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Acoustic streaming in homogenized deformable porous media

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1. Introduction

In microfluidic devices, nonlinear acoustic phenomena, namely the acoustic radiation, acoustic streaming are employed to manipulate particles and actuate fluid flow. These principles have attracted much interest of the research focused on developing new tissue engineering technologies since the acoustic wave are highly biocompatible, providing a non-contact controlable handle to manipulate bioparticles, or cells.

This paper deals with the acoustic streaming (AS) in periodic poroelastic media, see [2] where the AS in a bulk fluid was studied in response to vibrating walls of a channel. The fluid-structure interaction problem is imposed in elastic scaffolds. To capture the acoustic streaming phenomenon in response to propagating acoustic wave, nonlinearities originating in the divergence of the Reynolds stress, the advection acceleration term in the Navier Stokes equation, and the nonlinearity generated by deforming pore geometry must be retained. The perturbation with respect to a small parameter proportional to the inverse Strouhal number is applied. This yields the first and the second order sub-problem enabling to linearize the Navier-Stokes equations governing the barotropic viscous fluid dynamics in deforming scaffolds. Subsequent treatment by the asymptotic homogenization leads to a two scale problem where the macroscopic model provides the vibro-acoustic analysis in the Biot-type medium. It yields the AS source term for the second order problem which attains the form of the Darcy flow.

2. Micromodel of the heterogeneous structure

Flow of the barotropic viscous fluid (parameterized by the 1st and the 2nd viscosities, μ_f and η_f, and the reference sound speed c_0) is described by the velocity, pressure and density $(\boldsymbol{v}^f, p, \rho^f)$ satisfying the Navier-Stokes equations in the pores Ω_f. The fluid interacts with the deforming solid skeleton Ω_s, such that its displacement field $\boldsymbol{u}$ is governed by the elastodynamic (wave) equation involving the elasticity tensor $\mathbb{D}_s$ and the density ρ_s. A nonlinear problem is constituted by the following system of equations:

$$
\begin{aligned}
\rho_s \partial_{tt}^2 \boldsymbol{u} - \nabla \cdot \mathbb{D}_s \boldsymbol{e}(\boldsymbol{u}) = 0 \quad & \text{in } \Omega_S , \\
\rho_f \left(\partial_t \boldsymbol{v}^f + \boldsymbol{v}^f \cdot \nabla \boldsymbol{v}^f \right) = -\nabla p + \nabla \cdot \mathbb{D}^f \nabla \boldsymbol{v}^f + \boldsymbol{f}^f \quad & \text{in } \Omega_f , \\
\partial_t \rho_f + \nabla \cdot (\rho \boldsymbol{v}^f) = 0 \quad & \text{in } \Omega_f
\end{aligned}
\tag{1}
$$

with the state equation $p = c_0^2 \rho_f + c_0 c_0' (\rho_f)^2$ and the standard continuity of the stress tractions and velocities considered on the interface Γ_{fs},

$$
\dot{\boldsymbol{u}} = \boldsymbol{v}^f, \quad \text{and} \quad \boldsymbol{n} \cdot \mathbb{D}_s \boldsymbol{e}(\boldsymbol{u}) = \boldsymbol{n} \cdot (\mathbb{D}_f \boldsymbol{e}(\boldsymbol{v}^f) - p\boldsymbol{I}) \quad \text{on } \Gamma_{fs} . \tag{2}
$$

It is of advantage to introduce a smooth extension $\widetilde{\boldsymbol{u}}^f$ of the solid displacements to Ω_f, such that $\boldsymbol{v}^f$ can be expressed using the seepage $\boldsymbol{w}$, as follows: $\boldsymbol{v}^f = \boldsymbol{w} + \dot{\widetilde{\boldsymbol{u}}}^f$, whereby $\boldsymbol{w} = 0$ on Γ_{fs}.

Following the approach suggested by Nyborg [1] based on the successive approximations, the nonlinear problem represented by (1)–(2) can be decomposed into 2 subproblems:

- 1st order: Fast time–periodic dynamics: acoustic waves propagating in the two-phase medium, $(\boldsymbol{u}, \boldsymbol{w}_1, p_1)$ satisfy (with boundary conditions imposing an incident acoustic wave; recall the "tilde" notation denoting the displacement extension to Ω_f)

$$
\begin{aligned}
\rho_s \partial_{tt}^2 \boldsymbol{u} - \nabla \cdot \mathbb{D}_s \boldsymbol{e}(\boldsymbol{u}) &= 0 \quad \text{in } \Omega_s, \\
\partial_t \rho_1 + \rho_0 \nabla \cdot (\boldsymbol{w}_1 + \dot{\widetilde{\boldsymbol{u}}}^f) &= 0 \quad \text{in } \Omega_f, \\
\rho_0 (\partial_t \boldsymbol{w} + \partial_t \dot{\widetilde{\boldsymbol{u}}}^f) + \nabla p_1 - [\mu \nabla^2 + (\mu/3 + \eta) \nabla (\nabla \cdot)](\boldsymbol{w}_1 + \dot{\widetilde{\boldsymbol{u}}}^f) &= 0 \quad \text{in } \Omega_f, \\
p_1 = c_0^2 \rho_1 & \quad \text{in } \Omega_f \,.
\end{aligned}
\tag{3}
$$

