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Postupné vlny v kvazilineárních reakčně-difuzních rovnicích

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Travelling Waves in Quasilinear Reaction-Diffusion Equations

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Dedication

I would like to dedicate this work to the memory of prof. RNDr. Josef Daněček, CSc., whose encouragement and support inspired my academic pursuits.

Declaration

I hereby declare that this thesis is my original work. I affirm that the contributions presented in this thesis are based on my independent research. Any external sources or ideas are duly acknowledged in the references.

Some parts of this work have been previously included in my rigorous thesis [44], which was successfully defended at the University of West Bohemia in April 2022.

Plzeň, ...

Michaela Zahradníková

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I would like to thank my advisor, prof. RNDr. Pavel Drábek, DrSc., for his time, patience, inspiring discussions and unwavering support, especially during difficult times. I am thankful for the privilege of working under his guidance.

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Abstrakt

V této práci se zabýváme skalárními reakčně-difuzními rovnicemi s p-Laplaciánem v difuzním členu a různými typy spojitých reakcí. V našem obecném pojetí připouštíme, aby měl difuzní koeficient (závislý na hustotě) degenerace nebo singularity v ekvilibriích 0 a 1, jakož i konečně mnoho skokových nespojitostí mezi nimi. Za těchto předpokladů studujeme existenci a vlastnosti postupných vln, které propojují stacionární stavy 0 a 1. Zavedením nového typu zobecněného řešení transformujeme úlohu druhého řádu na reálné ose pro neznámý profil a jeho rychlost na úlohu prvního řádu na omezeném intervalu, kterou pak studujeme ve smyslu Carathéodoryho. Výsledky pro tuto úlohu prvního řádu mají samostatný význam vzhledem k jejich uplatnitelnosti i mimo rámec našich předpokladů pro úlohu druhého řádu.

Uvádíme postačující podmínky pro existenci postupných vln v případě bistabilní a monostabilní reakce, které rozšiřují klasické výsledky získané za silnějších předpokladů. Klíčovou roli v existenci či neexistenci postupných vln a jejich vlastnostech přitom hraje společný vliv reakčního a difuzního členu. Za předpokladu mocninného chování těchto členů v blízkosti 0 a 1 pak studujeme, jak spolu s hodnotou p ovlivňují asymptotické vlastnosti řešení.

Hlavní část této práce vychází z našich tří publikovaných článků, věnovaných studiu reakčně-difuzních rovnic bez konvekce, v nichž jsme se zabývali zvlášť stacionárními vlnami, postupnými vlnami v bistabilních rovnicích a postupnými vlnami v monostabilních rovnicích. Naše metody a výsledky prezentujeme jednotným způsobem, abychom zdůraznili jak společný základ všech zmíněných případů, tak rozdíly mezi nimi.

Závěrečná kapitola je věnována úloze s konvekcí a reakčním členem vyskytujícím se v modelech spalování. Existenci a neexistenci řešení zde dokazujeme za silnějších předpokladů na reakční a difuzní členy než u rovnice bez konvekce a přímo tak zobecňujeme známé výsledky v případě difuze bez p-Laplaciánu. Náš článek na toto téma je v současné době recenzován.

Klíčová slova: reakčně-difuzní rovnice, postupné vlny, profil vlny, difuze s p-Laplaciánem, difuze se singularitami a degeneracemi, nespojitá difuze, bistabilní reakce, monostabilní reakce, nelipschitzovské diferenciální rovnice, řešení ve smyslu Carathéodoryho, nehladký profil, konvekce

Abstract

This thesis concerns scalar reaction-diffusion equations with p-Laplacian type diffusion and different types of continuous reaction. In our general setting, the density-dependent diffusion coefficient allows for degenerations and singularities at equilibria 0 and 1 as well as finitely many jump discontinuities between them. Under these assumptions, we study the existence and properties of travelling wave solutions which connect the steady states 0 and 1. Introducing a new concept of generalized solution, we transform the second-order problem on the real line for the unknown profile and its speed into a first-order problem on a bounded interval, which we then study in the sense of Carathéodory. The results for this first-order problem are of independent interest due to their applicability outside of our framework.

We present sufficient conditions for the existence of travelling wave solutions in the case of bistable and monostable reaction, which extend the classical results obtained in more regular settings. It is revealed to be the combined influence of the reaction and diffusion terms which plays a key role in the existence and non-existence of travelling waves as well as their properties. Assuming power-type behaviour of these terms near 0 and 1, we then study how they affect, together with the value of p, the asymptotic properties of solutions.

The main part of this work is based on our three published papers, devoted to the study of reaction-diffusion equations without convective effects, in which we focused separately on stationary waves, travelling waves in bistable equations and travelling waves in monostable equations. We present our methods and results in a unified manner to emphasize both the shared foundation and the differences among these cases.

The final chapter concerns reaction-diffusion-convection equation with combustion-type reaction. Existence and non-existence results are derived under stronger regularity assumptions on the reaction and diffusion terms than in the reaction-diffusion case, generalizing established results for density-dependent diffusion without the p-Laplacian. Our findings have been submitted as a paper and are currently under review.

Keywords: reaction-diffusion equation, travelling waves, wave profile, p-Laplacian type diffusion, singular and degenerate diffusion, discontinuous diffusion, bistable reaction, monostable reaction, non-Lipschitz ODE, solutions in the sense of Carathéodory, non-smooth profile, convection

Zusammenfassung

Diese Dissertation behandelt skalare Reaktions-Diffusionsgleichungen mit p-Laplace-Diffusion und verschiedenen Typen stetiger Reaktionen. In unserer allgemeinen Einstellung ermöglicht der dichtabhängige Diffusionskoeffizient Degenerationen und Singularitäten an den Gleichgewichtspunkten 0 und 1 sowie endlich viele Sprungunstetigkeiten dazwischen. Unter diesen Annahmen untersuchen wir die Existenz und Eigenschaften von Wanderwellenlösungen, die die Gleichgewichtszustände 0 und 1 verbinden. Durch Einführung eines neuen Konzepts der verallgemeinerten Lösung transformieren wir das Problem zweiter Ordnung auf der reellen Linie für das unbekannte Profil und seine Geschwindigkeit in ein Problem erster Ordnung auf einem begrenzten Intervall, das wir dann im Sinne von Carathéodory studieren. Die Ergebnisse für dieses Problem erster Ordnung sind aufgrund ihrer Anwendbarkeit außerhalb unseres Rahmens von unabhängigem Interesse.

Wir präsentieren ausreichende Bedingungen für die Existenz von Wanderwellenlösungen im Fall einer bistabilen und monostabilen Reaktion, die die in regelmäßigeren Einstellungen erzielten klassischen Ergebnisse erweitern. Es zeigt sich, dass der kombinierte Einfluss der Reaktions- und Diffusionsterme eine Schlüsselrolle bei der Existenz und Nichtexistenz von Wanderwellen sowie ihren Eigenschaften spielt. Unter Annahme eines Potenzverhaltens dieser Terme nahe 0 und 1 untersuchen wir dann, wie sie zusammen mit dem Wert von p die asymptotischen Eigenschaften der Lösungen beeinflussen.

Der Hauptteil dieser Arbeit basiert auf unseren drei veröffentlichten Papers, die der Untersuchung von Reaktionsdiffusionsgleichungen ohne konvektive Effekte gewidmet sind. Dabei haben wir uns separat auf stationäre Wellen, Wanderwellen in bistabilen Gleichungen und Wanderwellen in monostabilen Gleichungen konzentriert. Wir präsentieren unsere Methoden und Ergebnisse auf einheitliche Weise, um sowohl die gemeinsame Grundlage als auch die Unterschiede zwischen diesen Fällen zu betonen.

Das abschließende Kapitel behandelt die Reaktions-Diffusions-Konvektionsgleichung mit reaktionstypischer Verbrennung. Existenz- und Nichtexistenzresultate werden unter stärkeren Regularitätsannahmen für die Reaktions- und Diffusionsterme als im Fall der Reaktions-Diffusionsgleichung abgeleitet und verallgemeinern etablierte Ergebnisse für dichtedependente Diffusion ohne p-Laplace. Unsere Ergebnisse wurden als Artikel eingereicht und befinden sich derzeit im Begutachtungsprozess.

Schlüsselwörter: Reaktions-Diffusionsgleichung, Wanderwellen, Wellenprofil, p-Laplace-Diffusion, singuläre und degenerierte Diffusion, diskontinuierliche Diffusion, bistabile Reaktion, monostabile Reaktion, nicht-Lipschitz ODE, Lösungen im Sinne von Carathéodory, nicht-glatte Profile, Konvektion

Contents

Chapter 1

Introduction

The occurrence of wave phenomena in many natural reaction-diffusion processes has inspired the study of travelling waves almost a century ago and since then became an essential part of the analysis of reaction-diffusion equations. Prototypes of such problems arise from various fields of applications, such as population genetics, signal propagation, combustion theory, insect dispersal models and others. Apart from providing a tool for finding explicit solutions via comparison principles, the analysis of travelling waves is significant also for the investigation of long-term behaviour of solutions.

Extensive studies of travelling waves in numerous types of equations inspired us to consider a quasilinear reaction-diffusion equation on the real line

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(d(u)\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) + g(u), \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \quad p > 1. \tag{1.1}$$

It comprises as particular cases many of the above mentioned models, which will be explored shortly in Section 1.1. Our assumptions on the functions $d : [0,1] \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$ are motivated by classical as well as more special instances that arise in applications. Our aim is to provide a broad theoretical background for the mathematical treatment of travelling waves in rather general models. Such task presents challenges otherwise absent in more regular settings. Consequently, we focus exclusively on the existence of travelling waves and their properties without delving into the study of initial value problems or stability.

Customarily, a *travelling wave* is a non-constant bounded solution which maintains its shape while propagating at a constant speed. This means that the shape of the wave, referred to as *wave profile* or simply *profile*, remains constant over time, but it is not a constant function itself. Since the speed of propagation does not change in time as well, from a reference frame moving with the same speed this wave would appear stationary. Expressing this mathematically, a travelling wave solution is of the form

$$u(x,t) = U(x-ct) = U(z), \quad z := x - ct, \tag{1.2}$$

where U is the profile of the wave and c denotes its *wave speed*. Both U and c need to be determined, making the task of finding travelling wave solutions akin to an eigenvalue problem. The variable z is usually referred to as the moving coordinate or *wave variable*. In accordance with the predominant modelling origins of reaction-diffusion equations, we consider travelling waves only with nonnegative values.

Assuming $c > 0$, the wave (1.2) moves in the positive x-direction, while waves moving in the negative x-direction have the form $u(x,t) = U(x+ct)$. Altering the usual notion of speed to include negative values as well, both types of waves can be described by (1.2) with the sign of c now determining the direction. An example of a strictly monotone travelling wave is depicted in Figure 1.1. Waves with $c = 0$ are stationary solutions of the partial differential equation under consideration and we will refer to them as *stationary waves*.

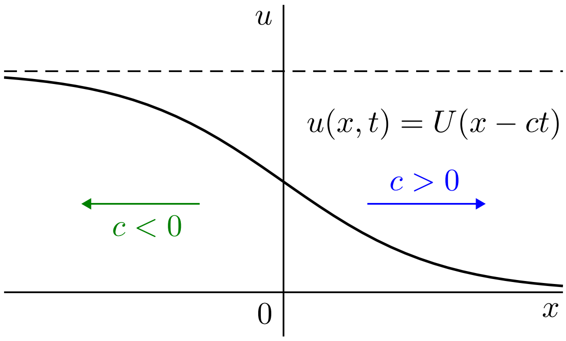

Figure 1.1: Travelling wave with a speed of propagation $c \in \mathbb{R}$

1.1 Motivation

In the pioneering work [24] from 1937, R. A. Fisher proposed a model for the spatial spread of an advantageous gene in a population. He considered a population uniformly distributed in a one-dimensional habitat and suggested that if a beneficial mutation occurs, there will be a wave of increase in the frequency u of the mutant gene at the expense of the allele previously occupying the same locus. Assuming that the parent allele is the only one present, its frequency equals $1 - u$. Using the analogy of physical diffusion, Fisher considers that the rate of diffusion per generation across any boundary is given by

$$-d\,\frac{\partial u}{\partial x},$$

where $d > 0$. The frequency u then satisfies the differential equation

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + mu(1-u). \tag{1.3}$$

Here $m > 0$ denotes the intensity of selection in favour of the mutant gene, independent of u. Fisher's model specifies the simplest possible conditions, such as constant diffusion coefficient $d > 0$, with respect to which variations might be discussed. The assumed independence of u and m is attributed to the reasonable expectation that there is no dominance in respect to the advantageous mutation. After suitable rescaling, the equation (1.3) can be rewritten as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u).$$

Independently of Fisher's work, Kolmogorov, Petrovsky and Piskunov [30] studied in the same year travelling wave solutions to the semilinear equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(u), \tag{1.4}$$

considering a class of suitable reaction terms $g \in C^1[0,1]$. They assumed

$$g(u) > 0 \quad \text{for } u \in (0,1), \quad g(0) = g(1) = 0,$$
$$g'(0) > 0, \quad g'(u) \le g'(0) \quad \text{for } u \in (0,1),$$

of which Fisher's reaction $g(u) = u(1-u)$ is a special case. In tribute to both of these seminal works, the equation (1.4) is often regarded as the *Fisher-KPP equation*.

The genetical context envisaged by Fisher was later explored in detail by Aronson and Weinberger [1]. The authors consider a population of diploid individuals and derive equation (1.4) as a simplified model of the genetic processes. Assuming that a gene at a specific locus in a specific chromosome pair occurs in two forms, denoted by a and A, the population then consists of three different genotypes, denoted aa, AA and aA. Homozygotes (aa or AA) carry only one kind of allele, while heterozygotes (aA) carry one of each allele. The following three cases are distinguished based on the properties of the function $g \in C^1[0,1]$, $g(0) = g(1) = 0$. In the *heterozygote intermediate case*, the viability of the heterozygote is between the viabilities of the homozygotes, and g satisfies

$$g(u) > 0 \quad \text{in } (0,1), \quad g'(0) > 0.$$

This is the case that was considered in the classical studies mentioned above. *Heterozygote superiority* occurs when the heterozygote is more viable than the homozygotes, the relevant properties of g now being

$$g(u) > 0 \quad \text{in } (0, s_*), \quad g(u) < 0 \quad \text{in } (s_*, 1) \quad \text{for some } s_* \in (0,1)$$
$$g'(0) > 0, \quad g'(1) > 0.$$

If, on the other hand, the viability of the homozygotes exceeds that of the heterozygote, we have *heterozygote inferiority*. The characteristic features of g are

$$\begin{gathered} g(u) < 0 \quad \text{in } (0, s_*), \quad g(u) > 0 \quad \text{in } (s_*, 1) \quad \text{for some } s_* \in (0,1) \\ g'(0) < 0, \quad \int_0^1 g(u)\,\mathrm{d}u > 0. \end{gathered} \tag{1.5}$$

As we can see, in the latter two cases g changes sign in $(0,1)$ exactly once.

Besides population genetics, equations of the form (1.4) are relevant in other contexts as well. We mention the following notable examples, selected from the overview in [25].

The *Newell-Whitehead equation* or *amplitude equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u^2),$$

which arises in the study of thermal convection of a fluid heated from below.

The *Zeldovich equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^2(1-u)$$

which arises in combustion theory. The unknown u represents temperature and the reaction term $g(u) = u^2(1-u)$ corresponds to the generation of heat by combustion.

The *Nagumo equation* or *bistable equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-s_*), \quad 0 < s_* < 1,$$

suggested in [11] as a model for a nerve which has been treated with certain toxins. A rescaled version has been used by Nagumo, Yoshizawa and Arimoto [39] as a model for bistable transmission lines.

Combustion models with ignition thresholds

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \begin{cases} 0 & \text{for } 0 \le u < s_* \\ g(u) & \text{for } s_* \le u \le 1 \end{cases}$$

where $g(u) > 0$ for $u > s_*$ and $g(s_*) \ge 0$. Such equations describe flame propagation with one reactant involved in a single-step chemical reaction. Here u represents the normalized temperature and s_* is a critical temperature at which the reaction starts.

The applications motivate the typical sign conditions for the reaction term g, considered in the mathematical treatment of (1.4). The following terminology from [6] refers to the general shape of a function and it can be used not only for the reaction g.

Definition 1.1. We shall say that a function $a : [0,1] \to \mathbb{R}$, $a(0) = a(1) = 0$, is a *type* A *function* if

$$a(s) > 0 \quad \text{for all } s \in (0,1),$$

type B *function* if there exists $s_* \in (0,1)$ such that

$$a(s) = 0 \quad \text{for all } s \in [0, s_*], \quad a(s) > 0 \quad \text{for all } s \in (s_*, 1),$$

type C *function* if there exists $s_* \in (0,1)$ such that

$$a(s) < 0 \quad \text{for all } s \in (0, s_*), \quad a(s) > 0 \quad \text{for all } s \in (s_*, 1),$$

see Figure 1.2.

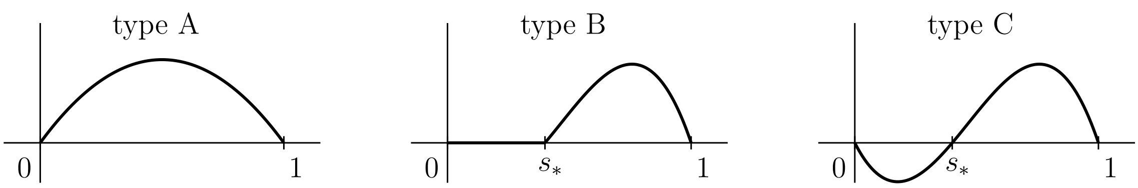

Figure 1.2: Classification of functions according Definition 1.1

In literature, other terminologies are commonly used for reaction terms satisfying the conditions from Definition 1.1, which refer to their occurrence in classical models or standard stability results for the equilibria 0 and 1. Specifically, reaction terms of type A are also known as Fisher-KPP or monostable reactions, those of type B as combustion or ignition type, and those of type C as Nagumo or bistable reaction. The notion of monostable and bistable indicates whether one or both of the stationary points 0 and 1 are stable.

Some ambiguity regarding these notions might stem from the fact that, apart from the general shape, the original works concerned a particular function g or a more specific set of assumptions on its derivative as well. Later generalizations extended the results also to less regular settings, but it is important to note that even minor differences might render some standard techniques inapplicable. Therefore, it will be our understanding that all of the above mentioned terminologies refer simply to the sign conditions and any other assumptions will be stated explicitly if necessary. In Chapter 2, we will, for the purposes of this work, redefine the notion of bistable reaction, which encompasses and generalizes the concept of type B and type C functions.

Density-dependent dispersal has been observed in many populations, such as insects or small rodents (cf. [38, 41]), due to biological and physical factors. Its introduction into the derivation of the corresponding models leads to equations with a non-constant coefficient $d = d(u)$:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[d(u)\frac{\partial u}{\partial x}\right] + g(u), \tag{1.6}$$

where $d = d(u)$ is a positive (or at least non-negative) function. An example from [38] is a model with $d(u) = u$, which foresees a dispersal of individuals to regions of lower density becoming more rapid as the population gets more crowded. Diffusion coefficients that vanish at some points, typically at 0, are called *degenerate*.

King and McCabe [29] study the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[u^{-n}\,\frac{\partial u}{\partial x}\right] + u(1-u), \quad n > 0, \tag{1.7}$$

as the simplest model of situations in which low concentration disperse very rapidly. Notice that in this case, the diffusion coefficient is a decreasing function of the density with a *singularity* at 0. A relevant (multi-dimensional) example arises from observations concerning the dispersal of Palaeoindian peoples in North America, see [29] and the references therein. The rapidity of the southward spread, suggested by archaeological records, is not consistent with the predictions of the standard semilinear (Fisher's) model. This has lead to the suggestion that early Palaeoindians adapted to low-density mate distributions through exogamy and travelling large distances to find eligible mates, implying that dispersal driven by mate searching is responsible for accelerating range expansion. This phenomenon, represented in (1.7) by the singular diffusivity $d(u) = u^{-n}$, $n > 0$, is in a certain sense opposite to the case $n < 0$ (degenerate diffusivity), which corresponds to the avoidance of crowding.

From other fields of study, we mention the well known *porous medium equation* with absorption or with a source term:

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\left(u^m\right) \mp u^q,$$

where $m > 0$ and $q > 0$. This equation can be written in an equivalent form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[m u^{m-1}\frac{\partial u}{\partial x}\right] \mp u^q.$$

Without the last term on the right-hand side, it reduces to the linear heat equation in the particular case $m = 1$. Berestycki [5] investigated a combustion model with a temperature-dependent diffusion and a type B reaction, interpreted as deflagration wave problem for a compressible reacting gas, with one reactant involved in a single step chemical reaction. In [42], the authors consider discontinuous density-dependent coefficient which can be used to describe phenomena involving a sudden change in the diffusion constant. Such problems include polymer dynamics, in which the diffusivity drops abruptly by several orders of magnitude beyond the gelation critical density, and processes related to hydrogen storage as a source of energy, cf. [42] and the references therein.

More recently, reaction-diffusion equations involving the p-Laplacian operator

$$u \mapsto \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right)$$

have been considered in literature, see e.g. [2, 3, 22, 27]. The p-Laplacian operator arises for example in models derived from the power-type Darcy's law, cf. [4].

The selection of reaction-diffusion equations presented in this section is by no means exhaustive. Our aim was to provide an overview of applications and historically notable examples which motivated the study of travelling waves, as well as models which provide reasonable foundation for the assumptions considered in this thesis. Before we proceed to the existence results for reaction-diffusion equations relevant to our research, we briefly mention problems with convection. In general, such equations usually take the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[d(u)\frac{\partial u}{\partial x}\right] + h(u)\frac{\partial u}{\partial x} + g(u), \tag{1.8}$$

where the second term on the right-hand side represents a convective or advective phenomenon, with $h = h(u)$ denoting the convective velocity function. Equivalently, this problem can be written as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[d(u)\frac{\partial u}{\partial x}\right] + \frac{\partial H(u)}{\partial x} + g(u),$$

where $H = H(u)$ can be viewed as a convective flux function with H' the corresponding velocity. Various applications can be found in the monograph [25], which is dedicated to the study of travelling wave solutions to (1.8) via integral equations. The introduction of convective processes can significantly affect the usual results derived in the absence of convection.

