha = u_2\,,\quad \beta = u_3\,,\quad \dot\beta = u_4\,,\quad \dot\varphi = \omega + u_5.$$

The first integrals of the movement with perturbations take the form

$$\begin{aligned} J_1(\mathbf{u}) &= \frac{1}{2}I_x(u_2^2 + u_4^2\cos^2 u_1) + \frac{1}{2}I_z(\dot\varphi_o + u_5 - u_4\sin u_1)^2 + Mgl\cdot\cos u_1\cos u_3 = C_1\,, \\ J_2(\mathbf{u}) &= \omega + u_5 - u_4\sin u_1 = C_2\,, \\ J_3(\mathbf{u}) &= I_x(-u_2\sin u_3 + u_4\cos u_1\sin u_1\cos u_3) + \\ &\qquad + I_z(\dot\varphi_o + u_5 - u_4\sin u_1)\cos u_1\cos u_3 = C_3\,, \end{aligned} \tag{11}$$

where only J_1 is positive definite. Therefore, the LF has the form of (5) with $b_i = 0$ and $a_1 = 1$. Assuming small values of perturbations, i.e., keeping only two terms of the $\sin, \cos$ Taylor expansions, the (approximate) LF is introduced as follows:

$$\begin{aligned} \lambda(\mathbf{u}) = &-\frac{1}{2}(Mgl + a_3\omega I_z)u_1^2 + \frac{1}{2}I_x u_2^2 - \frac{1}{2}(Mgl + a_3\omega I_z)u_3^2 + \frac{1}{2}I_x u_4^2 + \frac{1}{2}I_z u_5^2 + \\ &(\omega I_z + a_2 + a_3 I_z)u_5 - (\omega I_z + a_2 + a_3 I_z - a_3 I_x)u_1 u_4 - a_3 I_x u_2 u_3\,, \end{aligned} \tag{12}$$

where coefficients a_2, a_3 remain to be determined. To ensure that the function in (12) is positive definite, it is necessary to set the coefficient of the first power of u_5 to zero, i.e.:

$$\omega I_z + a_2 + a_3 I_z = 0\,. \tag{13}$$

Such assumption changes (12) to the form

$$\lambda(\mathbf{u}) = \frac{1}{2}\underbrace{\left(\delta u_1^2 + 2a_3 I_x u_1 u_4 + I_x u_4^2\right)}_{=\lambda_1(\mathbf{u})} + \frac{1}{2}\underbrace{\left(\delta u_3^2 - 2a_3 I_x u_2 u_3 + I_x u_2^2\right)}_{=\lambda_2(\mathbf{u})} + \frac{1}{2}\underbrace{I_z u_5^2}_{=\lambda_3(\mathbf{u})}\,, \tag{14}$$

where $\delta = -(Mgl + a_3\omega I_z)$. The last term $\lambda_3(\mathbf{u})$ is positive definite in the variable u_5. Functions $\lambda_1(\mathbf{u})$ and $\lambda_2(\mathbf{u})$ are the quadratic forms which are positive, according to Sylvester's conditions, for non-vanishing u_1, u_4 and u_2, u_3, when

$$\Delta_1 = \delta > 0\,, \quad \Delta_2 = \begin{vmatrix} \delta & \pm a_3 I_x \\ \pm a_3 I_x & I_x \end{vmatrix} = I_x(\delta - a_3^2 I_x) > 0\,. \tag{15}$$

This implies

$$a_3 < -\frac{Mgl}{\omega I_z} \;\wedge\; \left|2a_3 + \omega\frac{I_z}{I_x}\right| < \sqrt{\left(\frac{I_z}{I_x}\right)^2 \omega^2 - \frac{Mgl}{I_x}}\,. \tag{16}$$

Therefore, e.g., for a sufficiently high angular velocity of the gyroscope, that is, for $|\omega| > 2\sqrt{MglI_x}/I_z$, there exist real coefficients a_3 and a_2 that satisfy (16) and (13), respectively, so that the function $\lambda(\mathbf{u})$ defined in (12) is positive definite and can be used as the Lyapunov function.

The further analysis continues by assembly of the FP equation, see Eq. (1), using the drift and diffusion coefficients defined in (2). The stochastic equation form follows from the Lagrange equations based on the energy balance. This approach shows that when investigating the stochastic stability of the system, one can start from the characteristics of the deterministic system and examine only the characteristics of the last term of the FP operator according to (1) with respect to its contribution to the positive or negative values of the function $\psi(\mathbf{u})$.

5. Conclusion

The Lyapunov function constructed on the basis of first integrals provides a possibility to work with the stochastic part of the problem with a much greater overview and to construct mathematical models with regard to the stabilizing or destabilizing effects of parametric random noises. This type of analysis is applicable to a variety of dynamic stability problems, including naturally the problem of signal and noise separation in structural health monitoring and various indirect measuring methods.

Acknowledgements

The kind support of Czech Science Foundation project No. 21-32122J and of the RVO 68378297 institutional support are gratefully acknowledged.

References

[1] Bolotin, V. V., Random vibrations of elastic systems, Netherlands, Springer, 1984.
[2] Eiserman, M. A., Classical mechanics, Moscow, Nauka, 1980. (in Russian)