- 2nd order: Slow flow streaming in Ω_f described by $(\boldsymbol{w}_2, p_2 . \rho_2)$ with given $\boldsymbol{v}_1 := \boldsymbol{w}_1 + \dot{\widetilde{\boldsymbol{u}}}^f$,

$$
\begin{aligned}
\partial_t \bar{\rho}_2 + \rho_0 \nabla \cdot \bar{\boldsymbol{w}}_2 &= -\nabla \cdot \overline{(\rho_1 \boldsymbol{v}_1)}, \\
\rho_0 \partial_t \bar{\boldsymbol{w}}_2 + \nabla \bar{p}_2 - \mu \nabla^2 \bar{\boldsymbol{w}}_2 + (\mu/3 + \eta) \nabla (\nabla \cdot \bar{\boldsymbol{w}}_2) &= -\rho_0 \left(\overline{(\boldsymbol{v}_1 \cdot \nabla) \boldsymbol{v}_1} + \overline{\boldsymbol{v}_1 (\nabla \cdot \boldsymbol{v}_1)} \right), \\
\bar{p}_2 &= c_0^2 \bar{\rho}_2 + c_0 c_0' \overline{(\rho_1)^2},
\end{aligned}
\tag{4}
$$

  where all the "over-bar" designates the time-average over the time period of the acoustic waves described by the 1st order problem.

3. Homogenized model

We consider periodic porous structures, such that the characteristic pore scale is proportional to a small parameter $\varepsilon = \ell / L$ defined by the ratio of the micro- and macroscopic characteristic lengths, denoted by ℓ and L, respectively. The periodic structures of the poroelastic medium are generated by the so-called representative periodic cell (RPC) $Y = Y_s \cup Y_f \cup \Gamma_{fs}^Y$ which consists of the solid and fluid parts correspondingly to the decomposition of Ω, see Fig. 1. The homogenization procedure applied to derive an effective model of the two-phase medium consists in the asymptotic analysis $\varepsilon \to 0$ of the micro-models (3) and (4) presented above, where all the unknowns and model parameters depending on the scale ε. Homogenization of (3) yields the standard Biot model of the poro-elastodynamics, cf. [3] For a given frequency ω, the acoustic waves are represented by the amplitudes $(\boldsymbol{u}_1^0, \boldsymbol{w}_1^0, p_1^0)$ which satisfy the following

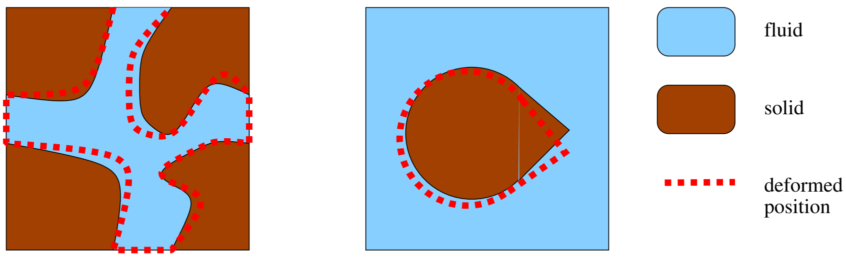

Fig. 1. The representative cells Y for the case of solid skeleton Y_s constituting a connected domain Ω_s^ε (left) and the case of suspended particles Y_s which can be rigid (right). In the latter case, the "suspension" can be realized due to a very thin (negligible) elastic network

set of equations which hold in Ω:

$$
\begin{aligned}
-\omega^2\hat{r}\boldsymbol{u}_1^0 + \mathrm{i}\omega\boldsymbol{w}_1^0 - \nabla\cdot\left(\widehat{\mathbb{D}}^H \boldsymbol{e}_x(\boldsymbol{u}_1^0) - p_1^0\widehat{\boldsymbol{B}}^H\right) &= 0,\\
\mathrm{i}\omega\widehat{\boldsymbol{B}}^H : \boldsymbol{e}_x(\boldsymbol{u}_1^0) + \nabla\cdot\boldsymbol{w}_1^0 + \mathrm{i}\omega\widehat{M}^H p_1^0 &= 0,\\
\boldsymbol{w}_1^0 + \widehat{\mathring{\boldsymbol{\mathcal{K}}}}(\nabla_x p_1^0 - \omega^2\omega^2\boldsymbol{u}_1^0) &= 0.
\end{aligned}
$$