Although this thesis focuses primarily on the reaction-diffusion case, we discuss the influence of the convective term on solutions to (1.1) with a type B reaction in Chapter 6. An overview of previously established results in the case $p = 2$ will be provided there.

1.2 State of the art

In this section, we mention some of the basic results concerning the existence of travelling wave solutions to reaction-diffusion equations on the real line with standard types of reaction, which are relevant to our research.

When looking for travelling wave solutions

$$u(x,t) = U(x-ct) = U(z), \quad z = x - ct, \tag{1.9}$$

we have

$$\frac{\partial u}{\partial x} = \frac{\mathrm{d}U}{\mathrm{d}z}, \quad \frac{\partial u}{\partial t} = -c\frac{\mathrm{d}U}{\mathrm{d}z},$$

hence partial differential equations in x and t can be written as ordinary differential equations in the wave variable z. For example, the semilinear reaction-diffusion

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(u) \tag{1.10}$$

becomes

$$U'' + cU' + g(U) = 0, \tag{1.11}$$

where primes denote differentiation with respect to z. The standard approach is to study solutions of this problem in the (U, V) phase plane where

$$U' = V, \quad V' = cV - g(U)$$

which gives phase plane trajectories as solutions of

$$\frac{\mathrm{d}V}{\mathrm{d}U} = -c - \frac{g(U)}{V}.$$

Notice that if $g \equiv 0$, (1.11) does not admit any solutions U that satisfy the conventional requirements imposed on wave profiles. Indeed, in this case U takes the general form

$$U(z) = A + B\mathrm{e}^{-cz}, \quad A, B \in \mathbb{R},$$

i.e., it is unbounded or constant, and hence not considered a wave solution. The appearance of travelling waves is therefore tied to the presence and particular form of the nonlinear reaction term. In this work, we focus on reaction terms that vanish at both endpoints 0 and 1. This implies that $u \equiv 0$ and $u \equiv 1$ are stationary states of (1.10). Consequently, it is natural to consider travelling waves which connect these states, i.e., as $z \to -\infty$ they are at one steady state and at the other as $z \to +\infty$. For definiteness, we may assume

$$U(-\infty) := \lim_{x\to-\infty} U(z) = 1, \quad U(+\infty) := \lim_{x\to+\infty} U(z) = 0.$$

A fundamental result concerning the existence of such travelling waves is the following (see [1, Theorem 4.2]):

If $g \in C^1[0,1]$, $g(0) = g(1) = 0$, is a type A function or it satisfies

$$\begin{gathered} g(s) \le 0 \quad \text{in } (0, s_*), \quad g(s) > 0 \quad \text{in } (s_*, 1) \qquad \text{for some } s_* \in (0,1), \\ \int_0^1 g(s)\,\mathrm{d}u > 0 \end{gathered} \tag{1.12}$$

(notice that this condition contains functions of type B and C as particular cases), there exists a travelling wave solution $u(x,t) = U(x-ct)$, $c > 0$, of (1.10). Moreover, the travelling wave is strictly decreasing, i.e., $U'(z) < 0$ for finite $z = x - ct$. In fact, the monotonicity property can be proven assuming only $g \in C^1[0,1]$, $g(0) = g(1) = 0$, see [23, Lemma 2.1].

In the case of type A (monostable) reaction terms g, the proof of this assertion can be extended to show that if $g'(0) > 0$, there exists a travelling wave solution for every wave speed $c \geq c^*$, where

$$2\sqrt{g'(0)} \leq c^* \leq 2\sqrt{\sup_{s\in(0,1)} \frac{g(s)}{s}}\,.$$

In other words, there exists a half-line $[c^*, +\infty)$ of admissible wave speeds for which (1.10) admits a travelling wave solution. This was also proven by Kolmogorov, Petrovsky and Piskunov [30] under the additional assumption $g'(u) \leq g'(0)$ for $u \in (0,1)$, and by Fisher [24] in the case $g(u) = u(1-u)$. In the Fisher-KPP setting, a straightforward observation yields that $c^* = 2\sqrt{g'(0)}$.

For reactions g satisfying (1.12), the situation differs considerably. Instead of infinitely many travelling waves with different wave speeds, there now exists only one wave with a unique speed $c = c_*$, see e.g. [23, 28]. Equation (1.10) with this type of reaction may also admit stationary wave solutions if $\int_0^1 g(u)\,\mathrm{d}u = 0$ instead of $\int_0^1 g(u)\,\mathrm{d}u > 0$ (this excludes functions of type B). In special cases the solution can be written down explicitly. For example, if

$$g(s) = s(1-s)\left(s - \frac{1}{2}\right),$$

there exists (cf. [40]) a decreasing stationary wave

$$u(x) = -\frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}}\right) + \frac{1}{2}.$$

An important topic related to the study of travelling wave solutions is the behaviour of solutions $u(x,t)$ of initial value problems for (1.10) as $t \to +\infty$. It was in fact shown already in the early works by Kolmogorov et al. [30] in the monostable case and Kanel' [28] in the generalized bistable case that solutions to special initial value problems converge, in a certain sense, to travelling wave solutions. The problem of determining conditions under which a solution does or does not approach a travelling wave has since been subject to extensive studies, from which we mention the now classical papers [1, 23] and [15] for more recent findings. Despite the significance of this subject, we omit a more comprehensive overview, as it is well beyond the scope of our research.

Considering a positive density-dependent diffusion coefficient instead of a constant one does not affect the existence of travelling waves in a substantial way. In the wave variable z, the equation (1.6) now reads

$$(d(U)U')' + cU' + g(U) = 0 \tag{1.13}$$

together with boundary conditions $U(-\infty) = 1$, $U(+\infty) = 0$. By means of suitable change of variables, this problem can be reduced to the previous case with reaction given by the product $d(u)g(u)$. For functions g of type A with $g'(0) > 0$ and $d \in C^1[0,1]$ strictly positive in $[0,1]$, Engler [21] and Hadeler [26] proved that (2.4) is solvable if and only if $c \geq c^*$ where

$$2\sqrt{d(0)g'(0)} \leq c^* \leq 2\sqrt{\sup_{s\in(0,1)} \frac{d(s)g(s)}{s}}\,.$$

Also in this case, each solution is strictly decreasing on $\mathbb{R}$. Generalization of this result is due to [31] where weaker regularity assumptions were imposed on the functions d and g. In particular, the authors assume only $d \in C[0,1]$ and $g \in C[0,1]$ and the existence is guaranteed provided that the lower right Dini derivative $D_+(dg)(0)$ is finite, otherwise there is no solution to (1.13).

In [35], existence result for the equation (1.13) with g satisfying (1.12) is obtained assuming $g \in C[0,1]$ and strictly positive $d \in C^1[0,1]$. Similarly as in the constant diffusion case, if

$$\int_0^1 d(s)g(s)\,\mathrm{d}s > 0$$

there exists a unique value of $c_* > 0$ for which (1.13) admits a solution. However, the profile U need not be strictly monotone on the whole real line $\mathbb{R}$, only in the open interval $\{z \in \mathbb{R} : 0 < U(z) < 1\}$, implying that both equilibria 0 and 1 can be attained. This does not occur if the Dini derivatives $D_-(dg)(1)$ and $D_+(dg)(0)$ are finite, see [35, Proposition 2].

An interesting phenomena associated with non-constant diffusion is the appearance of new type solutions if $d(0) = 0$, which are referred to as "sharp-type". More precisely, if $d(0) = 0$, $d'(0) > 0$ and g is of type A, then the travelling wave solution corresponding to the minimal wave speed $c^* > 0$ reaches 0 in a finite $z^* \in \mathbb{R}$ with a negative slope $U'(z^*) = -c/d'(0)$, see e.g. [33]. For other values of $c > c^*$, the equilibrium 0 is not attained and the solution is strictly decreasing on $\mathbb{R}$. The first systematic treatment of degenerate diffusion problems was given in [41] under rather strong regularity assumptions, generalized by Marcelli and Malaguti [33]. In the particular case of $d(s) = s$ and $g(s) = s(1-s)$, $s \in [0,1]$, Murray [38] calculates the value $c^* = 1/\sqrt{2}$ and finds the explicit solution

$$U(z) = \begin{cases} 1 - \exp\left(\dfrac{z - z^*}{\sqrt{2}}\right) & z < z^*, \\ 0 & z > z^*, \end{cases} \tag{1.14}$$

depicted in Figure 1.3.

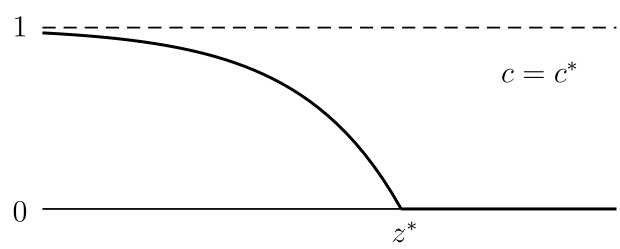

Figure 1.3: The solution (1.14) of a degenerate diffusion problem

The equation (1.1) with $p > 1$, which combines density-dependent diffusion coefficient with the p-Laplacian operator, has been studied e.g. in [14, 22]. Some of the above results for $p = 2$ extend also to the more general case $p > 1$. In particular, the number of travelling waves obtained for monostable and generalized bistable reaction remains the same. However, a complete analogy that takes into account also the properties of solutions is not possible. Most notably, it is well known that solutions to problems involving the p-Laplacian do not have second derivatives in general, hence we cannot expect U to be a classical solution. Furthermore, the lack of uniqueness of the associated Cauchy problem for U at 0 when $p > 2$ implies that the solution might reach 0 in a finite z.

In [22], the authors study an equivalent first-order problem to obtain a theory of admissible wave speeds c. They then interpret the results in connection to the second-order problem under stronger regularity assumptions than those required by the equivalent problem, which allows them to generalize results derived by Marcelli and Malaguti [31, 33] for monostable reaction to the case $1 < p \leq 2$. The case of continuous type C reaction and positive diffusion is studied in detail in [14].

The thesis is further organized as follows. Chapters 2–5 concern the study of travelling and stationary waves for the reaction-diffusion equation (1.1) with two types of reaction. Presented results and methodology are based on our published articles [17], [18] and [19], which are now merged into a comprehensive study accompanied by further discussions and more detailed reasoning. In Chapter 2 we introduce a new definition of a generalized wave profile and show how the problem of finding travelling waves can be reduced to an equivalent first-order boundary value problem. The investigation of this b.v.p. in Chapter 3 serves as the main tool for proving the existence and non-existence of travelling wave profiles, which is summarized in Chapter 4. Asymptotic behaviour of profiles is discussed in Chapter 5.

In Chapter 6 we present our recent findings concerning travelling waves to (1.1) with an additional convective term and type B reaction. As in the previous case, we study the existence of solutions via an equivalent first-order ODE. Our paper [20] devoted to this subject is, at the time of submission of this thesis, under review.

Chapter 2

Quasilinear reaction-diffusion equation

We are concerned with travelling wave solutions to the quasilinear reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(d(u)\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) + g(u), \quad (x,t) \in \mathbb{R} \times [0,+\infty), \quad p > 1. \tag{2.1}$$

We study the existence and properties of these travelling waves under very general assumptions on the reaction and diffusion terms, which comprise special cases arising in applications.

Classical models with $p = 2$ typically deal with smooth functions d and g that satisfy certain sign conditions in $[0,1]$. As mentioned in the previous chapter, existence results can be obtained under weaker regularity assumptions as well by employing different techniques of proof. Our approach allows us to relax these assumptions even further and treat discontinuous diffusion coefficient d with possible degenerations as well as singularities near 0 and 1. However, some properties of solutions such as "finiteness" and "sharpness" cannot be recovered in general. To do so, we assume power-type behaviour of d and g in the neighbourhood of 0 and 1, cf. Chapter 5.

2.1 The second-order ODE

First, let us specify our hypotheses on the diffusion coefficient d and the reaction term g. We assume that

(H1) $d : [0,1] \to [0,+\infty)$ is a lower semi-continuous function with $d(s) > 0$ for all $s \in (0,1)$. There exist $0 = s_0 < s_1 < s_2 < \cdots < s_n < s_{n+1} = 1$ such that d has discontinuity of the first kind (finite jump) at s_i, $i = 1,\ldots,n$, and

$$d|_{(s_i,s_{i+1})} \in C(s_i,s_{i+1}), \quad i = 0,\ldots,n;$$

(H2) $g : [0,1] \to \mathbb{R}$ is a continuous function (not necessarily smooth or Lipschitz) with $g(0) = g(1) = 0$.

Note that $d(0)$, $d(1)$ are always defined and possibly zero, but we do not require $d(0+) = d(0)$, $d(1-) = d(1)$. In particular, the limits $d(0+)$, $d(1-)$ may not exist or they may be infinite. Examples of admissible diffusion coefficients with qualitatively different properties

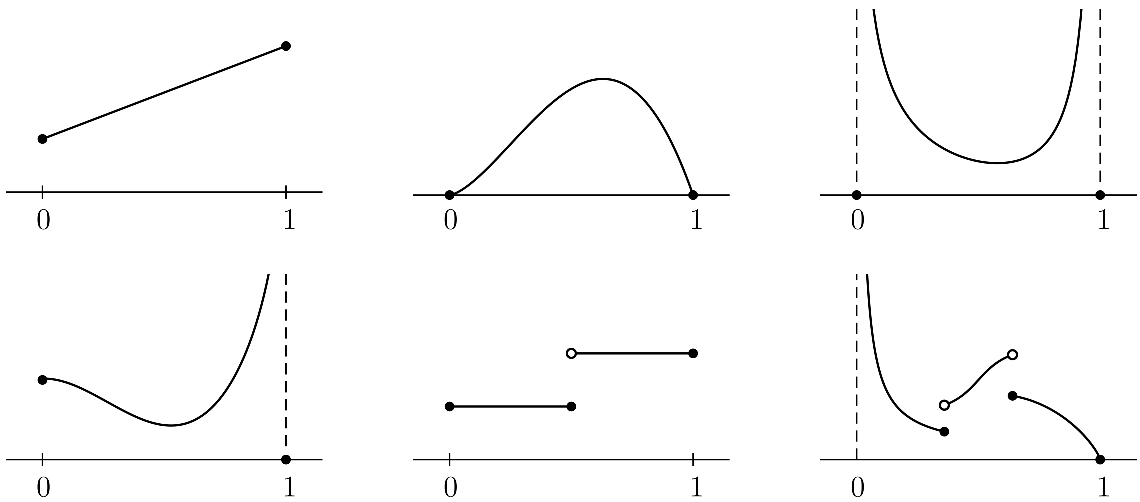

Figure 2.1: Admissible functions $d = d(s)$

are shown in Figure 2.1. In particular, **(H1)** implies that if $s \in (0, 1)$ then $\lim_{\sigma \to s} d(\sigma) > 0$. Therefore, the diffusion coefficient cannot exhibit the behaviour depicted in Figure 2.2.

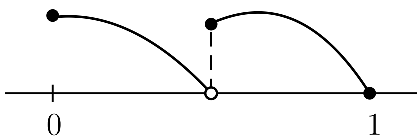

Figure 2.2: Inadmissible function $d = d(s)$

Concerning the reaction term g, apart from **(H2)** we further assume that $g = g(s)$ satisfies either

$$g(s) \leq 0 \ \text{ if } \ s \in (0, s_*), \quad g(s_*) = 0, \quad g(s) > 0 \ \text{ if } \ s \in (s_*, 1) \tag{2.2}$$

for some fixed $s_* \in (0, 1)$, or

$$g(s) > 0 \ \text{ for all } s \in (0, 1). \tag{2.3}$$

In what follows, we shall refer to these two alternatives as *bistable* and *monostable* case, respectively. While (2.3) is also commonly known as the Fisher–KPP type reaction term, the terminology for nonlinearity (2.2) is not established in literature. It contains as particular instances the classical bistable reaction, which has exactly one intermediate zero (at s_*), and the combustion nonlinearity, when $g(s) = 0$ for $s \in [0, s_*]$. If distinction should be necessary, we will adopt the terminology from Definition 1.1, represented in Figure 1.2. Finally, let us point out that our general assumption $g \in C[0, 1]$ also allows $g'(0+)$ and $g'(1-)$ to be zero or infinite.

Both in the bistable and monostable case, constant functions $u_0 \equiv 0$ and $u_1 \equiv 1$ are stationary solutions of (2.1) called *equilibria*. We look for travelling waves which connect these equilibria, i.e., functions of the form $u(x, t) = U(x - ct)$ with $U : \mathbb{R} \to [0, 1]$ and unequal limits $U(\pm\infty) \in \{0, 1\}$. The real parameter c stands for the wave speed and it is also an unknown parameter of the problem. Using the wave variable $z = x - ct$, we can (formally) rewrite (2.1) as an ordinary differential equation on the real line

$$\Big(d(U(z)) \, |U'(z)|^{p-2} \, U'(z)\Big)' + cU'(z) + g(U(z)) = 0, \quad z \in \mathbb{R}, \tag{2.4}$$

where $U(z) = U(x-ct)$ and $'$ denotes differentiation with respect to z. It is clear from our assumptions on $d = d(s)$ that we cannot expect (2.4) to hold pointwise except for some special cases. Therefore, we introduce a new concept of solution based on the first integral of (2.4).

Definition 2.1. We say that a function $U \in C(\mathbb{R})$ is *piecewise* C^1, denoted $U \in \widehat{C}^1(\mathbb{R})$, if there is a set $D_U \subset \mathbb{R}$ of isolated points such that $U \in C^1(\mathbb{R} \setminus D_U)$.

We are now ready to formulate our definition in which the term *solution* refers to the unknown profile U. Recall, however, that we also deal with a problem to determine the value (or values) of c such that there exists a function $U = U(z)$ which satisfies (2.4) in the following sense.

Definition 2.2 (Definition of solution)**.** Let $U : \mathbb{R} \to [0,1]$, $U \in \widehat{C}^1(\mathbb{R})$. We denote

$$M_U := \{z \in \mathbb{R} : U(z) = s_i,\ i = 1, 2, \dots, n\}, \quad N_U := \{z \in \mathbb{R} : U(z) = 0 \text{ or } U(z) = 1\}.$$

Then U is called a solution of (2.4) if the following holds:

(a) $\partial M_U \cup \partial N_U = D_U$.

(b) For any $z \in \partial M_U$ there exist finite one-sided derivatives $U'(z-)$, $U'(z+)$ and

$$L(z) := |U'(z-)|^{p-2}\, U'(z-) \lim_{\xi \to z-} d(U(\xi)) = |U'(z+)|^{p-2}\, U'(z+) \lim_{\xi \to z+} d(U(\xi)).$$

(c) Function $v : \mathbb{R} \to \mathbb{R}$ defined by

$$v(z) := \begin{cases} d(U(z))\, |U'(z)|^{p-2}\, U'(z), & z \notin \partial M_U \cup \partial N_U, \\ 0, & z \in \partial N_U, \\ L(z), & z \in \partial M_U \end{cases}$$

is continuous and for any $z, \hat{z} \in \mathbb{R}$

$$v(\hat{z}) - v(z) + c\,(U(\hat{z}) - U(z)) + \int_z^{\hat{z}} g(U(\xi))\, \mathrm{d}\xi = 0. \tag{2.5}$$

Moreover, $\lim_{z \to \pm\infty} v(z) = 0$ if either $\lim_{z \to -\infty} U(z) = 1$ and $\lim_{z \to +\infty} U(z) = 0$ or else $\lim_{z \to -\infty} U(z) = 0$ and $\lim_{z \to +\infty} U(z) = 1$.

Let us now explain the main idea behind the definition. Some technical details will be addressed in the subsequent remarks. The profile $U = U(z)$ is defined on the whole real line and it is a non-smooth function in general. The lack of differentiability is caused by the discontinuities of the diffusion coefficient and possibly also by its behaviour near 0 and 1. In particular, if the profile reaches one or both equilibria in a finite z, it can do so with a non-zero slope. Therefore, we introduce the sets M_U, N_U to account for such phenomena.

Observe that part (a) holds for any *monotone* profile, since both sets ∂M_U and ∂N_U consist of isolated points. We will prove in the next section that if g is a monostable reaction term and a solution U of (2.4) satisfies boundary conditions $U(-\infty) = 1$, $U(+\infty) = 0$, then U is necessarily nonincreasing on $\mathbb{R}$. As for the bistable reaction term, an analogous result cannot be obtained in general. However, it is reasonable to expect some solutions to have this property, cf. [35, Proposition 2]. We will discuss this topic in more detail at the end of this chapter.

For a more intuitive understanding of parts (b) and (c) of the definition, suppose for now that $c = 0$, i.e., let us look for non-constant *stationary solutions* $u(x,t) = u(x)$ of (2.1), which satisfy the equation

$$\Big(d(u(x))\,|u'(x)|^{p-2}\,u'(x)\Big)' = -g(u(x)), \quad x \in \mathbb{R}, \quad ' := \frac{\mathrm{d}}{\mathrm{d}x}\,. \tag{2.6}$$

Since $g \in C[0,1]$, it is natural to require that the function $d(u(x))|u'(x)|^{p-2}u'(x) =: v(x)$ is continuously differentiable. Hence it is sufficient to assume that only one-sided derivatives of $u = u(x)$ exist for $x \in \partial M_u$ as long as they are properly "compensated" by the discontinuities of d. The resulting product v then attains one value, but the individual terms taken as one-sided limits can be unequal. In other words, the *transition condition*

$$|u'(x-)|^{p-2}\,u'(x-)\lim_{\xi\to x-} d(u(\xi)) = |u'(x+)|^{p-2}\,u'(x+)\lim_{\xi\to x+} d(u(\xi))$$

from part (b) of the definition must hold. To visualize this condition, let us assume for simplicity that d has only one point of discontinuity $s_1 \in (0,1)$ with $d(s_1-) > d(s_1+)$, $N_u = \emptyset$ and the solution $u = u(x)$ is a decreasing function. Then $M_u = \{\xi_1\}$, $u(\xi_1) = s_1$ and $u'(\xi_1-) < u'(\xi_1+)$, as illustrated in Figure 2.3. The stationary case $c = 0$ is special

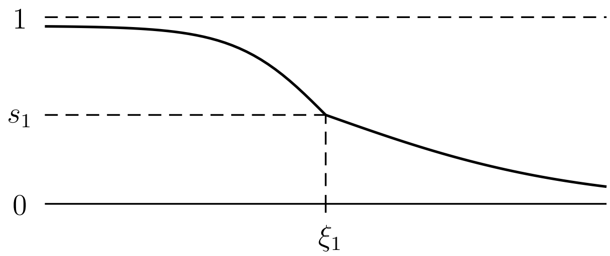

Figure 2.3: Profile of a nonincreasing solution $u = u(x)$ for d discontinuous at s_1

in the sense that the continuous function $v = v(x)$, defined in part (c) of the definition, is indeed differentiable for all $x \in \mathbb{R}$, cf. Remark 2.4 below. A similar reasoning for

$$\Big(d(U(z))\,|U'(z)|^{p-2}\,U'(z)\Big)' + cU'(z) = -g(U(z))$$

with $c \neq 0$ becomes more involved due to the additional term $cU'(z)$ on the left-hand side, but the general idea remains the same, suggesting that at some points we can now expect only the existence of one-sided derivatives $v'(z\pm)$.