The effective coefficients $\widehat{\mathbb{D}}^H$, $\widehat{\boldsymbol{B}}^H$, $\widehat{M}^H$ and $\mathring{\boldsymbol{\mathcal{K}}}$ are computed using the characteristic responses of the so-called cell problems solved in Y_s and Y_f. The dynamic permeability $\widehat{\mathring{\boldsymbol{\mathcal{K}}}}$ depending on ω can be expressed in terms of eigenvalues $\{\eta_r\}_r$ and eigenfunctions $\{\boldsymbol{w}^r\}_r$ of the Stokes flow problem in Y_f.

In order to evaluate the AS source (force), the fluid velocity must be reconstructed at the microscopic level (for any $x \in \Omega$). For this, the decomposition using the extended velocity of the solid and the fluid seepage are employed, as follows

$$
\begin{aligned}
\boldsymbol{v}_1^{\mathrm{mic}} &= \hat{\boldsymbol{w}}_1 + \widetilde{\dot{\boldsymbol{u}}_1^{\mathrm{mic}}}^f \quad \text{in the fluid } Y_f \times \Omega,\\
\dot{\boldsymbol{u}}_1^{\mathrm{mic}} &= \dot{\boldsymbol{u}}_1^1 + \boldsymbol{\Pi}^{rs} e_{rs}^x(\dot{\boldsymbol{u}}_1^0) \quad \text{in the solid } Y_s \times \Omega\\
\text{with} \quad \dot{\boldsymbol{u}}_1^1 &= \boldsymbol{\chi}^{rs} e_{rs}^x(\dot{\boldsymbol{u}}_1^0) + \boldsymbol{\chi}^P \dot{p}_1^0 \,,
\end{aligned}
$$

where $\dot{\boldsymbol{u}}_1^1$ is the solid velocity corrector (the two-scale function defined in $Y_s\times\Omega$,being expressed in terms of $\boldsymbol{\chi}^{rs}$ and $\boldsymbol{\chi}^P$, the characteristic responses of the microstructure, i.e., the skeleton displacements with respect to the unit strain modes and the unit pore pressure, respectively. Further, $\boldsymbol{\Pi}^{rs} e_{rs}^x(\boldsymbol{u}_1^0)$ provides the affine displacement in Y due to the macroscopic strain $e_{rs}^x(\boldsymbol{u}_1^0)$. Then, the seepage velocity is expressed using the convolution integral with the kernel associated with the dynamic permeability through the eigenpairs $\{\eta_k, \boldsymbol{w}^k\}_k$ and $\boldsymbol{\beta}^k$ denoting the mean of $\boldsymbol{w}^k$ in Y_f, so that

$$
\begin{aligned}
\hat{\boldsymbol{w}}_1(t,y,x) &= -\sum_k \rho_0^{-1}\int_0^t \hat{\alpha}_k \boldsymbol{w}^k(y)\otimes\boldsymbol{\beta}^k \exp\{-\eta_k(t-\tau)\}\boldsymbol{p}_1^{\mathrm{dyn}}(\tau,x)\mathrm{d}\tau \quad \text{in fluid}\\
\text{with} \quad \boldsymbol{p}_1^{\mathrm{dyn}} &= \nabla_x \dot{p}_1^0 - \rho_s \ddot{\boldsymbol{u}}_1^0 \,.
\end{aligned}
$$

The streaming source $\boldsymbol{\mathcal{S}}$ (also called the "acoustic force") is defined by virtue of the 2nd ordered problem (4)

$$
\boldsymbol{\mathcal{S}}(\boldsymbol{v}_1^{\mathrm{mic}}) = (\hat{\boldsymbol{w}}_1 + \widetilde{\dot{\boldsymbol{u}}_1^{\mathrm{mic}}}^f)\cdot\nabla_y(\hat{\boldsymbol{w}}_1 + \widetilde{\dot{\boldsymbol{u}}_1^{\mathrm{mic}}}^f) + (\nabla_y\cdot\widetilde{\dot{\boldsymbol{u}}_1^{\mathrm{mic}}}^f)(\hat{\boldsymbol{w}}_1 + \widetilde{\dot{\boldsymbol{u}}_1^{\mathrm{mic}}}^f)\,.
$$

Thus, in the limit, $\boldsymbol{\mathcal{S}}(\boldsymbol{v}_1^{\mathrm{mic}})$ depends on the characteristic responses Ξ of the fluid and solid parts of the microstructure, and on the 1st order macroscopic response — the acoustic waves described by $\mathbf{U}(x,t) := (\ddot{\boldsymbol{u}}_1^0, e_{rs}^x(\dot{\boldsymbol{u}}_1^0), \dot{p}_1^0, \nabla_x\dot{p}_1^0)$. In general, the streaming force (acceleration) is defined by the mapping $\mathcal{F}_{AS} : (\Xi, \mathbf{U}) \mapsto \boldsymbol{\mathcal{S}}$.