Finally, the requirement concerning limits $v(\pm\infty)$, provided certain boundary conditions are satisfied, is motivated by the case of a smooth diffusion coefficient. According to [22, Lemma 6.1], if g is a monostable reaction term and $d \in C^1(0,1)$ is positive and bounded, these limits are equal to zero. Since our diffusion coefficient d need not be bounded, we incorporate this property explicitly into the definition.

We now proceed with some more detailed remarks concerning Definition 2.2.

Remark 2.3. Constant functions $U(z) = k$, $z \in \mathbb{R}$, where $k \in [0,1]$ is such that $g(k) = 0$, are solutions of (2.4). In particular, $U_0 \equiv 0$ and $U_1 \equiv 1$ are solutions. Here $M_{U_0} = M_{U_1} = \emptyset$, $N_{U_0} = N_{U_1} = \mathbb{R}$. In the monostable case, those are the only constant solutions, while in the bistable case we have at least one more constant solution, namely $U_* \equiv s_*$.

Remark 2.4. Non-constant stationary solutions $u(x,t) = u(x)$ of (2.1) satisfy the equation

$$\Big(d(u(x))\,|u'(x)|^{p-2}\,u'(x)\Big)' + g(u(x)) = 0$$

in the sense of Definition 2.2 with $c = 0$. In particular, the first integral (2.5) becomes

$$v(\hat{x}) - v(x) + \int_x^{\hat{x}} g(u(\xi))\,\mathrm{d}\xi = 0$$

for all $x, \hat{x} \in \mathbb{R}$. Multiplying both sides of the above equation by $\frac{1}{h}$ and passing to the limit for $h \to 0$, we obtain that v is continuously differentiable and

$$v'(x) + g(u(x)) = 0$$

holds for all $x \in \mathbb{R}$.

Remark 2.5. Let $z \notin \partial M_U \cup \partial N_U$, $\hat{z} = z + h$, $h \neq 0$. Since ∂M_U and ∂N_U are closed sets, we can choose $|h|$ so small that $\hat{z} \notin \partial M_U \cup \partial N_U$. Divide (2.5) by h and let $h \to 0$. Then, by Definition 2.2, the derivative $U'(z)$ exists and

$$v'(z) + cU'(z) + g(U(z)) = 0. \tag{2.7}$$

In particular, v is differentiable in $z \notin \partial M_U \cup \partial N_U$.
Proceeding similarly for $z \in \partial M_U$ and taking the limits as $h \to 0-$ and $h \to 0+$, we obtain

$$v'(z-) + cU'(z-) + g(U(z)) = 0$$

and

$$v'(z+) + cU'(z+) + g(U(z)) = 0,$$

respectively. In particular, both $v'(z-)$, $v'(z+)$ exist and they are finite.

Remark 2.6. Let $p = 2$, $d \equiv 1$ and let $U = U(z)$ be a solution of (2.4) in the sense of Definition 2.2. Then $M_U = \emptyset$ and $U \in C^1(\mathbb{R} \setminus \partial N_U)$. It follows from Remark 2.5 that $v(z) = U'(z)$ is differentiable for any $z \notin \partial N_U$. In particular, if $N_U = \emptyset$ then $U \in C^2(\mathbb{R})$ and the equation (2.4) holds pointwise.

Remark 2.7. Let $U = U(z)$ be a solution of (2.4) such that $N_U \neq \emptyset$. Clearly $U'(z) = 0$ if $z \in \operatorname{int} N_U$. The existence of $U'(z)$ (and also $U'(z-)$, $U'(z+)$) for $z \in \partial N_U$ depends on the behaviour of d near 0 and 1. In particular, if

$$\liminf_{s \to 0+} d(s) > 0 \quad \text{and} \quad \liminf_{s \to 1-} d(s) > 0$$

then it follows from the continuity of v that $U'(z) = 0$ for any $z \in \partial N_U$. On the other hand, if $\lim_{s \to 0+} d(s) = 0$ ($\lim_{s \to 1-} d(s) = 0$) then $U'(z\pm)$ need not be equal or even finite for $z \in \partial N_U$ such that $U(z) = 0$ ($U(z) = 1$).

Remark 2.8. Since the equation (2.4) is autonomous, then if $U = U(z)$ is a solution of (2.4), given any fixed $\xi \in \mathbb{R}$, the function $z \mapsto U(z + \xi)$ is also a solution of (2.4).

2.2 Monotonicity of solutions

Consider the boundary value problem (b.v.p. for short)

$$\begin{cases} \left(d(U(z))\,|U'(z)|^{p-2}\,U'(z)\right)' + cU'(z) + g(U(z)) = 0, & z \in \mathbb{R} \\ \lim\limits_{z\to-\infty} U(z) = 1, \quad \lim\limits_{z\to+\infty} U(z) = 0. \end{cases} \tag{2.8}$$

A typical result established for $p = 2$ states that each solution $U = U(z)$ of (2.8) is nonincreasing on $\mathbb{R}$ and strictly decreasing in the open interval $J := \{z \in \mathbb{R} : 0 < U(z) < 1\}$, see e.g. [23, 33, 35]. Moreover, the derivative of U does not vanish in J. As shown in [16, Proposition 3.4] for monostable reaction, the profile U maintains this property even if $d \in C(0,1)$ has integrable singularities near 0 and 1. Below we extend this result to the case $p > 1$ within our functional setting.

Let $U = U(z)$ be a solution of (2.8) with a monostable reaction term g. Passing to the limit for $z \to -\infty$ in (2.5) and writing z in place of $\hat{z}$, we obtain that

$$v(z) + c\,(U(z) - 1) + \int_{-\infty}^{z} g(U(\sigma))\,\mathrm{d}\sigma = 0 \tag{2.9}$$

holds for any $z \in \mathbb{R}$. On the other hand, passing to the limit for $\hat{z} \to +\infty$ in (2.5), we obtain that

$$v(z) + cU(z) - \int_{z}^{+\infty} g(U(\sigma))\,\mathrm{d}\sigma = 0 \tag{2.10}$$

holds for any $z \in \mathbb{R}$. Taking now the limit for $z \to -\infty$ in (2.10) yields

$$c = \int_{-\infty}^{+\infty} g(U(\sigma))\,\mathrm{d}\sigma$$

and it follows from $g > 0$ in $(0,1)$ that $c > 0$. Similarly, for the opposite boundary conditions $U(-\infty) = 0$, $U(+\infty) = 1$ we would arrive at $c < 0$.

Lemma 2.9. *Let $U = U(z)$, $z \in \mathbb{R}$, be a solution of the b.v.p.* (2.8) *with monostable reaction term $g \in C[0,1]$ and assume $\xi \in N_U$. Then the following two alternatives occur:*

(i) *if $U(\xi) = 0$ then $U(z) = 0$ for every $z \geq \xi$;*

(ii) *if $U(\xi) = 1$ then $U(z) = 1$ for every $z \leq \xi$.*

Proof. (i) Let $U(\xi) = 0$ and assume that there exists $\xi_* > \xi$ such that $U(\xi_*) > 0$. Taking ξ_* closer to ξ if necessary, we may assume that also $U(\xi_*) < 1$. Then $g(U(\xi_*)) > 0$ and therefore $\int_\xi^{+\infty} g(U(\sigma))\,\mathrm{d}\sigma > 0$. From the definition of v we get $v(\xi) = 0$ and from (2.10) with $z = \xi$ we deduce $\int_\xi^{+\infty} g(U(\sigma))\,\mathrm{d}\sigma = 0$, a contradiction.
(ii) Let $U(\xi) = 1$ and assume that there exists $\xi_* < \xi$ such that $U(\xi_*) < 1$. Taking ξ_* closer to ξ if necessary, we can guarantee also $U(\xi_*) > 0$. Hence $g(U(\xi_*)) > 0$ and so $\int_{-\infty}^{\xi} g(U(\sigma))\,\mathrm{d}\sigma > 0$. From the definition of v we have $v(\xi) = 0$ and from (2.9) with $z = \xi$ we deduce $\int_{-\infty}^{\xi} g(U(\sigma))\,\mathrm{d}\sigma = 0$, a contradiction. □

Proposition 2.10. *Let $U = U(z)$, $z \in \mathbb{R}$, be a solution of the b.v.p. (2.8) with monostable reaction term $g \in C[0,1]$. Then U is nonincreasing on $\mathbb{R}$ and strictly decreasing in the open interval $J = \{z \in \mathbb{R} : 0 < U(z) < 1\}$. Moreover, for $z \in J$ we have $U'(z) < 0$ if $z \notin M_U$ and $U'(z-) < 0$, $U'(z+) < 0$ if $z \in M_U$.*

Proof. Clearly, the set J is open and from Lemma 2.9 we conclude that $J = (z_0, z_1)$ where $-\infty \le z_0 < z_1 \le +\infty$, $U(z_0) = 1$ if $z_0 \in \mathbb{R}$ and $U(z_1) = 0$ if $z_1 \in \mathbb{R}$.

Let $\xi \in J$ be such that $U'(\xi-) = 0$. Then it follows from Remark 2.5 (for both alternatives $z \notin \partial M_U$ and $z \in \partial M_U$) that

$$v'(\xi-) = -g(U(\xi)) < 0.$$

Since $v(\xi) = 0$, there exists a left neighbourhood $\mathcal{U}_-(\xi)$ of the point ξ such that for all $z \in \mathcal{U}_-(\xi)$ we have $v(z) > 0$. Taking $\mathcal{U}_-(\xi)$ smaller if necessary, we may assume that $N_U \cap \mathcal{U}_-(\xi) = \emptyset$. Since $d(U(z)) > 0$, $z \in \mathcal{U}_-(\xi)$, from $v(z) > 0$ we deduce that for any $z \in \mathcal{U}_-(\xi)$ we have also $U'(z-) > 0$, $U'(z+) > 0$. However, this implies that $U(z) < U(\xi)$, $z \in \mathcal{U}_-(\xi)$. Since, by Definition 2.2, $U'(\xi+) = 0$, we deduce similarly as above that there is also a right neighbourhood $\mathcal{U}_+(\xi)$ of ξ such that $U(z) < U(\xi)$, $z \in \mathcal{U}_+(\xi)$. Therefore, ξ is the point of strict local maximum for U. Since $U(z) \to 1$ as $z \to -\infty$ and $U(\xi) < 1$, there is $\xi_* \in (-\infty, \xi)$ such that $U(\xi) \le U(\xi_*) < 1$. Let $\xi^* \in [\xi_*, \xi]$ be a global minimizer for U over the compact interval $[\xi_*, \xi]$. Then $U(\xi^*) < U(\xi) \le U(\xi_*) < 1$ and therefore $\xi^* \in (\xi_*, \xi)$. In particular, ξ^* is also a local minimizer for U. If $\xi^* \notin \partial M_U$ then $U'(\xi^*)$ exists and hence $U'(\xi^*) = 0$ (ξ^* is a local minimizer for U). We can prove as above that ξ^* is a strict local maximizer for U, a contradiction. Finally, if $\xi^* \in \partial M_U$ then from Definition 2.2 (b) and $d(U(\xi^*)) > 0$ we conclude $\operatorname{sgn} U'(\xi^*_-) = \operatorname{sgn} U'(\xi^*_+)$. But ξ^* being local minimizer for U implies that $U'(\xi^*_-) \le 0$ and $U'(\xi^*_+) \ge 0$. Hence, $U'(\xi^*_-) = U'(\xi^*_+) = 0$, i.e., $U'(\xi^*) = 0$ and we proceed as above. This concludes the proof. $\square$

Remark 2.11. In the case of bistable reaction term g we are not able to prove Lemma 2.9 and Proposition 2.10 as it is possible for $p = 2$, smooth d and g, see e.g. [23]. The proof relies on the uniqueness of the solution of the initial value problem for the second-order equation in (2.8). However, in our general setting of the problem this uniqueness result is not available. Therefore, in the bistable case we will always deal a priori with monotone travelling wave profiles $U = U(z)$ with properties from Proposition 2.10.

2.3 Reduction to a first-order problem

Let $U = U(z)$ be a solution of (2.8) which is nonincreasing on $\mathbb{R}$ and there exists an open interval $(z_0, z_1) \subset \mathbb{R}$, $-\infty \le z_0 < z_1 \le +\infty$, such that U is strictly decreasing in (z_0, z_1),

$$\begin{aligned} &\lim_{z \to z_0+} U(z) = 1 \quad \text{and} \quad U(z) = 1 \qquad \text{if} \quad -\infty < z \le z_0, \\ &\lim_{z \to z_1-} U(z) = 0 \quad \text{and} \quad U(z) = 0 \qquad \text{if} \quad z_1 \le z < +\infty. \end{aligned}$$

Moreover, $M_U = \{\xi_1, \xi_2, \ldots, \xi_n\}$ where $U(\xi_i) = s_i$, $i = 1, 2, \ldots, n$. In particular, $\operatorname{int} M_U = \emptyset$ and $M_U = \partial M_U$. For all $z \notin M_U \cup N_U$ we have $U'(z) < 0$ and for all $z \in M_U$ we have $U'(z-) < 0$ and $U'(z+) < 0$. The function U is continuous and piecewise C^1 in the sense that $U|_{(\xi_i, \xi_{i+1})} \in C^1(\xi_i, \xi_{i+1})$. Therefore, there exists a strictly decreasing inverse function

$U^{-1} : (0,1) \to (z_0, z_1)$, $z = U^{-1}(U)$, such that $U^{-1}|_{(s_i,s_{i+1})} \in C^1(s_i, s_{i+1})$, $i = 0, 1, \dots, n$ and the limits

$$\lim_{U\to s_i-} \frac{\mathrm{d}z}{\mathrm{d}U} = \left(\lim_{z\to \xi_i+} \frac{\mathrm{d}U}{\mathrm{d}z}\right)^{-1}, \quad \lim_{U\to s_i+} \frac{\mathrm{d}z}{\mathrm{d}U} = \left(\lim_{z\to \xi_i-} \frac{\mathrm{d}U}{\mathrm{d}z}\right)^{-1}$$

exist finite, $i = 1, 2, \dots, n$. Set

$$w(U) = v(z(U)), \quad U \in (0,1). \tag{2.11}$$

Then $w = w(U)$ is a piecewise C^1-function in $(0,1)$,

$$w|_{(s_i,s_{i+1})} \in C^1(s_i, s_{i+1}), \quad i = 0, 1, \dots, n,$$

with finite limits $\lim_{U\to s_i-} w'(U)$, $\lim_{U\to s_i+} w'(U)$, $i = 1, 2, \dots, n$. Therefore, for any $z \in (\xi_i, \xi_{i+1})$ and $U \in (s_i, s_{i+1})$, $i = 0, 1, \dots, n$, we have

$$\frac{\mathrm{d}}{\mathrm{d}z} v(z) = \frac{\mathrm{d}}{\mathrm{d}z} w(U(z)) = \frac{\mathrm{d}w}{\mathrm{d}U} (U(z))\, U'(z). \tag{2.12}$$

From $v(z) = -d(U(z))\, |U'(z)|^{p-1}$ we deduce that

$$U'(z) = -\left|\frac{v(z)}{d(U(z))}\right|^{p'-1}, \qquad p' = \frac{p}{p-1}. \tag{2.13}$$

From (2.11), (2.12) and (2.13),

$$\frac{\mathrm{d}v}{\mathrm{d}z} = -\frac{\mathrm{d}w}{\mathrm{d}U}(U(z)) \left|\frac{v(z)}{d(U(z))}\right|^{p'-1} = -\frac{\mathrm{d}w}{\mathrm{d}U} \left|\frac{w(U)}{d(U)}\right|^{p'-1}.$$

Therefore, the equation (2.7) for $z \in (\xi_i, \xi_{i+1})$ becomes

$$-\frac{\mathrm{d}w}{\mathrm{d}U} \left|\frac{w(U)}{d(U)}\right|^{p'-1} - c \left|\frac{w(U)}{d(U)}\right|^{p'-1} + g(U) = 0, \quad U \in (s_i, s_{i+1}),$$

$i = 0, 1, \dots, n$. This is equivalent to

$$|w|^{p'-1} \frac{\mathrm{d}w}{\mathrm{d}U} = -c\, |w|^{p'-1} + (d(U))^{p'-1}\, g(U), \tag{2.14}$$

or

$$\frac{1}{p'} \frac{\mathrm{d}}{\mathrm{d}U} |w|^{p'} = c\, |w|^{p'-1} - (d(U))^{p'-1}\, g(U). \tag{2.15}$$

Writing t instead of U, we set

$$f(t) = (d(t))^{\frac{1}{p-1}}\, g(t) \tag{2.16}$$

and $y(t) = |w(t)|^{p'}$. Then (2.15) becomes

$$y'(t) = p' \left[c\,(y(t))^{\frac{1}{p}} - f(t)\right], \quad t \in (0,1) \setminus \bigcup_{i=1}^{n} \{s_i\}. \tag{2.17}$$

From the boundary conditions in (2.8) and Definition 2.2 (c) we deduce that $v(z) \to 0$ as $z \to z_0+$ or $z \to z_1-$ which is equivalent to $\lim_{U\to 0+} w(U) = \lim_{U\to 1-} w(U) = 0$. Therefore, $y = y(t)$ satisfies the boundary conditions

$$y(0) = y(1) = 0. \tag{2.18}$$

On the other hand, let us suppose that $y = y(t)$, $y \in C[0,1]$, is a positive solution of (2.17), (2.18). Set $w(s) := -(y(s))^{\frac{1}{p'}}$. Then w satisfies (2.14) and (2.15). For $U \in (0,1)$ set

$$z(U) = -\int_{\frac{1}{2}}^{U} \left|\frac{d(s)}{w(s)}\right|^{\frac{1}{p-1}} \mathrm{d}s, \tag{2.19}$$

where $w(s) = -(y(s))^{\frac{1}{p'}}$. Then the function $z = z(U)$ is continuous strictly decreasing in $(0,1)$, $z(\frac{1}{2}) = 0$ and maps the interval $(0,1)$ onto (z_0, z_1), where $-\infty \le z_0 < z_1 \le +\infty$. Let us denote by $U : (z_0, z_1) \to (0,1)$ the inverse function to $z = z(U)$. Then $U(0) = \frac{1}{2}$, U is continuous strictly decreasing,

$$\lim_{z\to z_0+} U(z) = 1 \quad \text{and} \quad \lim_{z\to z_1-} U(z) = 0.$$

Let $z \in (\xi_i, \xi_{i+1})$, $i = 0, 1, \dots, n$, where $U(\xi_i) = s_i$, $i = 0, 1, \dots, n, n+1$. Then from (2.19) we deduce

$$\frac{\mathrm{d}U}{\mathrm{d}z} = \frac{1}{\frac{\mathrm{d}z(U)}{\mathrm{d}U}} = -\left|\frac{w(U)}{d(U)}\right|^{\frac{1}{p-1}}, \quad U \in (s_i, s_{i+1}), \tag{2.20}$$

i.e., $U \in C^1(\xi_i, \xi_{i+1})$, $U'(z) < 0$ and

$$-d(U(z))\left|\frac{\mathrm{d}U(z)}{\mathrm{d}z}\right|^{p-1} = w(U(z)) =: v(z), \tag{2.21}$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[d(U(z))\left|\frac{\mathrm{d}U}{\mathrm{d}z}\right|^{p-2}\frac{\mathrm{d}U}{\mathrm{d}z}\right] = \frac{\mathrm{d}}{\mathrm{d}z}w(U(z)) = \frac{\mathrm{d}w}{\mathrm{d}U}\frac{\mathrm{d}U(z)}{\mathrm{d}z}. \tag{2.22}$$

From (2.14), (2.21) we deduce

$$\begin{aligned}
\frac{\mathrm{d}w}{\mathrm{d}U} &= |w(U)|^{-(p'-1)}\left(-c\,|w(U)|^{p'-1} + (d(U))^{p'-1}g(U)\right)\\
&= -c + |w(U)|^{-(p'-1)}(d(U))^{p'-1}g(U)\\
&= -c + (d(U(z)))^{-(p'-1)}\left|\frac{\mathrm{d}U(z)}{\mathrm{d}z}\right|^{-(p-1)(p'-1)}(d(U(z)))^{p'-1}g(U(z))\\
&= -c + \left|\frac{\mathrm{d}U(z)}{\mathrm{d}z}\right|^{-1}g(U(z)).
\end{aligned}$$

Let us substitute this into (2.22):

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[d(U(z))\left|\frac{\mathrm{d}U}{\mathrm{d}z}\right|^{p-2}\frac{\mathrm{d}U}{\mathrm{d}z}\right] = \left[-c + \left|\frac{\mathrm{d}U(z)}{\mathrm{d}z}\right|^{-1}g(U(z))\right]\frac{\mathrm{d}U(z)}{\mathrm{d}z} = -c\frac{\mathrm{d}U(z)}{\mathrm{d}z} - g(U(z)),$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[d(U(z))\left|\frac{\mathrm{d}U}{\mathrm{d}z}\right|^{p-2}\frac{\mathrm{d}U}{\mathrm{d}z}\right] + c\frac{\mathrm{d}U(z)}{\mathrm{d}z} + g(U(z)) = 0, \quad z \in (\xi_i, \xi_{i+1}), \quad i = 0, 1, \dots, n.$$

It follows from (2.20) that

$$\lim_{z\to\xi_i\pm} U'(z) = -\left|\frac{w(s_i)}{\lim\limits_{s\to s_i\pm} d(s)}\right|^{\frac{1}{p-1}}, \quad i=1,2,\dots,n.$$

Notice that $\lim_{s\to s_i\pm} d(s) \neq 0$ due to **(H1)**. From (2.21) and the continuity of U we then have

$$\lim_{z\to z_0+} d(U(z))\,|U'(z)|^{p-2}\,U'(z) = \lim_{z\to z_1-} d(U(z))\,|U'(z)|^{p-2}\,U'(z) = 0$$

and the following one-sided limits are finite

$$\lim_{z\to\xi_i-} d(U(z))\,|U'(z)|^{p-2}\,U'(z) = \lim_{z\to\xi_i+} d(U(z))\,|U'(z)|^{p-2}\,U'(z), \quad i=1,2,\dots,n.$$

Since U is monotone decreasing in (z_0,z_1), we have

$$\lim_{z\to\xi_i-} d(U(z)) = \lim_{s\to s_i+} d(s) \qquad\text{and}\qquad \lim_{z\to\xi_i+} d(U(z)) = \lim_{s\to s_i-} d(s), \quad i=1,2,\dots,n.$$

Therefore, U satisfies the transition condition

$$|U'(\xi_i-)|^{p-2}\,U'(\xi_i-) \lim_{s\to s_i+} d(s) = |U'(\xi_i+)|^{p-2}\,U'(\xi_i+) \lim_{s\to s_i-} d(s), \quad i=1,2,\dots,n.$$

We may summarize the above reasoning in the following equivalence.

Proposition 2.12. *Let g be a monostable reaction term. Then $U=U(z)$, $U\in\widehat{C}^1(\mathbb{R})$, is a unique solution (up to translation) of* (2.8) *if and only if $y:[0,1]\to\mathbb{R}$, $y\in C[0,1]$ is a unique positive solution of* (2.17)*,* (2.18)*.*

Let g be a bistable reaction term. Then $U=U(z)$, $U\in\widehat{C}^1(\mathbb{R})$, is a unique (up to translation) nonincreasing solution of (2.8) *if and only if $y:[0,1]\to\mathbb{R}$, $y\in C[0,1]$ is a unique positive solution of* (2.17)*,* (2.18)*.*

Thanks to this proposition, we can study the first-order problem (2.17), (2.18) to derive the existence and uniqueness of solution to the second-order b.v.p. (2.8). Let us recall that there are two “unknowns” in this problem. Indeed, besides the positive solution $y=y(t)$ we also look for unknown speed of propagation $c>0$. Therefore, (2.17), (2.18) is not overdetermined.

Next, we discuss the sign of the speed of propagation c. Let $y(t)>0$, $t\in(0,1)$ be a positive solution of (2.17), (2.18). Integrating (2.17) and using (2.18) we obtain

$$0 = y(1)-y(0) = \int_0^1 y'(t)\,\mathrm{d}t = p'\left[c\int_0^1 (y(t))^{\frac{1}{p}}\,\mathrm{d}t - \int_0^1 f(t)\,\mathrm{d}t\right]$$

and hence

$$c = \frac{\int_0^1 f(t)\,\mathrm{d}t}{\int_0^1 (y(t))^{\frac{1}{p}}\,\mathrm{d}t}, \tag{2.23}$$

where f is given by (2.16). It follows immediately that the sign of c is ultimately determined by the sign of

$$\int_0^1 f(t)\,\mathrm{d}t = \int_0^1 (d(t))^{\frac{1}{p-1}}\,g(t)\,\mathrm{d}t,$$

justifying the following lemma.

Lemma 2.13. *Let us assume that*

$$\int_0^1 (d(t))^{\frac{1}{p-1}} g(t)\,\mathrm{d}t > 0 \quad (< 0)$$

and BVP (2.17), (2.18) *has a positive solution. Then* $c > 0$ (< 0)*.*

Remark 2.14. Suppose that the following balanced condition holds

$$\int_0^1 (d(s))^{\frac{1}{p-1}} g(s)\,\mathrm{d}s = 0.$$

Then $c = 0$ and

$$y(t) = -p' \int_0^t (d(s))^{\frac{1}{p-1}} g(s)\,\mathrm{d}s, \quad t \in (0,1) \tag{2.24}$$

is a unique positive solution of (2.17), (2.18) with $c = 0$ (cf. Theorem 3.10). The solution given by (2.24) leads to the stationary wave. Its profile $u = u(x)$ satisfies the equation

$$\big(d(u(x))|u'(x)|^{p-2}u'(x)\big)' + g(u(x)) = 0, \quad x \in \mathbb{R}.$$

Remark 2.15. If we were to look for nondecreasing solutions instead of nonincreasing ones, the procedure leading up to the first-order problem would be the same. Let us denote the speed of propagation of a nondecreasing travelling wave by C. Since $U'(z) > 0$ whenever U' exists and $U'(z-) > 0$, $U'(z+) > 0$ if $z = \xi_i$, $i = 1, 2, \ldots, n$, instead of equation (2.15) we would arrive at

$$\frac{1}{p'} \frac{\mathrm{d}}{\mathrm{d}U} |w|^{p'} = -C\,|w|^{p'-1} - (d(U))^{p'-1}\, g(U),$$

where $|w| = w$. The corresponding first order equation can be written in the form (2.17) if we set $c = -C$. Therefore, the existence results regarding nonincreasing solutions also hold for nondecreasing travelling waves which travel in the opposite direction.

Chapter 3

The first-order ODE

In this chapter, we study the first-order boundary value problem

$$\begin{cases} y'(t) = p' \left[c \left(y^+(t) \right)^{\frac{1}{p}} - f(t) \right], & t \in (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{3.1}$$

where $p > 1$, $p' = \frac{p}{p-1}$, $c \in \mathbb{R}$ is a parameter, $f \in L^1(0,1)$ and $y^+(t) = \max\{y(t), 0\}$.

For $f(t) = (d(t))^{\frac{1}{p-1}} g(t)$, where d and g satisfy hypotheses **(H1)**, **(H2)**, respectively, positive solutions of (3.1) correspond via Proposition 2.12 to nonincreasing solutions of the second-order b.v.p. (2.8). However, it is important to note that the results presented in this chapter apply to even more general functions f and are therefore of independent interest. In particular, existence and uniqueness for the associated initial value problems (cf. Section 3.1) hold for any $f \in L^1(0,1)$. Specifying the sign conditions on f, we then derive existence and non-existence results for the b.v.p. (3.1) in two qualitatively different cases.

In [22], the b.v.p. (3.1) was studied for functions $f \in C[0,1]$ of types A, B and C with $f(0) = f(1) = 0$. The lack of continuity of f in our setting requires the employment of different techniques to establish fundamental results and consequently prove the existence of solutions to (3.1). Due to the assumption $f \in L^1(0,1)$, we consider solutions *in the sense of Carathéodory*, i.e., functions which are absolutely continuous in $[0,1]$ and satisfy the differential equation for a.e. $t \in [0,1]$. In what follows, we will refer to a solution in the sense of Carathéodory simply as "solution".

3.1 Initial value problems

Throughout this section we assume $f \in L^1(0,1)$. For $(t, y, c) \in [0,1] \times \mathbb{R}^2$ we set

$$h(t, y, c) := p' \left[c \left(y^+ \right)^{\frac{1}{p}} - f(t) \right]$$

and study the following two initial value problems on $[0,1]$, which depend on the parameter $c \in \mathbb{R}$:

$$y'(t) = h(t, y(t), c), \quad y(0) = 0 \tag{3.2}$$

and

$$y'(t) = h(t, y(t), c), \quad y(1) = 0. \tag{3.3}$$

Using the terminology from [16], (3.2) will be referred to as *forward initial value problem*, while (3.3) will be referred to as *backward initial value problem*. Our aim is to determine whether for some values of $c \in \mathbb{R}$, the corresponding solution $y_c = y_c(t)$, $t \in [0,1]$, of either (3.2) or (3.3) also vanishes at the other endpoint of $[0,1]$.

First, let us note that $f \in L^1(0,1)$ implies that $h = h(t,y,c)$ satisfies *Carathéodory conditions*, i.e., for almost every $t \in [0,1]$ fixed, $h(t,\cdot,\cdot)$ is continuous with respect to y and c and for every $y \in \mathbb{R}$ and $c \in \mathbb{R}$ fixed, $h(\cdot,y,c)$ is measurable with respect to t.

Lemma 3.1. *For any $c \in \mathbb{R}$ there exists at least one solution $y_c = y_c(t)$ of the froward i.v.p.* (3.2) *defined on the entire interval* $[0,1]$. *The same holds for the backward i.v.p.* (3.3).

Proof. Let $c \in \mathbb{R}$ and $f \in L^1(0,1)$ be fixed. Since $h = h(t,y,c)$ satisfies Carathéodory conditions, then according to [43, §10.XVIII, p. 121], it is sufficient to show that there exists $m \in L^1(0,1)$ such that $|h(t,y,c)| \le m(t)$ for $(t,y) \in [0,1] \times \mathbb{R}$. However, the function $y \mapsto (y^+)^{\frac{1}{p}}$ is not bounded from above, hence we cannot apply this result directly. Therefore, we first show that all solutions of (3.2), if they exist, are a priori bounded by a constant $K > 0$.

Integrating (3.2), we obtain

$$y(\sigma) = p'\left(c\int_0^\sigma \left(y^+(\tau)\right)^{\frac{1}{p}}\,\mathrm{d}\tau - \int_0^\sigma f(\tau)\,\mathrm{d}\tau\right), \quad \sigma \in (0,1). \tag{3.4}$$

For $t \in (0,1)$ set

$$\varrho(t) := \max_{\sigma\in[0,t]} |y(\sigma)|.$$

It follows from (3.4) that for $\sigma \in [0,t]$

$$|y(\sigma)| \le p'\left(|c|\int_0^\sigma \left(y^+(\tau)\right)^{\frac{1}{p}}\,\mathrm{d}\tau + \|f\|_{L^1(0,1)}\right)$$

and therefore

$$\begin{aligned}
\varrho(t) &\le p'\left(|c|\max_{\sigma\in[0,t]}\int_0^\sigma \left(y^+(\tau)\right)^{\frac{1}{p}}\,\mathrm{d}\tau + \|f\|_{L^1(0,1)}\right)\\
&\le p'\left(|c|\int_0^1 \max_{\sigma\in[0,t]}\left(y^+(\sigma)\right)^{\frac{1}{p}}\,\mathrm{d}\tau + \|f\|_{L^1(0,1)}\right)\\
&\le p'\left(|c|\left(\max_{\sigma\in[0,t]}|y(\sigma)|\right)^{\frac{1}{p}} + \|f\|_{L^1(0,1)}\right) = p'\left(|c|\left(\varrho(t)\right)^{\frac{1}{p}} + \|f\|_{L^1(0,1)}\right).
\end{aligned}$$

Since $\frac{1}{p} < 1$, the last inequality yields that there exists a constant $K > 0$ such that $\varrho(t) < K$ for all $t \in [0,1]$.

Let us set

$$\widehat{h}(t,y,c) = \begin{cases} h(t,y,c) & \text{for } |y| < K,\\ p'\left(c\,K^{\frac{1}{p}} - f(t)\right) & \text{for } y \ge K,\\ -p'f(t) & \text{for } y \le -K. \end{cases}$$

Then solutions of the modified problem

$$y'(t) = \widehat{h}(t,y(t),c), \quad y(0) = 0 \tag{3.5}$$

are bounded by the same constant $K > 0$. Indeed, following the same procedure as above, we obtain

$$\begin{aligned}
\varrho(t) &= \max_{\sigma\in[0,t]} |y(\sigma)| \\
&\le p' \left(|c| \max_{\sigma\in[0,t]} \int\limits_{\{\tau\in(0,\sigma)\,:\,|y(\tau)|<K\}} \left(y^+(\tau)\right)^{\frac{1}{p}} \,\mathrm{d}\tau + |c| \max_{\sigma\in[0,t]} \int\limits_{\{\tau\in(0,\sigma)\,:\,y(\tau)\ge K\}} K^{\frac{1}{p}} \,\mathrm{d}\tau + \|f\|_{L^1(0,1)} \right) \\
&\le p' \left(|c| \max_{\sigma\in[0,t]} \int_0^\sigma \left(y^+(\tau)\right)^{\frac{1}{p}} \,\mathrm{d}\tau + \|f\|_{L^1(0,1)} \right) \\
&\le p' \left(|c| \int_0^1 \max_{\sigma\in[0,t]} \left(y^+(\sigma)\right)^{\frac{1}{p}} \,\mathrm{d}\tau + \|f\|_{L^1(0,1)} \right) \\
&\le p' \left(|c| \left(\max_{\sigma\in[0,t]} |y(\sigma)| \right)^{\frac{1}{p}} + \|f\|_{L^1(0,1)} \right) = p' \left(|c| \left(\varrho(t)\right)^{\frac{1}{p}} + \|f\|_{L^1(0,1)} \right).
\end{aligned}$$

Therefore, $\varrho(t) < K$ for all $t \in [0,1]$ and the set of solutions of the modified problem (3.5) coincides with the set of solutions of (3.2). But $\widehat{h}$ satisfies Carathéodory conditions and there is a function $m \in L^1(0,1)$ such that $|\widehat{h}(t,y,c)| \le m(t)$ for $(t,y) \in [0,1] \times \mathbb{R}$. Hence, (3.5) (and thus also (3.2)) has at least one solution in $[0,1]$. Similarly we proceed in the case of the backward i.v.p. (3.3). □

Remark 3.2. The solution y_c in the above lemma is not unique in general due to the fact that the function

$$y \mapsto c\left(y^+\right)^{\frac{1}{p}}, \quad y \in \mathbb{R},$$

does not satisfy Lipschitz condition at 0. However, this function is nondecreasing for $c \ge 0$ and nonincreasing for $c \le 0$. Therefore, it satisfies one-sided Lipschitz condition in either case and we can derive uniqueness results separately for the forward and backward initial value problems.

Lemma 3.3. *If $c \le 0$, then the forward i.v.p.* (3.2) *has exactly one solution $y_c = y_c(t)$, $t \in [0,1]$. If $c \ge 0$, then the backward i.v.p.* (3.3) *has exactly one solution $y_c = y_c(t)$, $t \in [0,1]$.*

Proof. Since the idea of the proof is the same for both alternatives, we only prove the latter. Let $c \ge 0$ and $y_1 = y_1(t)$, $y_2 = y_2(t)$ be two solutions of (3.3) in $[0,1]$. Set

$$\delta(t) = \left(y_1(t) - y_2(t)\right)^2.$$

Then $\delta(1) = 0$, $\delta(t) \ge 0$ and

$$\begin{aligned}
\delta'(t) &= 2\left(y_1'(t) - y_2'(t)\right)\left(y_1(t) - y_2(t)\right) \\
&= 2p'c\left[\left(y_1^+(t)\right)^{\frac{1}{p}} - \left(y_2^+(t)\right)^{\frac{1}{p}}\right]\left(y_1(t) - y_2(t)\right) \ge 0.
\end{aligned}$$

(the functions positive part and $1/p$-th power are both nondecreasing). Hence $\delta(t) = 0$ for a.e. $t \in [0,1]$ and $y_1(t) = y_2(t)$, $t \in [0,1]$. □

Thanks to the uniqueness result, we also have continuous dependence of solutions on the parameter c.

Lemma 3.4. *Let $c_0 \geq 0$. Then $c \to c_0 > 0$ or $c \to 0+$ if $c_0 = 0$ implies that solutions $y_c = y_c(t)$ of the backward i.v.p.* (3.3) *converge uniformly in* $[0,1]$ *(i.e., in the topology of* $C[0,1]$*) to y_{c_0}. A corresponding statement holds for $c_0 \leq 0$ and the forward i.v.p.* (3.2).

Proof. The proof follows from the uniqueness result in Lemma 3.3 and [9, Theorems 4.1 and 4.2 in Chapter 2]. □

In Section 2.2 we have shown that when looking for solutions $U = U(z)$ of the b.v.p. (2.8), monostable reaction term g yields $c > 0$. As for the bistable case, the sign of the wavespeed c is given by the sign of

$$\int_0^1 f(t)\,\mathrm{d}t = \int_0^1 (d(t))^{\frac{1}{p-1}} g(t)\,\mathrm{d}t,$$

cf. Lemma 2.13. Therefore, we further focus on parameters $c \in [0, +\infty)$ and the backward i.v.p. (3.3).

Let us introduce the notion of *defect* $P_c\varphi$ of a function φ with respect to the differential equation $y' = h(t, y, c)$, see [43, §9.II, p. 90]:

$$P_c\varphi := \varphi'(t) - h(t, \varphi(t), c).$$

The defect indicates "how close" is φ to being a solution of the differential equation. In particular, the defect of a solution is 0. The following comparison argument is one of our basic tools.

Lemma 3.5. *Let $c \geq 0$ and assume that the functions φ, $\psi \in AC[0,1]$ satisfy $\varphi(1) \leq \psi(1)$, $P_c\varphi \geq P_c\psi$ a.e. in* $[0,1]$. *Then $\varphi \leq \psi$ in* $[0,1]$.

Proof. Set $w = \psi - \varphi$. Then

$$w' = \psi' - \varphi' = P_c\psi + p'c(\psi^+)^{\frac{1}{p}} - P_c\varphi - p'c(\varphi^+)^{\frac{1}{p}} \leq p'c\left((\psi^+)^{\frac{1}{p}} - (\varphi^+)^{\frac{1}{p}}\right) \tag{3.6}$$

a.e. in $[0,1]$. Assume that there is $t_0 \in (0,1)$ such that $w(t_0) < 0$. Let $t_1 \in (t_0, 1]$ be such that $w(t) \leq 0$, $t \in (t_0, t_1]$. It follows from (3.6) that $w' \leq p'c[(\psi^+)^{\frac{1}{p}} - (\varphi^+)^{\frac{1}{p}}] \leq 0$ a.e. in $(t_0, t_1]$, i.e., $w(t_1) \leq w(t_0) < 0$. By using the same argument repeatedly if necessary, we conclude that $w(1) < 0$, a contradiction with $\varphi(1) \leq \psi(1)$. □

Corollary 3.6. *Let $0 \leq c_1 < c_2$. Let y_{c_1} and y_{c_2} be the solutions of the backward i.v.p.* (3.3) *with $c = c_1$ and $c = c_2$, respectively. Then*

$$y_{c_1}(t) \geq y_{c_2}(t), \quad t \in [0,1].$$

In particular, $y_{c_1}(0) \geq y_{c_2}(0)$.

Proof. We have

$$\begin{aligned} P_{c_2}y_{c_1} &= y'_{c_1} - h(t, y_{c_1}, c_2) = \underbrace{y'_{c_1} - h(t, y_{c_1}, c_1)}_{=0} + h(t, y_{c_1}, c_1) - h(t, y_{c_1}, c_2) \\ &= p'(c_1 - c_2)\left(y^+_{c_1}\right)^{\frac{1}{p}} \leq 0 = y'_{c_2} - h(t, y_{c_2}, c_2) = P_{c_2}y_{c_2} \qquad \text{a.e. in } [0,1]. \end{aligned}$$

Since $y_{c_1}(1) = y_{c_2}(1) = 0$, Lemma 3.5 yields that $y_{c_1} \geq y_{c_2}$ in $[0,1]$. □

So far we have shown that for each $c \geq 0$ there exists a unique solution of (3.3) and the solutions $y_c = y_c(t)$ decrease (not strictly) with c. Note that these results hold for any $f \in L^1(0,1)$.

In order to prove the existence of solution to the b.v.p. (3.1), we restrict ourselves to two cases according to the sign of the function f on $[0,1]$. For simplicity, we denote these cases as bistable and monostable, indicating which type of reaction term g will lead to the desired properties of f given by $f(t) = (d(t))^{\frac{1}{p-1}} g(t)$.

3.2 Bistable case

Let $f \in L^1(0,1)$ have the following property: there exists $s_* \in (0,1)$ such that

$$f(t) \leq 0 \quad \text{if } t \in (0, s_*), \qquad f(t) > 0 \quad \text{if } t \in (s_*, 1). \tag{3.7}$$

In this section, we prove that if (3.7) holds and

$$\int_0^1 f(t)\,\mathrm{d}t \geq 0, \tag{3.8}$$

there exists a unique $c_* \geq 0$ such that the b.v.p. (3.1) possesses a unique positive solution $y_{c_*} = y_{c_*}(t)$, $t \in [0,1]$. More precisely, strict inequality in (3.8) leads to $c_* > 0$, while in the case of equality we obtain that $c_* = 0$.

First, we mention the following two corollaries of Lemma 3.5.

Corollary 3.7. *Assume that $f \in L^1(0,1)$ satisfies (3.7) and $\tilde{f} \in L^1(0,1)$ is such that*

$$\tilde{f}(t) = 0 \quad \textit{for } t \in (0, s_*), \qquad \tilde{f}(t) = f(t) \quad \textit{for } t \in (s_*, 1).$$

Let $c \geq 0$ and $\tilde{y}_c = \tilde{y}_c(t)$, $t \in [0,1]$, be a solution of the backward i.v.p. (3.3) with f replaced by $\tilde{f}$. Then $y_c \leq \tilde{y}_c$ in $[0,1]$.