Homogenization of the 2nd order problem describing the AS in the fluid (obviously, no streaming in the solid) leads to the same result as the one obtained for the case of a rigid solid. Micro-response in Y_f is presented by the couple $(\boldsymbol{w}_2, p_2^1)$ satisfying

$$
\begin{aligned}
&-\rho_0\frac{\partial}{\partial t}\boldsymbol{w}_2 + \nabla_y\cdot(\bar{\mu}\nabla_y\otimes\boldsymbol{w}_2) - \nabla_y p_2^1 = \nabla_x p_2^0 + \rho_0\boldsymbol{\mathcal{S}}^*(\boldsymbol{v}_1^{\mathrm{mic}}),\\
&\nabla_y\cdot\boldsymbol{w}_2 = 0 \quad \text{in } Y_f\,, \quad \boldsymbol{w}_2 = 0 \quad \text{on } \Gamma_{fs}\,.
\end{aligned}
$$

This problem can be solved using the spectral decomposition, which leads to the same spectral problem as the one solved to solve the the 1st order problem. It enables to express the macroscopic "streaming source", velocity $\boldsymbol{w}^{AS}$ which constitutes the driving force in the AS equation governing the pressure field $\bar{p}_2^0$,

$$-\nabla_x \cdot \overline{\boldsymbol{\mathcal{K}}} \nabla_x \bar{p}_2^0 = \nabla_x \cdot \boldsymbol{w}^{AS} \quad \text{in } \Omega.$$

As for the boundary conditions for $\bar{p}_2^0$ on $\partial\Omega$, clearly a fixed wall Γ_w of a waveguide yields $\boldsymbol{n} \cdot \overline{\boldsymbol{\mathcal{K}}} \nabla_x \bar{p}_2^0 = 0$, whereas $\bar{p}_2^0 = 0$ can be considered on the open surfaces $\Gamma_o = \partial\Omega \setminus \Gamma_w$. In Fig. 2, the AS induced flow around the suspended rigid obstacles is reconstructed at two macroscopic positions $x \in \Omega$ of the waveguide.

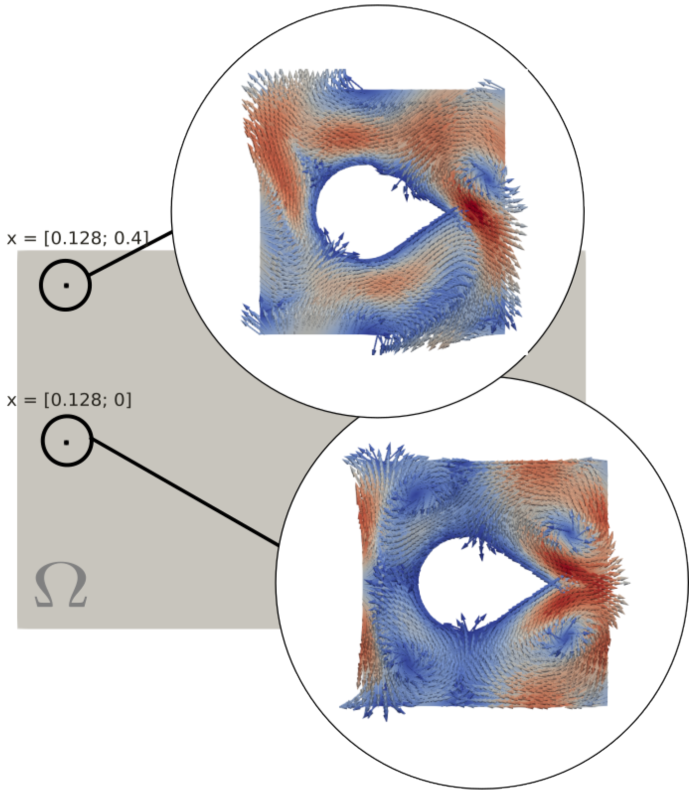

Fig. 2. Illustration of the AS in response to the acoustic waves in the fluid saturating rigid scaffolds (periodically distributed droplet-shaped obstacles). The AS flow is displayed in terms of streamlines (arrows) and $|\boldsymbol{w}_2|$ (color) at two macroscopic position $x \in \Omega$

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