Proof. Set $\tilde{h}(t,y,c) := p' \left[c(y^+(t))^{\frac{1}{p}} - \tilde{f}(t) \right]$. Then $\tilde{h} \leq h$ and so

$$\begin{aligned} P_c \tilde{y}_c = \tilde{y}_c' - h(t, \tilde{y}_c, c) &= \underbrace{\tilde{y}_c' - \tilde{h}(t, \tilde{y}_c, c)}_{=0} + \tilde{h}(t, \tilde{y}_c, c) - h(t, \tilde{y}_c, c) \leq 0 = y_c' - h(t, y_c, c) \\ &= P_c y_c \quad \text{a.e. in} \quad [0,1]. \end{aligned}$$

It then follows from Lemma 3.5 that $y_c \leq \tilde{y}_c$ in $[0,1]$. $\square$

Corollary 3.8. *Let $f \in L^1(0,1)$ be such that (3.7) holds and f is lower semicontinuous in $(s_*, 1)$. Let $y_c = y_c(t)$ be a solution of the backward i.v.p. (3.3) with $c \geq 0$. Then $y_c(t) > 0$ for $t \in (s_*, 1)$.*

Proof. We have

$$P_c 0 = 0 - h(t, 0, c) = p' f(t) \geq 0 = y_c' - h(t, y_c, c) = P_c y_c \quad \text{a.e. in} \quad [s_*, 1].$$

It follows from Lemma 3.5 with 0 replaced by s_* that $y_c \geq 0$ in $[s_*, 1]$. We prove that $y_c > 0$ in $(s_*, 1)$. Indeed, assume the contrary, i.e., there is $t_0 \in (s_*, 1)$ such that $y_c(t_0) = 0$. Since f is positive and lower semicontinuous in $(s_*, 1)$, given arbitrarily small $\varepsilon > 0$ there exists $\varrho > 0$ such that $f(t) \geq \varrho > 0$ for all $t \in [t_0, 1-\varepsilon]$. Integrating the equation in (3.3) from t_0 to $t \in (t_0, 1-\varepsilon]$ and using $y_c(t_0) = 0$, we get

$$y_c(t) = p' \left(c \int_{t_0}^{t} \left(y_c^+(\tau)\right)^{\frac{1}{p}} \,\mathrm{d}\tau - \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau \right),$$

$$\frac{y_c(t)}{t-t_0} = p' \left(c \, \frac{\int_{t_0}^{t} \left(y_c^+(\tau)\right)^{\frac{1}{p}} \,\mathrm{d}\tau}{t-t_0} - \frac{\int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau}{t-t_0} \right). \tag{3.9}$$

Since

$$\frac{\int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau}{t-t_0} \geq \varrho, \tag{3.10}$$

and, due to continuity of y_c at t_0 also

$$\lim_{t\to t_0+} \frac{\int_{t_0}^{t} \left(y_c^+(\tau)\right)^{\frac{1}{p}} \,\mathrm{d}\tau}{t-t_0} = 0, \tag{3.11}$$

we conclude from (3.9)–(3.11) that for t and t_0 close enough,

$$\frac{y_c(t)}{t-t_0} < 0,$$

a contradiction. Therefore, $y_c > 0$ in $(s_*, 1)$. $\square$

Below we present the main assertions of this section. In Theorem 3.9, we assume that $\int_0^1 f(t)\,\mathrm{d}t > 0$, while Theorem 3.10 deals with the case of $\int_0^1 f(t)\,\mathrm{d}t = 0$. We refer to these cases as unbalanced and balanced, respectively.

Theorem 3.9 (Unbalanced case)**.** *Let $f \in L^1(0,1)$ be such that (3.7) holds, f is lower semicontinuous in $(s_*, 1)$ and*

$$\int_0^1 f(t)\,\mathrm{d}t > 0.$$

Then there exists a number $c_* > 0$ such that the b.v.p. (3.1) has a unique positive solution if and only if $c = c_*$.

Proof. Let $y_0 = y_0(t)$ be the solution of the backward i.v.p. (3.3) with $c = 0$. It follows from our assumptions on f that

$$y_0(t) = p' \int_t^1 f(s)\,\mathrm{d}s > 0 \quad \text{for all } \; t \in [0,1). \tag{3.12}$$

In particular, $y_0(0) > 0$.

Set

$$c_* := \sup\left\{c > 0 : y_c(t) > 0 \; \text{ for all } \; t \in (0,1)\right\}.$$

From (3.12), continuous dependence on parameter (Lemma 3.4) and Corollary 3.8, we conclude that the set $\{c > 0 : y_c(t) > 0 \; \text{ for all } \; t \in (0,1)\}$ is non-empty and therefore $c_* > 0$. Next we prove that $c_* < +\infty$. Indeed, if $c_* = +\infty$, then by the definition of c_* there exist $c_n \to +\infty$ and corresponding $y_{c_n} = y_{c_n}(t)$ satisfying $y_{c_n} > 0$ in $(0,1)$. Let $\tilde{f}$ and

$\tilde{y}_c = \tilde{y}_c(t)$ be as in Corollary 3.7. Since $\tilde{y}_{c_n} \geq y_{c_n}$ in $(0,1)$ by the same corollary, we have $\tilde{y}_{c_n} > 0$ in $(0,1)$ and $\tilde{y}_{c_n}$ satisfies

$$\tilde{y}'_{c_n}(t) = p' c_n \left(\tilde{y}_{c_n}(t)\right)^{\frac{1}{p}}, \quad t \in (0, s_*), \tag{3.13}$$

$$\tilde{y}'_{c_n}(t) = p' \left[c_n \left(\tilde{y}_{c_n}(t)\right)^{\frac{1}{p}} - f(t)\right], \quad t \in (s_*, 1). \tag{3.14}$$

Separating variables in (3.13) yields

$$\left(\tilde{y}_{c_n}(t)\right)^{\frac{1}{p'}} = \left(\tilde{y}_{c_n}(s_*)\right)^{\frac{1}{p'}} + c_n(t - s_*), \quad t \in (0, s_*). \tag{3.15}$$

On the other hand, from (3.14) we obtain

$$\tilde{y}_{c_n}(s_*) = p' \int_{s_*}^{1} f(t)\,\mathrm{d}t - p' c \int_{s_*}^{1} \left(\tilde{y}^{+}_{c_n}(t)\right)^{\frac{1}{p}}\,\mathrm{d}t.$$

It follows that for all $n \in \mathbb{N}$

$$\tilde{y}_{c_n}(s_*) \leq p' \int_{s_*}^{1} f(t)\,\mathrm{d}t < +\infty$$

and therefore for any $t \in (0, s_*)$ the right-hand side of (3.15) tends to $-\infty$ as $n \to +\infty$, a contradiction. Hence $c_* < +\infty$.

Now we prove that $y_{c_*}(0) = 0$ and $y_{c_*}(t) > 0$, $t \in (0,1)$. By continuous dependence on parameter (Lemma 3.4), Corollary 3.8 and the definition of c_*, the function $y_{c_*} = y_{c_*}(t)$ must vanish somewhere in $[0, s_*]$. Let $\eta \in [0, s_*]$ be the largest zero of y_{c_*}. If $\eta > 0$, then for $c < c_*$ and $t \in (0, \eta]$ we have $y_c(t) > 0$ and hence from

$$y'_c(t) = p' c \left[\left(y_c(t)\right)^{\frac{1}{p}} - f(t)\right] \geq p' c \left(y_c(t)\right)^{\frac{1}{p}}, \quad t \in (0, \eta),$$

separating variables we deduce

$$0 < \left(y_c(t)\right)^{\frac{1}{p'}} \leq \left(y_c(\eta)\right)^{\frac{1}{p'}} + c(t - \eta), \quad t \in (0, \eta). \tag{3.16}$$

Since for $c \to c_*$ we have $y_c(\eta) \to y_{c_*}(\eta) = 0$ by Lemma 3.4, for any fixed $t \in (0, \eta)$ there exists $c < c_*$, $(c_* - c)$ sufficiently small, such that

$$\left(y_c(\eta)\right)^{\frac{1}{p'}} + c(t - \eta) < 0,$$

which contradicts (3.16). Therefore $\eta = 0$.

Finally, we show that positive solutions of the backward i.v.p. (3.3) do not vanish at 0 for other values of c different from c_*. Assume by contradiction that there exists $\hat{c} \neq c_*$ such that $y_{\hat{c}} = y_{\hat{c}}(t) > 0$ and $y_{\hat{c}}(0) = 0$. It follows immediately from the definition of c_* that $\hat{c} < c_*$ and by Corollary 3.6 we have $y_{\hat{c}}(t) \geq y_{c_*}(t)$, $t \in (0,1)$. Since

$$y'_{\hat{c}}(t) = p' \left[\hat{c} \left(y_{\hat{c}}(t)\right)^{\frac{1}{p}} - f(t)\right], \tag{3.17}$$

$$y'_{c_*}(t) = p' \left[c_* \left(y_{c_*}(t)\right)^{\frac{1}{p}} - f(t)\right], \tag{3.18}$$

for $c \in [\hat{c}, c_*]$ we obtain

$$y'_{\hat{c}}(t) \leq p' \left[c \left(y_{\hat{c}}(t)\right)^{\frac{1}{p}} - f(t)\right] \tag{3.19}$$

and

$$y'_{c_*}(t) \geq p' \left[c\, (y_{\hat{c}}(t))^{\frac{1}{p}} - f(t) \right] \tag{3.20}$$

for a.e. $t \in (0,1)$. Set $z_1 = (y_{\hat{c}})^{\frac{1}{p'}} > 0$, $z_2 = (y_{c_*})^{\frac{1}{p'}} > 0$. Then $z_1 \geq z_2$ in $(0,1)$ and it follows from (3.19) and (3.20) that

$$z_1'(t) \leq c - \frac{f(t)}{(z_1(t))^{\frac{1}{p-1}}}, \tag{3.21}$$

$$z_2'(t) \geq c - \frac{f(t)}{(z_2(t))^{\frac{1}{p-1}}}, \tag{3.22}$$

for a.e. $t \in (0,1)$. Let us subtract (3.22) from (3.21) and restrict on the interval $(0, s_*)$ where $f(t) \leq 0$. Then

$$(z_1(t) - z_2(t))' \leq -f(t) \left(\frac{1}{(z_1(t))^{\frac{1}{p-1}}} - \frac{1}{(z_2(t))^{\frac{1}{p-1}}} \right)$$

and

$$(z_1(t) - z_2(t))\,(z_1(t) - z_2(t))' \leq -f(t) \left(\frac{1}{(z_1(t))^{\frac{1}{p-1}}} - \frac{1}{(z_2(t))^{\frac{1}{p-1}}} \right) (z_1(t) - z_2(t)) \leq 0$$

for a.e. $t \in (0, s_*)$. Hence

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} (z_1(t) - z_2(t))^2 \leq 0 \quad \text{for a.e. } t \in (0, s_*). \tag{3.23}$$

Since $z_1(0) = z_2(0) = 0$, it follows from (3.23) that $z_1(t) = z_2(t)$, $t \in (0, s_*)$, i.e., $y_{\hat{c}}(t) = y_{c_*}(t)$, $t \in (0, s_*)$. Equations (3.17), (3.18) then hold for both $y_{\hat{c}}$, y_{c_*} on $(0, s_*)$ and by subtraction we conclude that

$$0 = p'(\hat{c} - c_*)\,(y_{c_*}(t))^{\frac{1}{p}}, \quad t \in (0, s_*).$$

But this equality cannot hold unless $\hat{c} = c_*$. Therefore c_* is the unique value of c for which $y_c(0) = 0$ and $y_c > 0$ in $(0,1)$. The uniqueness of y_{c_*} as a solution of the b.v.p. (3.1) follows from the uniqueness result for the backward i.v.p. (3.3). This completes the proof. □

Theorem 3.10 (Balanced case)**.** *Let $f \in L^1(0,1)$ be such that* (3.7) *holds, $f < 0$ on $(0, \delta)$ for some $\delta \in (0, s_*)$ and*

$$\int_0^1 f(t)\,\mathrm{d}t = 0. \tag{3.24}$$

Then the b.v.p. (3.1) *has a unique positive solution if and only if $c = 0$.*

Proof. Let $y = y(t)$, $t \in [0,1]$, be a positive solution of (3.1). Integrating the equation in (3.1) from 0 to 1 and using the boundary conditions together with (3.24), we obtain

$$0 = y(1) - y(0) = \int_0^1 y'(t)\,\mathrm{d}t = p' \left[c \int_0^1 \left(y^+(t)\right)^{\frac{1}{p}}\,\mathrm{d}t - \int_0^1 f(t)\,\mathrm{d}t \right] = p'c \int_0^1 \left(y^+(t)\right)^{\frac{1}{p}}\,\mathrm{d}t.$$

Hence $c = 0$.

On the other hand, the backward i.v.p. (3.3) with $c = 0$ has a unique solution

$$y(t) = p' \int_t^1 f(s)\,\mathrm{d}s.$$

It follows from our assumptions on f that $y(t) > 0$ for all $t \in (0,1)$ and $y(0) = 0$. Therefore, it is also a unique positive solution of (3.1). $\square$

Remark 3.11. Unlike in the unbalanced case, Theorem 3.10 does not require the assumption of lower semicontinuity of f in $(s_*, 1)$. On the other hand, f must now be negative on some small neighbourhood of 0 to ensure that the integral of f is equal to 0 only when taken over the entire interval $(0,1)$. Although properties (3.7) and (3.24) imply that f must be negative on a set of non-zero measure, if $f = 0$ on $(0,\delta)$, then $y(t) = p' \int_t^1 f(s)\,\mathrm{d}s = 0$ (at least) on $(0,\delta)$.

Remark 3.12. The result in Theorem 3.10 remains valid under the following (more general) assumptions on f: (3.24) holds and there exist $\alpha, \beta \in (0,1)$, $\alpha < \beta$, such that $f < 0$ on $(0,\alpha)$, $f \leq 0$ on (α, s_*), $f \geq 0$ on (s_*, β) and $f > 0$ on $(\beta, 1)$. In particular, f might be equal to zero on (α, β).

3.3 Monostable case

We now consider the case when $f > 0$ in $(0,1)$. We formulate sufficient conditions under which there exist positive solutions of (3.1) for a continuum of admissible values $c > 0$. We also present a nonexistence result, in which the behaviour of f near 0 plays a crucial role.

The following result generalizes that from [22, Proposition 2].

Theorem 3.13 (Existence)**.** *Let f be lower semicontinuous, $f(t) > 0$, $t \in (0,1)$, and*

$$0 < \mu := \sup_{t\in(0,1)} \frac{f(t)}{t^{p'-1}} < +\infty. \tag{3.25}$$

Then there exists a number $c^* \in (0, (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \mu^{\frac{1}{p'}}]$ such that the b.v.p. (3.1) has a unique positive solution if and only if $c \geq c^*$.

Proof. It follows from (3.25) that f is bounded in $(0,1)$. In particular, $f \in L^1(0,1)$. For a solution $y_c = y_c(t)$ of (3.3) with $c \geq 0$ we have

$$P_c 0 = 0 - h(t,0,c) = p' f(t) \geq 0 = y_c' - h(t, y_c, c) = P_c y_c \quad \text{a.e. in } [0,1].$$

Then by Lemma 3.5 we have $y_c \geq 0$ in $[0,1]$. The same argument as in the proof of Corollary 3.8 applied on the entire interval $(0,1)$ yields that $y_c > 0$ in $(0,1)$.

Next, using the estimates similar to [22, p. 176], we prove that $y_c(0) = 0$ provided c is "large enough". Set $\phi_c(s) = cs^{\frac{1}{p}} - s$, $c > 0$, $s \in (0, c^{p'})$. Then $\phi_c > 0$ in $(0, c^{p'})$, $\phi_c(0) = \phi_c(c^{p'}) = 0$, and ϕ_c attains maximum value $M_c := (\frac{c}{p})^{p'}(p-1)$ at the point $k := (\frac{c}{p})^{p'} \in (0, c^{p'})$. Elementary calculation yields that $c \geq (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \mu^{\frac{1}{p'}}$ if and only if $M_c \geq \mu$, i.e., $\phi_c(k) \geq \mu$, or equivalently we have

$$ck^{\frac{1}{p}} - \mu \geq k\,. \tag{3.26}$$

Let $s(t) := kt^{p'}$. Then $s(1) > 0$ and thanks to (3.26),

$$\begin{aligned} P_c s = s'(t) - h(t,s,c) = kp't^{p'-1} - h(t,s,c) &\le \left(ck^{\frac{1}{p}} - \mu\right) p't^{p'-1} - p'\left[c(s(t))^{\frac{1}{p}} - f(t)\right] \\ &\le \left(ck^{\frac{1}{p}} - \mu\right) p't^{p'-1} - p'\left[c(s(t))^{\frac{1}{p}} - \mu t^{p'-1}\right] = 0 = P_c y_c \quad \text{a.e. in} \quad [0,1]. \end{aligned}$$

Then again by Lemma 3.5 we have

$$0 \le y_c(t) \le s(t), \quad t \in [0,1].$$

In particular,

$$0 = y_c(0) = s(0).$$

To summarize, we have proved that for any fixed $c \ge (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \mu^{\frac{1}{p'}}$ there exists unique positive solution $y_c = y_c(t)$ of the backward i.v.p. (3.3) satisfying $y_c(0) = 0$. In particular, $y_c = y_c(t)$ is a unique positive solution of (3.1).

By Corollary 3.6, $y_{c_1}(t) \ge y_{c_2}(t)$, $t \in (0,1)$, $y_{c_i}(0) = y_{c_i}(1) = 0$, $i = 1,2$, if $c_1 < c_2$. Set

$$c^* := \inf\{c > 0 : \text{ (3.1) has a unique positive solution}\}.$$

Then from above we get $c^* \le (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \mu^{\frac{1}{p'}}$. Let $c_n \to c^*+$, $y_{c_n} = y_{c_n}(t)$, $t \in [0,1]$, be solutions of (3.1) with $c = c_n$. Then, according to Lemma 3.4, solutions y_{c_n} converge uniformly to a solution y_{c^*} of (3.1) with $c = c^*$. Since $c^* \ge 0$, we have $y_{c^*}(t) > 0$, $t \in (0,1)$ by Corollary 3.8 applied on the entire interval $(0,1)$. Hence (3.1) has a unique positive solution if and only if $c \ge c^*$. For $c = 0$ we have

$$y_0(t) = p' \int_t^1 f(\tau)\,\mathrm{d}\tau, \quad t \in [0,1].$$

In particular, $y_0(0) > 0$ and therefore $c^* > 0$. □

We also have the following non-existence result.

Theorem 3.14 (Non-existence). *Let $f(t) > 0$, $t \in (0,1)$,*

$$0 < \nu := \liminf_{t \to 0+} \frac{f(t)}{t^{p'-1}}\,. \tag{3.27}$$

If

$$0 \le c < (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \nu^{\frac{1}{p'}} \tag{3.28}$$

then the b.v.p. (3.1) *has no positive solution. In particular, if*

$$\lim_{t \to 0+} \frac{f(t)}{t^{p'-1}} = +\infty, \tag{3.29}$$

then (3.1) *has no positive solution for any $c \ge 0$.*

Proof. The case $c = 0$ is obvious, see the end of the proof of Theorem 3.13. To prove the rest of the statement, we proceed by contradiction. Let $c > 0$ be fixed and satisfy (3.28). Assume that (3.1) has a positive solution $y_c = y_c(t) > 0$, $t \in (0,1)$. Since y_c is also a solution of the backward i.v.p. (3.3) and $c > 0$, by Lemma 3.3 function y_c is a also unique solution of (3.1). For $v \in C[0,1]$ fixed let $u \in C[0,1]$ be such that

$$u(t) = p' \int_0^t \left[c \left(v^+(\tau) \right)^{\frac{1}{p}} - f(\tau) \right] \mathrm{d}\tau.$$

Then $u = T(v)$ defines a monotone increasing operator from $C[0,1]$ into $C[0,1]$ and y_c is a fixed point of T. Indeed, let $v_1, v_2 \in C[0,1]$, $v_1(t) \le v_2(t)$, $t \in [0,1]$. Then

$$T(v_1)(t) - T(v_2)(t) = p' \int_0^t c \left[\left(v_1^+(\tau) \right)^{\frac{1}{p}} - \left(v_2^+(\tau) \right)^{\frac{1}{p}} \right] \mathrm{d}\tau \le 0.$$

Set $y_0(t) = c^{p'} t^{p'}$, $t \in [0,1]$. Then

$$T(y_0)(t) = y_0(t) - p' \int_0^t f(\tau)\,\mathrm{d}\tau \le y_0(t), \quad t \in [0,1],$$

i.e., y_0 is a supersolution of T (see e.g. [13, Definition 6.3.15]). We consider the following successive approximations

$$y_{n+1} = T(y_n), \quad n = 0, 1, 2, \dots$$

Since T is monotone increasing, we have

$$y_0(t) \ge y_1(t) \ge \dots \ge y_n(t) \ge \dots \tag{3.30}$$

For any $n \in \mathbb{N}$,

$$y_n(t) = T(y_{n-1})(t) \ge -p' \int_0^t f(\tau)\,\mathrm{d}\tau,$$

i.e., the sequence $\{y_n\}_{n=0}^\infty$ is bounded below in $C[0,1]$. By [13, Theorem 6.3.16], this sequence converges to the greatest fixed point of T. Therefore,

$$y_0(t) \ge y_1(t) \ge \dots \ge y_n(t) \ge \dots \ge y_c(t) > 0, \quad t \in (0,1). \tag{3.31}$$

It follows from (3.27), (3.28) that there exists $\delta \in (0,1]$ and $\tilde{\nu} \in \left(\frac{1}{p' p^{p'-1}}, 1 \right)$ such that

$$f(t) \ge \tilde{\nu} c^{p'} t^{p'-1} \quad \text{for all} \quad t \in (0, \delta). \tag{3.32}$$

Now, using (3.32) we deduce

$$\begin{aligned}
y_1(t) &= p' \left[c \int_0^t \left(y_0^+(\tau) \right)^{\frac{1}{p}} \mathrm{d}\tau - \int_0^t f(\tau)\,\mathrm{d}\tau \right] = p' \left[c \int_0^t c^{\frac{p'}{p}} \tau^{\frac{p'}{p}} \mathrm{d}\tau - \int_0^t f(\tau)\,\mathrm{d}\tau \right] \\
&\le p' c^{p'} \left[\frac{\tau^{p'}}{p'} \right]_0^t - p' \int_0^t \tilde{\nu} c^{p'} \tau^{p'-1}\,\mathrm{d}\tau = c^{p'} t^{p'} - \tilde{\nu} c^{p'} t^{p'} = c t^{p'} (1 - \tilde{\nu}), \quad t \in (0, \delta),
\end{aligned}$$

$$\begin{aligned}
y_2(t) &= p' \left[c \int_0^t \left(y_1^+(\tau) \right)^{\frac{1}{p}} \mathrm{d}\tau - \int_0^t f(\tau)\,\mathrm{d}\tau \right] \\
&\le p' c \int_0^t c^{\frac{p'}{p}} \tau^{\frac{p'}{p}} (1 - \tilde{\nu})^{\frac{1}{p}} \mathrm{d}\tau - p' \int_0^t \tilde{\nu} c^{p'} \tau^{p'-1}\,\mathrm{d}\tau = c^{p'} t^{p'} (1 - \tilde{\nu})^{\frac{1}{p}} - \tilde{\nu} c^{p'} t^{p'} \\
&= c^{p'} t^{p'} \left[(1 - \tilde{\nu})^{\frac{1}{p}} - \tilde{\nu} \right], \quad t \in (0, \delta).
\end{aligned}$$

Performing the iterative process, we get for $k = 1, 2, \dots$ that

$$y_k(t) \leq a_k c^{p'} t^{p'} \quad \text{for} \quad t \in (0, \delta), \tag{3.33}$$

where

$$a_0 = 1, \ a_k = (a_{k-1})^{\frac{1}{p}} - \tilde{\nu}. \tag{3.34}$$

It follows from (3.31), (3.33) that

$$0 < y_c(t) \leq \cdots \leq a_k c^{p'} t^{p'} \leq a_{k-1} c^{p'} t^{p'} \leq \cdots \leq a_1 c^{p'} t^{p'} \leq c^{p'} t^{p'} \tag{3.35}$$

for $t \in (0, \delta)$. Hence $\{a_k\}_{k=1}^{\infty}$ is a bounded and monotone decreasing sequence and therefore there exists its finite limit $a_\infty := \lim_{k\to\infty} a_k$. Then obviously $a_\infty < 1$ and due to (3.35) we infer $a_\infty > 0$. Passing to the limit for $k \to \infty$ in (3.34), we get

$$a_\infty = a_\infty^{\frac{1}{p}} - \tilde{\nu}, \quad \text{i.e.,} \quad \tilde{\nu} = a_\infty^{\frac{1}{p}}(1 - a_\infty^{\frac{1}{p'}}).$$

Since the function $x \mapsto x^{\frac{1}{p}}(1 - x^{\frac{1}{p'}})$, $x \in (0,1)$, attains its maximum $\frac{1}{p' p^{p'-1}}$ at the point $x = \frac{1}{p^{p'}}$, we necessarily have $\tilde{\nu} \leq \frac{1}{p' p^{p'-1}}$, a contradiction with the fact $\tilde{\nu} \in \left(\frac{1}{p' p^{p'-1}}, 1\right)$. Therefore (3.1) cannot have a positive solution. In particular, if (3.29) holds then $\nu = +\infty$ and (3.28) yields that (3.1) has no positive solution for any $c \geq 0$. $\square$

Remark 3.15. Let μ and ν be defined as in Theorems 3.13 and 3.14, respectively. Then we conclude from the existence and nonexistence results above that the minimal value of the “critical” speed $c^* > 0$ must satisfy

$$(p')^{\frac{1}{p'}} p^{\frac{1}{p}} \nu^{\frac{1}{p'}} \leq c^* \leq (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \mu^{\frac{1}{p'}}.$$

Chapter 4

Existence and non-existence of travelling wave profiles

Combining the results from Chapters 2 and 3, we can now formulate existence and non-existence results for the generalized profile U satisfying the second-order boundary value problem

$$\begin{cases} \left(d(U(z))\,|U'(z)|^{p-2}\,U'(z)\right)' + cU'(z) + g(U(z)) = 0, \quad z \in \mathbb{R}, \\ \lim\limits_{z\to-\infty} U(z) = 1, \quad \lim\limits_{z\to+\infty} U(z) = 0 \end{cases} \tag{4.1}$$

in the bistable unbalanced, bistable balanced and monostable cases. We also summarize the properties of the profile U.

In each of the theorems below, we assume that the diffusion coefficient d and reaction term g satisfy the hypotheses **(H1)**, **(H2)**, respectively, introduced in Section 2.1.

4.1 Bistable case

Theorem 4.1 (Unbalanced case). *Assume that $g = g(s)$ satisfies*

$$g(0) = g(s_*) = g(1) = 0 \ \textit{ for some } \ s_* \in (0,1)$$
$$g(s) \le 0 \ \textit{ if } \ s \in (0, s_*), \quad g(s) > 0 \ \textit{ if } \ s \in (s_*, 1),$$

and let

$$\int_0^1 (d(s))^{\frac{1}{p-1}}\, g(s)\,\mathrm{d}s > 0. \tag{4.2}$$

Then there is a unique value of $c = c_* > 0$ such that the b.v.p. (4.1) has a unique nonincreasing solution $U = U(z)$, $z \in \mathbb{R}$. Moreover, U has the following properties:

(i) $U(0) = s_*$

(ii) *There exist $-\infty \le z_0 < 0 < z_1 \le +\infty$ such that $U(z) = 1$ for $z \in (-\infty, z_0]$, $U(z) = 0$ for $z \in [z_1, +\infty)$.*

(iii) *U is strictly decreasing in (z_0, z_1).*

(iv) *For $i = 1, 2, \dots, n$ let $\xi_i \in (z_0, z_1)$ be such that $U(\xi_i) = s_i$, $\xi_0 = z_0$, $\xi_{n+1} = z_1$. Then $U \in \widehat{C}^1(\mathbb{R})$,*

$$U|_{(\xi_i,\xi_{i+1})} \in C^1(\xi_i, \xi_{i+1}), \quad i = 0, 1, \dots, n$$

and the limits $U'(\xi_i-) \coloneqq \lim\limits_{z\to\xi_i-} U'(z)$ *and* $U'(\xi_i+) \coloneqq \lim\limits_{z\to\xi_i+} U'(z)$ *exist finite for all* $i = 1, 2, \dots, n$.

(v) *For any* $i = 1, 2, \dots, n$*, the following transition condition holds:*

$$|U'(\xi_i-)|^{p-2}\, U'(\xi_i-) \lim_{s\to s_i+} d(s) = |U'(\xi_i+)|^{p-2}\, U'(\xi_i+) \lim_{s\to s_i-} d(s).$$

(vi) U *satisfies*

$$\lim_{z\to z_0+} d(U(z))\, |U'(z)|^{p-2}\, U'(z) = \lim_{z\to z_1-} d(U(z))\, |U'(z)|^{p-2}\, U'(z) = 0.$$

Proof. The assumptions on d and g imply that $f(t) = (d(t))^{\frac{1}{p-1}} g(t)$ satisfies the hypotheses of Theorem 3.9, hence there exists a unique constant $c = c_* > 0$ such that the b.v.p. (3.1) possesses a unique positive solution $y_{c_*} = y_{c_*}(t)$ in the sense of Carathéodory. Since $f|_{(s_i,s_{i+1})} \in C(s_i, s_{i+1})$, $i = 0, 1, \dots, n$, the solution y also satisfies (2.17) pointwise. The proof then follows from Proposition 2.12 and Theorem 3.9. The properties of the profile U follow from the reasoning in Section 2.3. Without loss of generality, the solution can be normalized by $U(0) = s_*$, cf. Remark 2.8. □

Remark 4.2. Property (ii) in the above theorem indicates that the solution U may not actually attain the values 0 and 1, but if it does, then it must be constant outside of (z_0, z_1). Adopting the terminology from [34, Definition 1.1], we distinguish the following types of solutions, illustrated in Figure 4.1: *front-type* if $(z_0, z_1) = \mathbb{R}$; *sharp of type I* if $z_0 = -\infty$ and $z_1 \in \mathbb{R}$; *sharp of type II* if $z_0 \in \mathbb{R}$ and $z_1 = +\infty$; *sharp of type III* if $z_0, z_1 \in \mathbb{R}$.

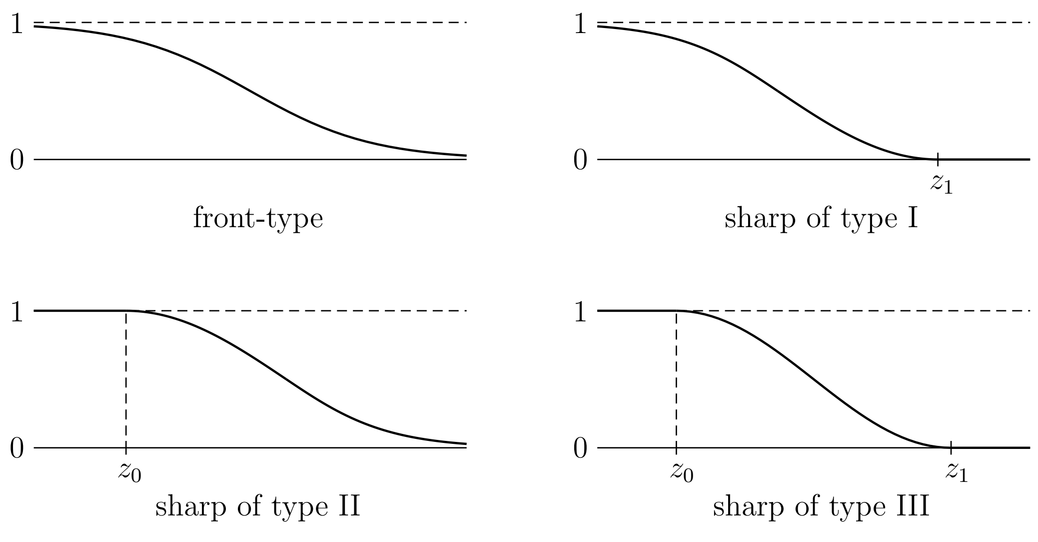

Figure 4.1: Classification of wave profiles based on the finiteness of z_0, z_1

Theorem 4.3 (Balanced case)**.** *Assume that* $g = g(s)$ *satisfies*

$$g(0) = g(s_*) = g(1) = 0 \;\; \textit{for some} \;\; s_* \in (0,1)$$
$$g(s) > 0 \;\; \textit{if} \;\; s \in (s_*, 1)$$

and there exists $\delta \in (0, s_*)$ such that

$$g(s) < 0 \ \text{ if } \ s \in (0, \delta), \quad g(s) \le 0 \ \text{ if } \ s \in (\delta, s_*).$$

Let

$$\int_0^1 (d(s))^{\frac{1}{p-1}}\, g(s)\, \mathrm{d}s = 0. \tag{4.3}$$

Then the b.v.p. (4.1) *has a unique nonincreasing solution $U = U(z)$, $z \in \mathbb{R}$, if and only if $c = 0$. Moreover, U has the properties* (i)–(vi) *from Theorem 4.1.*

Proof. The proof can be derived using the same reasoning as in the proof of Theorem 4.1, replacing the references to Theorem 3.9 with Theorem 3.10. □

Remark 4.4. Let us recall that solutions to (4.1) with $c = 0$ are stationary solutions of the partial differential equation (2.1), i.e., solutions of the form $u(x,t) = u(x)$, referred to as stationary waves. According to Theorem 3.10, condition (4.3) is sufficient for the existence of a monotone stationary wave solution, which is unique up to translation. On the other hand, if $c = 0$ in (4.1), it follows from Proposition 2.12 and Lemma 2.13 that this condition is also necessary. To summarize, each pair of the following statements implies the remaining one:

(a) $\int_0^1 (d(s))^{\frac{1}{p-1}}\, g(s)\, \mathrm{d}s = 0$;

(b) $c = 0$;

(c) there exists a unique (up to translation) nonincreasing solution of (4.1).

Remark 4.5 (Non-monotone solutions)**.** It follows from Lemma 2.13 and Remark 2.15 that the equation

$$\Big(d(u(x))\, |u'(x)|^{p-2}\, u'(x)\Big)' + g(u(x)) = 0, \quad x \in \mathbb{R} \tag{4.4}$$

possesses a pair of monotone stationary waves $u_1 = u_1(x)$, $u_2 = u_2(x)$, $x \in \mathbb{R}$, one nonincreasing and the other nondecreasing, which satisfy the boundary conditions

$$\lim_{x\to-\infty} u_1(x) = 1, \quad \lim_{x\to+\infty} u_1(x) = 0$$

and

$$\lim_{x\to-\infty} u_2(x) = 0, \quad \lim_{x\to+\infty} u_2(x) = 1,$$

respectively. If these waves are sharp of type III, we have many possibilities to "connect" 0 and 1 using suitable translations of u_1 and u_2. Indeed, let x_0, x_1 be associated with the nonincreasing solution u_1, $u_1'(x) < 0$ in (x_0, x_1), and assume that $x_0, x_1 \in \mathbb{R}$. Then $\widehat{x}_0$, $\widehat{x}_1$, associated with the nondecreasing solution u_2 must be also finite and we have $u_2'(x) > 0$ in $(\widehat{x}_0, \widehat{x}_1)$. Utilizing the translation invariance, we can normalize u_1 and u_2 so that $x_1 \le \widehat{x}_0$. Consequently, we can construct a solution $\widehat{u}$ as follows:

$$\widehat{u}(x) = u_1(x) \ \text{ for } \ (-\infty, x_1), \quad \widehat{u}(x) = 0 \ \text{ for } \ (x_1, \widehat{x}_0), \quad \widehat{u}(x) = u_2(x) \ \text{ for } \ (\widehat{x}_0, +\infty),$$

see Figure 4.2, which solves the equation (4.4) and satisfies boundary conditions

$$\lim_{x\to-\infty} \widehat{u}(x) = \lim_{x\to+\infty} \widehat{u}(x) = 1.$$

By the same reasoning, we can further extend $\widehat{u}$ to obtain a non-monotone solution $\widetilde{u}$ of the b.v.p. (4.1), see Figure 4.3.

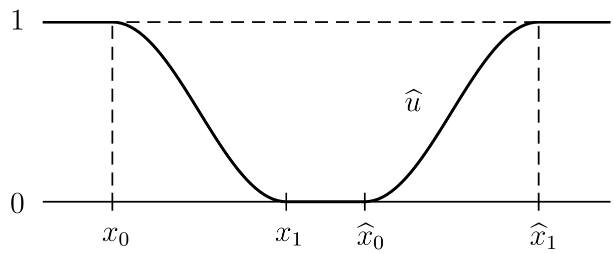

Figure 4.2: Non-monotone stationary wave $\widehat{u}$ satisfying $\widehat{u}(\pm\infty) = 1$

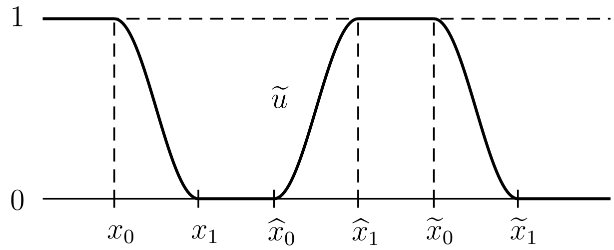

Figure 4.3: Non-monotone stationary wave $\widetilde{u}$ satisfying $\widetilde{u}(-\infty) = 1$, $\widetilde{u}(+\infty) = 0$

4.2 Monostable case

In this section we assume that $g = g(s)$ satisfies $g(0) = g(1) = 0$, $g(s) > 0$ for $s \in (0, 1)$.

Theorem 4.6 (Existence). *Let*

$$0 < \mu := \sup_{s \in (0,1)} \frac{(d(s))^{\frac{1}{p-1}} g(s)}{s^{p'-1}} < +\infty. \tag{4.5}$$

Then there exists a number $c^* \in (0, (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \mu^{\frac{1}{p'}}]$ such that the b.v.p. (4.1) *has a unique solution $U = U(z)$, $z \in \mathbb{R}$, if and only if $c \geq c^*$. Moreover, $U(0) = \frac{1}{2}$ and U has the properties* (ii)–(vi) *from Theorem 4.1.*

Proof. The assumptions on d and g imply that $f(t) = (d(t))^{\frac{1}{p-1}} g(t)$ satisfies the hypotheses of Theorem 3.13, hence the boundary value problem (3.1) has a unique positive solution in the sense of Carathéodory $y = y(t)$, $t \in [0, 1]$, if and only if $c \geq c^*$. Since $f|_{(s_i, s_{i+1})} \in C(s_i, s_{i+1})$, $i = 0, 1, \ldots, n$, the solution y also satisfies (2.17) pointwise. The proof then follows from Proposition 2.12 and Theorem 3.13. The properties of U, here normalized by $U(0) = \frac{1}{2}$, follow from Proposition 2.10 and the reasoning in Section 2.3. $\square$

Theorem 4.7 (Non-existence). *Let*

$$0 < \nu := \liminf_{s \to 0+} \frac{(d(s))^{\frac{1}{p-1}} g(s)}{s^{p'-1}}. \tag{4.6}$$

If

$$0 \leq c < (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \nu^{\frac{1}{p'}}$$

then there is no solution $U = U(z)$, $z \in \mathbb{R}$, of the b.v.p. (4.1)*. In particular,* (4.1) *has no solution for any $c \geq 0$ if*

$$\lim_{t \to 0+} \frac{(d(s))^{\frac{1}{p-1}} g(s)}{s^{p'-1}} = +\infty.$$

Proof. The proof follows from Proposition 2.12 and Theorem 3.14. $\square$

Chapter 5

Asymptotic analysis of the wave profile

With the existence and non-existence results established, we now turn our attention to investigating the behaviour of solutions near 0 and 1.

Let us recall that the reduction to a first-order b.v.p., discussed in Section 2.3, relies on the assumption that the profile U satisfying the b.v.p. (4.1) is strictly monotone whenever $0 < U(z) < 1$. More precisely, we assumed that $U = U(z)$ is nonincreasing on $\mathbb{R}$ and strictly decreasing in the open interval $(z_0, z_1) = \{z \in \mathbb{R} : 0 < U(z) < 1\}$. This property is granted in the monostable case, cf. Proposition 2.10.

Naturally, we are interested in whether the profile attains one, both or neither of the values 0 and 1, and how this depends on the properties of the reaction g and the diffusivity d. For instance, in the case of $p = 2$ and monostable reaction, degeneration of the diffusion coefficient d at 0 might cause the appearance of a sharp profile, which reaches 0 in a finite $z \in \mathbb{R}$ with a negative slope. This phenomenon also depends on the derivative of d at 0 and it only concerns the profile associated with the minimal wave speed c^*.

Due to our general assumptions on the functions d and g, a detailed discussion that directly correlates the finiteness of z_0 and z_1 to specific properties of d or g, such as degenerations, singularities or the speed of vanishing, is not feasible. We will show below that the outcome is determined by the combined influence of the diffusion and reaction terms as well as the value of p. For technical reasons, we will assume power-type behaviour of d and g near the equilibria 0 and 1.

We proceed by examining the inverse function to the profile $U = U(z)$, $U(0) = \frac{1}{2}$:

$$z(U) = -\int_{\frac{1}{2}}^{U} \frac{(d(t))^{\frac{1}{p-1}}}{(y_c(t))^{\frac{1}{p}}}\,\mathrm{d}t, \quad U \in (0,1), \tag{5.1}$$

as $U \to 1$ and $U \to 0$, respectively. In particular, we have

$$z_0 = -\int_{\frac{1}{2}}^{1} \frac{(d(t))^{\frac{1}{p-1}}}{(y_c(t))^{\frac{1}{p}}}\,\mathrm{d}t \quad \text{and} \quad z_1 = \int_{0}^{\frac{1}{2}} \frac{(d(t))^{\frac{1}{p-1}}}{(y_c(t))^{\frac{1}{p}}}\,\mathrm{d}t, \tag{5.2}$$

where y_c is the positive solution of the equivalent first-order problem (3.1). In the case of bistable reaction, the function $z = z(U)$ is customarily normalized by $z(s_*) = 0$ instead of $z(\frac{1}{2}) = 0$. This modification will be implemented automatically.

We focus separately on the bistable balanced, bistable unbalanced and monostable case. Where possible, we also discuss the one-sided derivatives $U'(z_0+)$, $U'(z_1-)$, provided $z_0, z_1 \in \mathbb{R}$, to obtain information about the smoothness of the solution at these "transition" points. Throughout this chapter, we consider d and g with properties **(H1)** and **(H2)**, respectively, introduced in Chapter 2. For the sake of notational simplicity, in what follows we will write

$$h_1(t) \sim h_2(t) \text{ as } t \to t_0 \in \mathbb{R} \quad \text{if and only if} \quad \lim_{t \to t_0} \frac{h_1(t)}{h_2(t)} \in (0, +\infty).$$

5.1 Bistable balanced case

Let us start with the asymptotic analysis of solutions in the bistable balanced case. Assume that d and g satisfy the hypotheses from Theorem 4.3 and

$$\int_0^1 (d(s))^{\frac{1}{p-1}} g(s) \, \mathrm{d}s = 0.$$

As discussed in Remark 4.4, the above condition is both necessary and sufficient for the existence of monotone stationary waves $u = u(x)$, characterized by zero wave speed c, which satisfy

$$\begin{cases} \left(d(u(x)) \, |u'(x)|^{p-2} \, u'(x)\right)' + g(u(x)) = 0, & x \in \mathbb{R}, \\ \lim\limits_{x \to -\infty} u(x) = 1, \quad \lim\limits_{x \to +\infty} u(x) = 0. \end{cases}$$

Due to $c = 0$, the unique solution $y = y(t)$ of the first-order b.v.p. (3.1) can be obtained via direct integration:

$$y(t) = -p' \int_0^t f(s) \, \mathrm{d}s, \quad t \in [0, 1]. \tag{5.3}$$

It then follows from (5.1) and (5.3) that the inverse function to the profile $u = u(x)$, $u(0) = s_*$, is given by

$$x(u) = -\left(\frac{1}{p'}\right)^{\frac{1}{p}} \int_{s_*}^{u} \frac{(d(s))^{\frac{1}{p-1}}}{\left(-\int_0^s (d(\sigma))^{\frac{1}{p-1}} g(\sigma) \, \mathrm{d}\sigma\right)^{1/p}} \, \mathrm{d}s,$$

with x_0 and x_1 now denoting the corresponding expressions in (5.2).

Assuming power-type behaviour of the reaction and diffusion terms, the asymptotic analysis of $x = x(u)$ as $u \to 0+$ yields the following result for the stationary wave $u = u(x)$ as $x \to +\infty$.

Theorem 5.1. *Let $\alpha > 0$, $\beta \in \mathbb{R}$ and $g(s) \sim (-s^\alpha)$, $d(s) \sim s^\beta$ as $s \to 0+$. Assume that*

$$\alpha + \frac{\beta}{p-1} > -1.$$

(i) *If $\alpha - \beta \geq p - 1$ then $x_1 = +\infty$. Moreover, for $\alpha - \beta = p - 1$ we have*

$$u(x) \sim \mathrm{e}^{-x} \to 0+ \qquad \textit{for } x \to +\infty$$

and for $\alpha - \beta > p - 1$ we have

$$u(x) \sim x^{\frac{p}{p-1-(\alpha-\beta)}} \to 0+ \qquad \textit{for } x \to +\infty.$$

(ii) *If $\alpha-\beta<p-1$ then $x_1<+\infty$ and for $x\to x_1-$ we have*

$$u(x)\sim(x_1-x)^{\frac{p}{p-1-(\alpha-\beta)}}.$$

As for the derivatives, we then have

(a) $\left.\dfrac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_1-}\sim-(x_1-x)^{\frac{\alpha-\beta+1}{p-1-(\alpha-\beta)}}\to 0$ *for* $x\to x_1-$ *if* $\alpha-\beta>-1$,

(b) $\left.\dfrac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_1-}\sim-(x_1-x)^{0}\to k<0$ *for* $x\to x_1-$ *if* $\alpha-\beta=-1$,

(c) $\left.\dfrac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_1+}\sim-(x_1-x)^{\frac{\alpha-\beta+1}{p-1-(\alpha-\beta)}}\to-\infty$ *for* $x\to x_1-$ *if* $\alpha-\beta<-1$.

Proof. Let $g(s)\sim(-s)^\alpha$, $d(s)\sim s^\beta$ as $s\to 0+$ for some $\alpha>0$, $\beta\in\mathbb{R}$. Then

$$f(s)=(d(s))^{\frac{1}{p-1}}g(s)\sim-s^{\alpha+\frac{\beta}{p-1}}\quad\text{as } s\to 0+$$

and our assumption $f\in L^1(0,1)$ implies that the parameters α, β and p must satisfy

$$\alpha+\frac{\beta}{p-1}>-1.$$

Since

$$-\int_0^s(d(\sigma))^{\frac{1}{p-1}}g(\sigma)\,\mathrm{d}\sigma\sim\int_0^s\sigma^{\alpha+\frac{\beta}{p-1}}\,\mathrm{d}\sigma\sim s^{\alpha+\frac{\beta}{p-1}+1}\quad\text{as } s\to 0+,$$

for $u\to 0+$ we can write

$$x(u)\sim-\int_{s_*}^{u}\frac{s^{\frac{\beta}{p-1}}}{\left(s^{\alpha+\frac{\beta}{p-1}+1}\right)^{1/p}}\,\mathrm{d}s=-\int_{s_*}^{u}s^{\frac{\beta}{p-1}-\frac{\alpha}{p}-\frac{\beta}{p(p-1)}-\frac{1}{p}}\,\mathrm{d}s=\int_u^{s_*}s^{\frac{\beta-\alpha-1}{p}}\,\mathrm{d}s. \tag{5.4}$$

From (5.4), we derive the fundamental distinction between two qualitatively different cases:

(i) If $\alpha-\beta\geq p-1$ then $x_1=+\infty$.

(ii) If $\alpha-\beta<p-1$ then $x_1<+\infty$.

Moreover, (5.4) provides additional insight into the asymptotic behaviour of $x=x(u)$, and hence $u=u(x)$, in both of these cases.
Case (i). Let $\alpha-\beta=p-1$. Then (5.4) implies that

$$x(u)\sim-\ln u\quad\text{as } u\to 0+$$

and therefore

$$u(x)\sim\mathrm{e}^{-x}\to 0+\quad\text{for } x\to+\infty.$$

On the other hand, if $\alpha-\beta>p-1$, we have

$$x(u)\sim u^{\frac{\beta-\alpha-1}{p}+1}=u^{\frac{p-1-(\alpha-\beta)}{p}}\quad\text{as } u\to 0+$$

and hence

$$u(x) \sim x^{\frac{p}{p-1-(\alpha-\beta)}} \to 0+ \qquad \text{for } x \to +\infty.$$

Case (ii). Let $\alpha - \beta < p - 1$. Then from (5.4) we conclude

$$x_1 - x(u) \sim u^{\frac{p-1-(\alpha-\beta)}{p}} \qquad \text{as } u \to 0+.$$

An inverse point of view then yields

$$u(x) \sim (x_1 - x)^{\frac{p}{p-1-(\alpha-\beta)}} \qquad \text{for } x \to x_1-.$$

Since $\frac{p}{p-1-(\alpha-\beta)} > 0$, we have

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_1-} \sim -(x_1 - x)^{\frac{\alpha-\beta+1}{p-1-(\alpha-\beta)}}$$

and the cases (a), (b) and (c) follow immediately. $\square$

Notice that apart from determining conditions for the parameters α, β that guarantee $x_1 = +\infty$, we are also able to distinguish whether the solution approaches 0 exponentially or at a power rate. If $x_1 < +\infty$, we obtain classification of profiles based on the one-sided derivative $u'(x_1-)$. Since $u'(x_1+) = 0$, the profile u is differentiable at x_1 only in case (a). In the other two cases (b) and (c), it reaches 0 with a negative slope (finite or infinite). The behaviour of u at x_1 is illustrated in Figure 5.1. The colours correspond to those used to depict the sets $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ (see Remark 5.2 below) in Figures 5.2, 5.3.

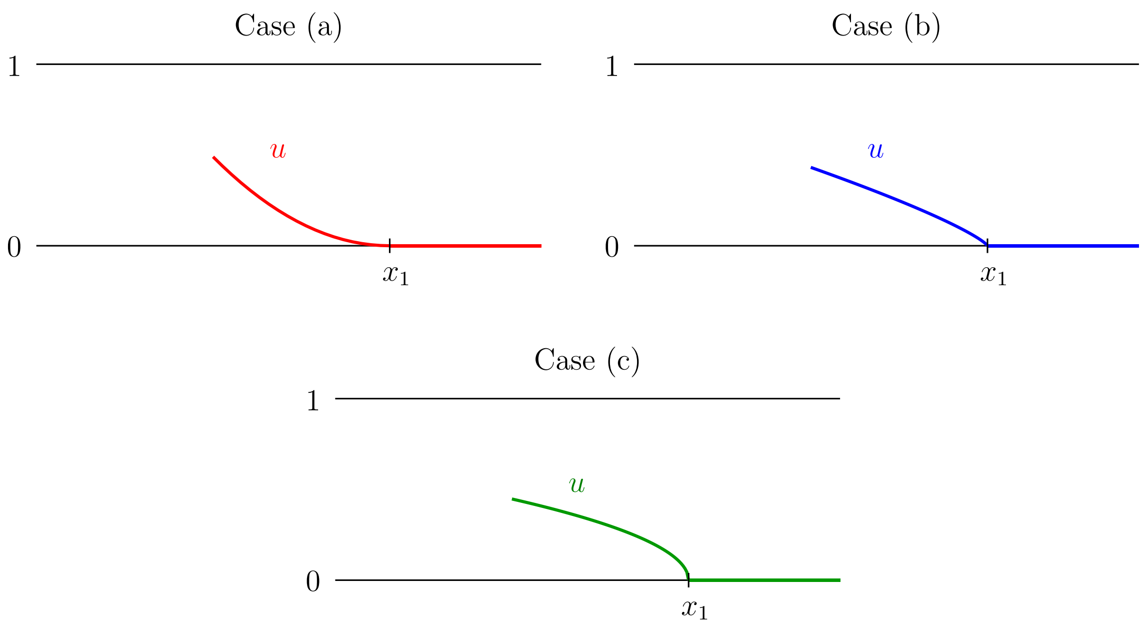

Figure 5.1: Behaviour of $u = u(x)$ at $x_1 \in \mathbb{R}$

Remark 5.2. To visualize conditions from Theorem 5.1, we introduce the sets

$$\mathcal{A}_1 := \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha > 0,\ \alpha - \beta = p - 1\},$$
$$\mathcal{A}_2 := \left\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha > 0,\ \alpha + \frac{\beta}{p-1} > -1,\ \alpha - \beta > p - 1\right\},$$

corresponding to case (i), in which $x_1 = +\infty$, and

$$\mathcal{B}_1 \coloneqq \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha > 0,\ -1 < \alpha - \beta < p - 1\},$$
$$\mathcal{B}_2 \coloneqq \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha > 0,\ \alpha - \beta = -1\},$$
$$\mathcal{B}_3 \coloneqq \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha > 0,\ \alpha - \beta < -1\},$$

corresponding to case (ii), in which $x_1 < +\infty$. For different values of p, these sets are depicted in Figures 5.2, 5.3.

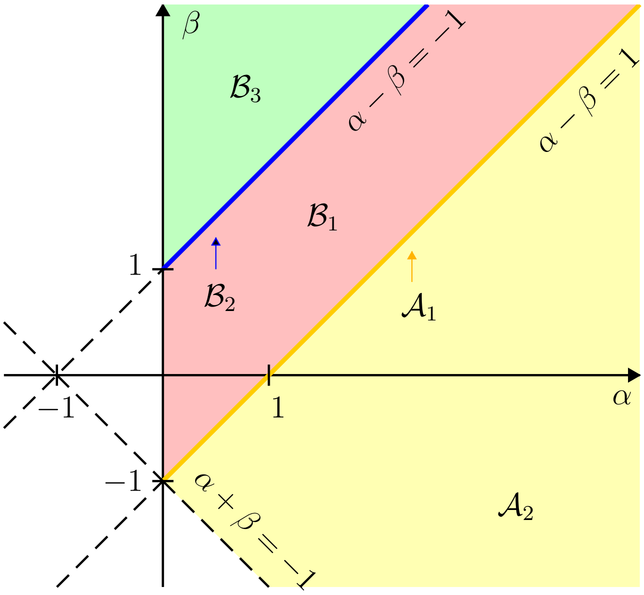

Figure 5.2: Visualization of the sets $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ for $p = 2$

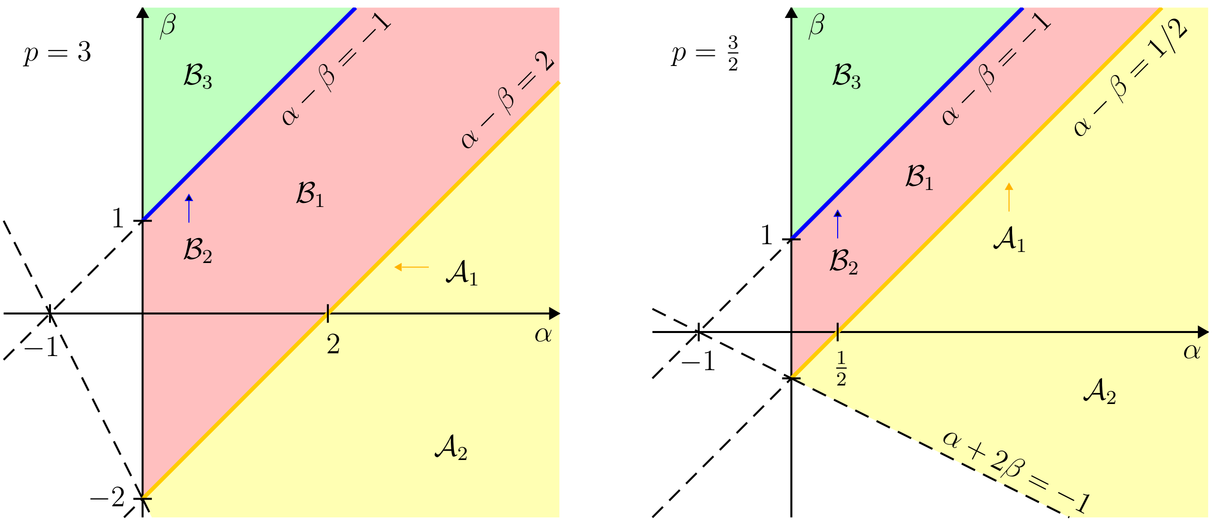

Figure 5.3: Geometric interpretation for values $p \neq 2$

We observe that both $x_1 = +\infty$ and $x_1 < +\infty$ occur for singular as well as degenerate diffusion coefficient d. In other words, the behaviour of d alone cannot be linked to a specific type of solution. This is due to the fact that the speed of vanishing of g at 0 also plays an important role. However, the smaller the value of p, the less likely becomes the appearance of sharp solutions if d has a singularity at 0, i.e., when $\beta < 0$.

Proceeding similarly as above, we derive the following result concerning the asymptotic behaviour of $u = u(x)$ near 1, i.e., for $x \to -\infty$.

Theorem 5.3. *Let $\gamma > 0$, $\delta \in \mathbb{R}$ and $g(s) \sim (1-s)^\gamma$, $d(s) \sim (1-s)^\delta$ as $s \to 1-$. Assume that*

$$\gamma + \frac{\delta}{p-1} > -1.$$

(i) *If $\gamma - \delta \geq p-1$ then $x_0 = -\infty$. Moreover, for $\gamma - \delta = p-1$ we have*

$$u(x) \sim 1 - \mathrm{e}^x \to 1- \qquad \text{for } x \to -\infty$$

and for $\gamma - \delta > p-1$ we have

$$u(x) \sim 1 - |x|^{\frac{p}{p-1-(\gamma-\delta)}} \to 1- \qquad \text{for } x \to -\infty.$$

(ii) *If $\gamma - \delta < p-1$ then $x_0 > -\infty$ and for $x \to x_0+$ we have*

$$u(x) \sim 1 - (x-x_0)^{\frac{p}{p-1-(\gamma-\delta)}}.$$

As for the derivatives, we then have

(a) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_0+} \sim -(x-x_0)^{\frac{\gamma-\delta+1}{p-1-(\gamma-\delta)}} \to 0 \quad \text{for } x \to x_0+ \qquad \text{if } \gamma-\delta > -1,$$

(b) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_0+} \sim -(x-x_0)^{0} \to k < 0 \quad \text{for } x \to x_0+ \qquad \text{if } \gamma-\delta = -1,$$

(c) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_0+} \sim -(x-x_0)^{\frac{\gamma-\delta+1}{p-1-(\gamma-\delta)}} \to -\infty \quad \text{for } x \to x_0+ \qquad \text{if } \gamma-\delta < -1.$$

Proof. The proof mirrors that of Theorem 5.1, now focusing on the behaviour of the inverse function $x = x(u)$ as $u \to 1-$. In particular, since

$$x(u) \sim -\int_{s_*}^{u} (1-s)^{\frac{\delta-\gamma-1}{p}}\,\mathrm{d}s \qquad \text{as } u \to 1-,$$

we derive the same conditions for γ and δ as previously for α and β, distinguishing between the cases $x_0 = -\infty$ and $x_0 > -\infty$. The asymptotic properties of $x = x(u)$, and consequently $u = u(x)$, follow again from elementary calculations. □

Remark 5.4. Visualizing the conditions from Theorem 5.3 in the (γ, δ)-plane yields the same geometric interpretation as in Figures 5.2, 5.3. In [17] we performed the asymptotic analysis for nondecreasing solutions, obtaining the same conditions for α, β, γ, δ and p as in Theorems 5.1 and 5.3.

Remark 5.5 (Classification of stationary waves)**.** Combining the results from Theorems 5.1 and 5.3, we arrive at the following classification of solutions, using the terminology from Remark 4.2. For the reader's convenience, below we include Figure 5.4 illustrating the basic characteristics (without taking into account the derivatives at z_0 and z_1). The stationary wave $u = u(x)$ is

front-type	if $\alpha-\beta \geq p-1$, $\gamma-\delta \geq p-1$;
sharp of type I	if $\alpha-\beta < p-1$, $\gamma-\delta \geq p-1$;
sharp of type II	if $\alpha-\beta \geq p-1$, $\gamma-\delta < p-1$;
sharp of type III	if $\alpha-\beta < p-1$, $\gamma-\delta < p-1$.

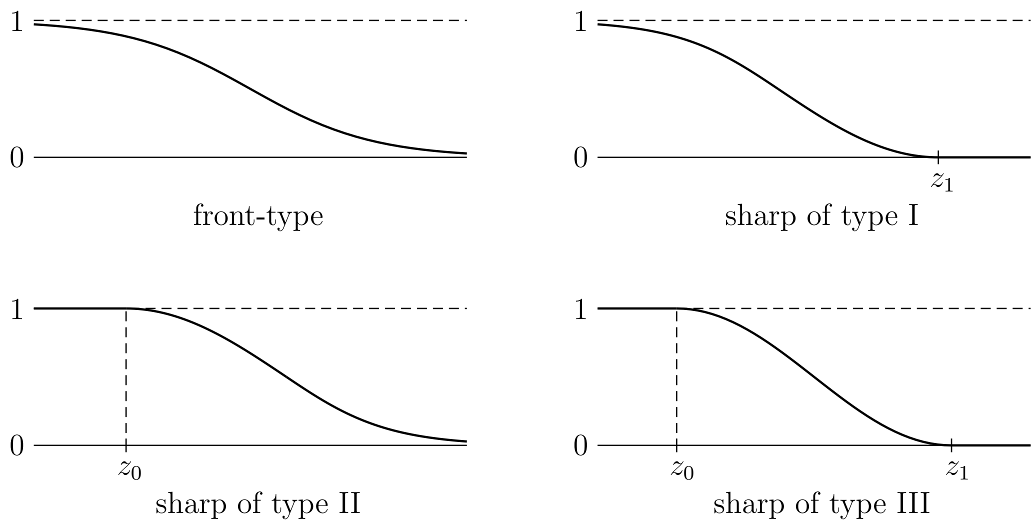

Figure 5.4: Classification of wave profiles from Remark 4.2

5.2 Bistable unbalanced case

Let us now assume that d and g satisfy the hypotheses of Theorem 4.1. In particular, we consider $g \in C[0,1]$ such that

$$g(0) = g(s_*) = g(1) = 0 \text{ for some } s_* \in (0,1)$$
$$g(s) \leq 0 \text{ if } s \in (0, s_*), \quad g(s) > 0 \text{ if } s \in (s_*, 1).$$

If (4.2) holds, then there exists a unique $c_* > 0$ such that the b.v.p. (4.1) possesses a unique monotone solution $U = U(z)$, $U(0) = s_*$ with $U'(z) < 0$ in (z_0, z_1).

In order to perform the asymptotic analysis of

$$z(U) = -\int_{s_*}^{U} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \,\mathrm{d}t, \quad U \in (0,1), \tag{5.5}$$

as $U \to 0+$ and $U \to 1-$, we first need to examine the behaviour of the unique solution $y_{c_*} = y_{c_*}(t)$ of the b.v.p. (3.1) as $t \to 0+$ and $t \to 1-$. Because of this, the analysis becomes more involved and less precise than in the stationary case $c = 0$, where we obtained the solution in a closed form. Our method consists in finding suitable upper and lower solutions of the initial value problems (3.2) and (3.3), which we use to estimate the values z_0 and z_1 with respect to $\pm\infty$.

5.2.1 Asymptotics near 1

For the purposes of this section, we first formulate a particular version of Lemma 3.5, which is essential for the upcoming proofs. We recall the notion of defect $P_c\varphi$ of an absolutely continuous function φ with respect to the differential equation in (3.1):

$$P_c\varphi = \varphi'(t) - p'\left[c(\varphi^+(t))^{\frac{1}{p}} - f(t)\right].$$

Lemma 5.6. *Let $y_{c_*} = y_{c_*}(t)$ be the solution of* (3.1) *with $c = c_*$ and consider a function $\varphi(t) = \kappa(1-t)^{\omega}$, $\kappa > 0$, $\omega > 0$.*

(i) *If* $P_{c_*}\varphi \geq 0$ *a.e. in* $[\varrho, 1]$, $0 \leq \varrho < 1$, *then* $\varphi \leq y_{c_*}$ *in* $[\varrho, 1]$.

(ii) *If* $P_{c_*}\varphi \leq 0$ *a.e. in* $[\varrho, 1]$, $0 \leq \varrho < 1$, *then* $\varphi \geq y_{c_*}$ *in* $[\varrho, 1]$.

Proof. Observe that both y_{c_*} and φ are absolutely continuous on $[0, 1]$ and recall that $P_{c_*}y_{c_*} = 0$. Since $\varphi(1) = y_{c_*}(1) = 0$, the statements (i) and (ii) follow immediately from Lemma 3.5. □

Assuming power-type behavior of the reaction and diffusion term, we obtain the following results.

Theorem 5.7. *Let* $\gamma > 0$, $\delta \in \mathbb{R}$ *and* $g(t) \sim (1-t)^\gamma$, $d(t) \sim (1-t)^\delta$ *as* $t \to 1-$. *Assume that*

$$-1 < \gamma + \frac{\delta}{p-1} \leq \frac{1}{p-1}\,. \tag{5.6}$$

If

$$\frac{\gamma - \delta + 1}{p} < 1$$

then $z_0 > -\infty$. *If*

$$\frac{\gamma - \delta + 1}{p} \geq 1$$

then $z_0 = -\infty$.

Proof. Set $f(t) = (d(t))^{\frac{1}{p-1}} g(t)$. Then $f(t) \sim (1-t)^{\gamma + \frac{\delta}{p-1}}$ as $t \to 1-$ and since $f \in L^1(0, 1)$, we have the following necessary condition for the parameters γ, δ and p:

$$\gamma + \frac{\delta}{p-1} > -1,$$

i.e., the first inequality in (5.6).

Due to our assumptions on d and g, there exists $\theta > 0$ (small enough) such that both d and g are continuous in $(1-\theta, 1)$. Therefore, $f = f(t)$ is also continuous in $(1-\theta, 1)$ and hence $f(t) \sim (1-t)^{\gamma + \frac{\delta}{p-1}}$ is equivalent to

$$f(t) = \eta(t)(1-t)^{\gamma + \frac{1}{p-1}}, \quad t \in (1-\theta, 1),$$

where $\eta = \eta(t)$ is a continuous function in $(1-\theta, 1)$ with $\lim_{t\to 1-} \eta(t) \in (0, +\infty)$.

Let $-1 < \gamma + \frac{\delta}{p-1} \leq \frac{1}{p-1}$. For $\kappa > 0$ we set

$$y_\kappa(t) = \kappa(1-t)^{\gamma + \frac{\delta}{p-1} + 1}, \quad t \in [1-\theta, 1].$$

Clearly $\gamma + \frac{\delta}{p-1} + 1 > 0$ and hence $y_\kappa(1) = 0$. Next we calculate the defect $P_{c_*}y_\kappa$:

$$\begin{aligned}
P_{c_*}y_\kappa &= y_\kappa' - p'\left[c_* (y_\kappa)^{\frac{1}{p}} - f(t)\right] \\
&= -\kappa\left(\gamma + \frac{\delta}{p-1} + 1\right)(1-t)^{\gamma + \frac{\delta}{p-1}} \\
&\qquad - p'\left[c_*\kappa^{\frac{1}{p}}(1-t)^{\frac{\gamma + \frac{\delta}{p-1} + 1}{p}} - \eta(t)(1-t)^{\gamma + \frac{\delta}{p-1}}\right] \\
&= (1-t)^{\gamma + \frac{\delta}{p-1}}\left[-\kappa\left(\gamma + \frac{\delta}{p-1} + 1\right) + p'\eta(t)\right] - (1-t)^{\frac{\gamma + \frac{\delta}{p-1} + 1}{p}} p' c_* \kappa^{\frac{1}{p}},
\end{aligned} \tag{5.7}$$

$t \in (1-\theta, 1)$. Our assumption $\gamma + \frac{\delta}{p-1} \leq \frac{1}{p-1}$ implies

$$\gamma + \frac{\delta}{p-1} \leq \frac{\gamma + \frac{\delta}{p-1} + 1}{p},$$

and therefore the power $(1-t)^{\gamma+\frac{\delta}{p-1}}$ dominates the power $(1-t)^{\frac{\gamma+\frac{\delta}{p-1}+1}{p}}$ near 1. It then follows from (5.7) that we may distinguish between two cases:

(i) There exists $\underline{\kappa} \ll 1$ so small that $P_{c_*} y_{\underline{\kappa}} \geq 0$ a.e. in $[1-\theta, 1]$.

(ii) There exists $\overline{\kappa} \gg 1$ so large that $P_{c_*} y_{\overline{\kappa}} \leq 0$ a.e. in $[1-\theta, 1]$.

Case (i). Let $\frac{\gamma-\delta+1}{p} < 1$. It follows from Lemma 5.6 with $\varrho = 1-\theta$, $\varphi(t) = y_{\underline{\kappa}}(t)$ that

$$y_{c_*}(t) \geq y_{\underline{\kappa}}(t), \quad \text{in } [1-\theta, 1]. \tag{5.8}$$

From (5.5) and (5.8) we conclude that there exists $c_1 > 0$ such that

$$\begin{aligned}
z_0 = & -\int_{s_*}^{1} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t \geq -\int_{s_*}^{1-\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t - \int_{1-\theta}^{1} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{\underline{\kappa}}(t))^{\frac{1}{p}}}\,\mathrm{d}t \\
\geq & -\int_{s_*}^{1-\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t - c_1 \int_{1-\theta}^{1} \frac{(1-t)^{\frac{\delta}{p-1}}}{(1-t)^{\frac{\gamma+\frac{\delta}{p-1}+1}{p}}}\,\mathrm{d}t \\
= & -\int_{s_*}^{1-\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t - c_1 \int_{1-\theta}^{1} \frac{\mathrm{d}t}{(1-t)^{\frac{\gamma-\delta+1}{p}}} > -\infty.
\end{aligned}$$

Case (ii). Let $\frac{\gamma-\delta+1}{p} \geq 1$. It follows from Lemma 5.6 with $\varrho = 1-\theta$, $\varphi(t) = y_{\overline{\kappa}}(t)$ that

$$y_{c_*}(t) \leq y_{\overline{\kappa}}(t) \quad \text{in } [1-\theta, 1]. \tag{5.9}$$

From (5.5) and (5.9) we conclude that there exists $c_2 > 0$ such that

$$\begin{aligned}
z_0 = & -\int_{s_*}^{1} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t \leq -\int_{s_*}^{1-\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t - \int_{1-\theta}^{1} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{\overline{\kappa}}(t))^{\frac{1}{p}}}\,\mathrm{d}t \\
\leq & -\int_{s_*}^{1-\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t - c_2 \int_{1-\theta}^{1} \frac{(1-t)^{\frac{\delta}{p-1}}}{(1-t)^{\frac{\gamma+\frac{\delta}{p-1}+1}{p}}}\,\mathrm{d}t \\
= & -\int_{s_*}^{1-\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t - c_2 \int_{1-\theta}^{1} \frac{\mathrm{d}t}{(1-t)^{\frac{\gamma-\delta+1}{p}}} = -\infty.
\end{aligned}$$

□

Theorem 5.8. *Let $\gamma > 0$, $\delta \in \mathbb{R}$ and $g(t) \sim (1-t)^{\gamma}$, $d(t) \sim (1-t)^{\delta}$ as $t \to 1-$. Assume that*

$$\gamma + \frac{\delta}{p-1} > \frac{1}{p-1}. \tag{5.10}$$

If $\gamma < 1$ then $z_0 > -\infty$. If $\gamma \geq 1$ then $z_0 = -\infty$.

Proof. We proceed similarly as in the proof of Theorem 5.7. In particular, thanks to our assumptions on d and g we have

$$f(t) = (d(t))^{\frac{1}{p-1}} g(t) = \eta(t)(1-t)^{\gamma+\frac{1}{p-1}}, \quad t \in (1-\theta, 1),$$

where $\eta = \eta(t)$ is a continuous function in $(1-\theta,1)$ with $\lim_{t\to 1-}\eta(t) \in (0,+\infty)$.

Let $\gamma + \frac{\delta}{p-1} > \frac{1}{p-1}$. For $\kappa > 0$ we set

$$y_\kappa(t) = \kappa(1-t)^{p\left(\gamma+\frac{\delta}{p-1}\right)}, \quad t \in [1-\theta, 1].$$

Clearly $p\left(\gamma+\frac{\delta}{p-1}\right) > 0$ and hence $y_\kappa(1) = 0$. As for the defect $P_{c_*}y_\kappa$, we now have

$$\begin{aligned} P_{c_*}y_\kappa &= y_\kappa' - p'\left[c_*\left(y_\kappa\right)^{\frac{1}{p}} - f(t)\right] \\ &= -\kappa p\left(\gamma+\frac{\delta}{p-1}\right)(1-t)^{p\left(\gamma+\frac{\delta}{p-1}\right)-1} \\ &\quad - p'\left[c_*\kappa^{\frac{1}{p}}(1-t)^{\gamma+\frac{\delta}{p-1}} - \eta(t)(1-t)^{\gamma+\frac{\delta}{p-1}}\right] \\ &= -\kappa p\left(\gamma+\frac{\delta}{p-1}\right)(1-t)^{p\left(\gamma+\frac{\delta}{p-1}\right)-1} - p'\left[c_*\kappa^{\frac{1}{p}} - \eta(t)\right](1-t)^{\gamma+\frac{\delta}{p-1}}, \end{aligned} \tag{5.11}$$

$t \in (1-\theta,1)$. Our assumption $\gamma(p-1)+\delta > 1$ implies $\gamma+\frac{\delta}{p-1} < p\left(\gamma+\frac{\delta}{p-1}\right) - 1$ and now the power $(1-t)t^{\gamma+\frac{\delta}{p-1}}$ dominates the power $(1-t)^{p\left(\gamma+\frac{\delta}{p-1}\right)-1}$ near 1. It follows from (5.11) that we may distinguish between two cases:

(i) There exists $\underline{\kappa} \ll 1$ so small that $P_{c_*}y_{\underline{\kappa}} \geq 0$ a.e. in $[1-\theta,1]$.

(ii) There exists $\overline{\kappa} \gg 1$ so large that $P_{c_*}y_{\overline{\kappa}} \leq 0$ a.e. in $[1-\theta,1]$.

<u>Case (i)</u>. Let $\gamma < 1$. It follows from Lemma 5.6 with $\varrho = 1-\theta$, $\varphi(t) = y_{\underline{\kappa}}(t)$ that

$$y_{c_*}(t) \geq y_{\underline{\kappa}}(t), \quad \text{in } [1-\theta,1],$$

and we conclude that there exists a constant $c_3 > 0$ such that

$$\begin{aligned} z_0 &= -\int_{s_*}^1 \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t \geq -\int_{s_*}^{1-\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t - c_3\int_{1-\theta}^1 \frac{(1-t)^{\frac{\delta}{p-1}}}{(1-t)^{\gamma+\frac{\delta}{p-1}}}\,\mathrm{d}t \\ &= -\int_{s_*}^{1-\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t - c_3\int_{1-\theta}^1 \frac{\mathrm{d}t}{(1-t)^\gamma} > -\infty. \end{aligned}$$

<u>Case (ii)</u>. Let $\gamma \geq 1$. From Lemma 5.6 with $\varrho = 1-\theta$, $\varphi(t) = y_{\underline{\kappa}}(t)$ we obtain

$$y_{c_*}(t) \leq y_{\overline{\kappa}}(t) \quad \text{in } [1-\theta,1].$$

Hence there exists $c_4 > 0$ such that

$$\begin{aligned} z_0 &\leq -\int_{s_*}^{1-\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t - \int_{1-\theta}^1 \frac{(d(t))^{\frac{1}{p-1}}}{(y_{\overline{\kappa}}(t))^{\frac{1}{p}}}\,\mathrm{d}t \\ &\leq -\int_{s_*}^{1-\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t - c_4\int_{1-\theta}^1 \frac{\mathrm{d}t}{(1-t)^\gamma} = -\infty. \end{aligned}$$

$\square$

Remark 5.9. To visualize conditions from Theorems 5.7 and 5.8, we introduce the sets

$$\mathcal{A}_1^1 := \left\{(\gamma,\delta)\in\mathbb{R}^2 : \gamma>0, -1<\gamma+\frac{\delta}{p-1}\le\frac{1}{p-1}, \gamma-\delta+1\ge p\right\},$$
$$\mathcal{B}_1^1 := \left\{(\gamma,\delta)\in\mathbb{R}^2 : \gamma>0, -1<\gamma+\frac{\delta}{p-1}\le\frac{1}{p-1}, \gamma-\delta+1< p\right\},$$
$$\mathcal{A}_1^2 := \left\{(\gamma,\delta)\in\mathbb{R}^2 : \gamma\ge 1, \gamma+\frac{\delta}{p-1}>\frac{1}{p-1}\right\},$$
$$\mathcal{B}_1^2 := \left\{(\gamma,\delta)\in\mathbb{R}^2 : 0<\gamma<1, \gamma+\frac{\delta}{p-1}>\frac{1}{p-1}\right\}.$$

Then $z_0=-\infty$ if $(\gamma,\delta)\in\mathcal{A}_1^1\cup\mathcal{A}_1^2$ and $z_0>-\infty$ if $(\gamma,\delta)\in\mathcal{B}_1^1\cup\mathcal{B}_1^2$. For different values of p, theses sets are depicted in Figures 5.5 and 5.6.

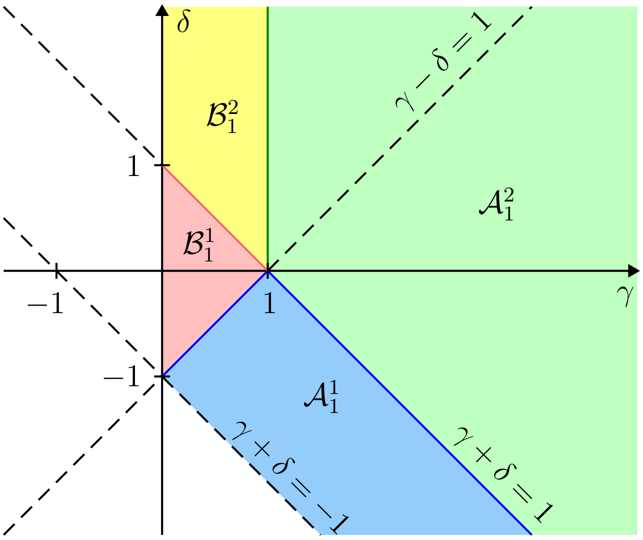

Figure 5.5: Visualization of the sets $\mathcal{A}_1^1$, $\mathcal{A}_1^2$ and $\mathcal{B}_1^1$, $\mathcal{B}_1^2$ for $p=2$

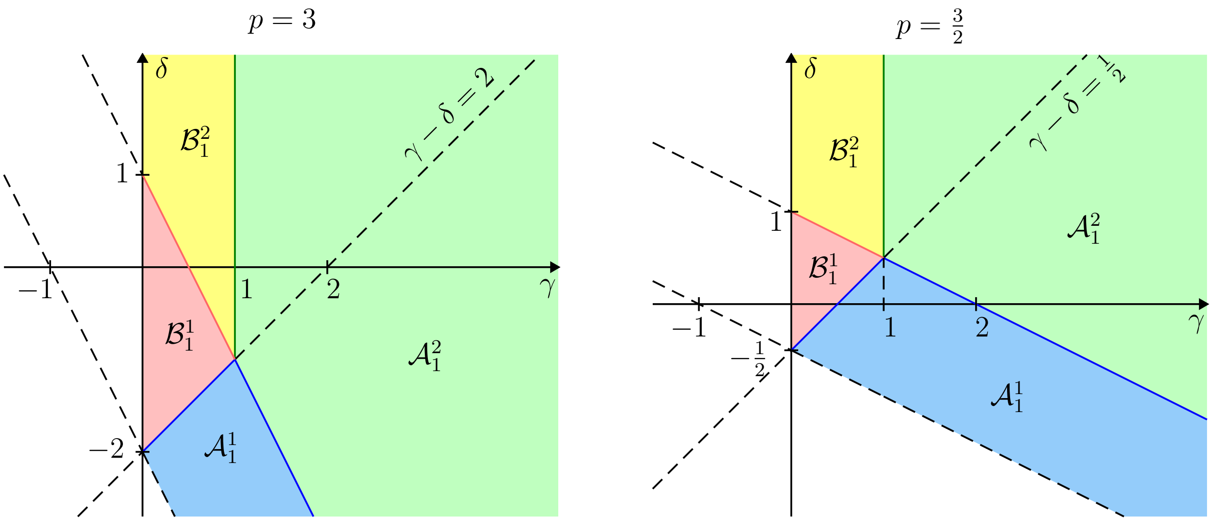

Figure 5.6: Geometric interpretation for values $p\neq 2$

It is interesting to observe how the value of p affects the layout of these sets and consequently types of solutions for given γ, δ. For large values of p, there exists a solution for almost any combination of γ and δ. Notice that sharp-type solution are always produced only by non-Lipschitz reaction g. As p increases, the regions corresponding to finite z_0 (depicted in red and yellow) expand further below the γ-axis. In other words, the parameter p helps to compensate singularity of d and the sharp-type solutions become more frequent.

On the other hand, small values of p seem to have the opposite effect. The lower dashed boundary line is getting less steep and, in the limit for $p \to 1+$, aligns with the γ-axis. Therefore, admissible singularities of d, which might still produce a solution, are more restricted and typically yield a front-type profile.

Remark 5.10. Let $(\gamma, \delta) \in \mathcal{B}_1^1 \cup \mathcal{B}_1^2$, i.e., $z_0 > -\infty$. Then it follows from Remark 2.7 that for $\delta \leq 0$ we have $\lim_{z \to z_0+} U'(z) = 0$. This implies that the wave profile $U = U(z)$ is a C^1-function in a neighbourhood of $z_0 \in \mathbb{R}$. The estimates in the proofs of Theorems 5.7 and 5.8 allows us to extend this result also for $\delta > 0$ in the following way.

Let $(\gamma, \delta) \in \mathcal{B}_1^1$. Then

$$y_{c_*}(t) \leq \overline{\kappa}(1-t)^{\gamma + \frac{\delta}{p-1} + 1}, \quad t \in [1-\theta, 1],$$

and, therefore, there exist a constant $c_5 > 0$ such that

$$\begin{aligned} z'(1-) &= \lim_{U \to 1-} \frac{\mathrm{d}z}{\mathrm{d}U} = \lim_{U \to 1-} -\frac{(d(U))^{\frac{1}{p-1}}}{(y_{c_*}(U))^{\frac{1}{p}}} \leq -c_5 \lim_{U \to 1-} \frac{(1-U)^{\frac{\delta}{p-1}}}{(1-U)^{\frac{\gamma + \frac{\delta}{p-1} + 1}{p}}} \\ &= -c_5 \lim_{U \to 1-} (1-U)^{-\frac{\gamma - \delta + 1}{p}} = -\infty, \end{aligned}$$

i.e., $U'(z_0+) = 0$ if $\delta < \gamma + 1$.

Let $(\gamma, \delta) \in \mathcal{B}_1^2$. Then

$$y_c(t) \leq \overline{\kappa}(1-t)^{p(\gamma + \frac{\delta}{p-1})}, \quad t \in [1-\theta, 1],$$

and, therefore, there is a constant $c_6 > 0$ such that

$$\begin{aligned} z'(1-) &= \lim_{U \to 1-} \frac{\mathrm{d}z}{\mathrm{d}U} = \lim_{U \to 1-} -\frac{(d(U))^{\frac{1}{p-1}}}{(y_{c_*}(U))^{\frac{1}{p}}} \leq -c_6 \lim_{U \to 1-} \frac{(1-U)^{\frac{\delta}{p-1}}}{(1-U)^{\gamma + \frac{\delta}{p-1}}} \\ &= -c_6 \lim_{U \to 1-} (1-U)^{-\gamma} = -\infty, \end{aligned}$$

i.e., $U'(z_0+) = 0$ if $\gamma > 0$.

To sum up the above discussion, the wave profile U is a C^1-function in a neighbourhood of $z_0 \in \mathbb{R}$ for any $(\gamma, \delta) \in \mathcal{B}_1^1 \cup \mathcal{B}_1^2$.

5.2.2 Asymptotics near 0

Let us now investigate the asymptotic behaviour of the profile U near 0. In order to employ the same technique used for asymptotics near 1, where we derived estimates for z_0 using upper and lower solutions in the form of suitable power functions, we first need to establish an analogue of Lemma 3.5 for functions with prescribed values at 0.

The main challenge arises from the lack of uniqueness for the forward i.v.p. (3.2) when $c > 0$. In particular, according to [43, Theorem §10.XXII],the forward i.v.p. (3.2) has a maximal solution y^* and a minimal solution y_* in $[0, 1]$ and for $\varphi, \psi \in AC[0, 1]$, the following holds true:

$$
\begin{aligned}
&\varphi' \le h(t, \varphi(t), c) \ \text{ a.e. in } [0,1]\,, \quad \varphi(0) \le 0 && \Rightarrow \quad \varphi \le y^* \ \text{ in } \ [0,1] \\
&\psi' \ge h(t, \psi(t), c) \ \text{ a.e. in } [0,1]\,, \quad \psi(0) \ge 0 && \Rightarrow \quad \psi \ge y_* \ \text{ in } \ [0,1]
\end{aligned}
$$

and $y_* \le y \le y^*$ in $[0, 1]$ for every solution y. Observe that the need to work with minimal and maximal solutions is not convenient for our purposes, and we are not able to compare functions φ and ψ relative to each other. However, special form of the equation in (3.2) guarantees uniqueness in the set of solutions which are positive in $(0, s_*)$.

Lemma 5.11. *Let f be as in Theorem 3.9. Then the forward i.v.p.* (3.2) *with $c > 0$ has a unique positive solution in* $(0, s_*)$.

Proof. Let $y = y(t)$, $t \in (0, s_*)$, be a solution of the forward i.v.p. (3.2) with $c > 0$, cf. Lemma 3.1. Then

$$y'(t) = p' \left[c \left(y^+(t)\right)^{\frac{1}{p}} - f(t) \right] \ge 0, \quad t \in (0, s_*)$$

and therefore

$$y(t) = y(0) + \int_0^t y'(\sigma)\, \mathrm{d}\sigma \ge 0, \quad t \in (0, s_*).$$

Assume that there are two positive solutions $y_1 = y_1(t)$, $y_2 = y_2(t)$, $t \in (0, s_*)$ of (3.2). Then $z_1 = (y_1)^{\frac{1}{p'}} > 0$, $z_2 = (y_2)^{\frac{1}{p'}} > 0$ solve the forward initial value problem

$$
\begin{aligned}
&z_i'(t) = c - \frac{f(t)}{(z_i(t))^{\frac{1}{p-1}}} \quad \text{for a.e. } t \in (0, s_*), \\
&z_i(0) = 0
\end{aligned}
$$

for $i = 1, 2$. It then follows that

$$(z_1(t) - z_2(t))' = -f(t) \left(\frac{1}{(z_1(t))^{\frac{1}{p-1}}} - \frac{1}{(z_2(t))^{\frac{1}{p-1}}} \right)$$

and

$$(z_1(t) - z_2(t))^+ (z_1(t) - z_2(t))' = -f(t) \left(\frac{1}{(z_1(t))^{\frac{1}{p-1}}} - \frac{1}{(z_2(t))^{\frac{1}{p-1}}} \right) (z_1(t) - z_2(t))^+$$

for a.e. $t \in (0, s_*)$. Since $f(t) \le 0$, $t \in (0, s_*)$, it follows from here that

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left[(z_1(t) - z_2(t))^+ \right]^2 \le 0, \quad \text{a.e. in } (0, s_*). \tag{5.12}$$

But $z_1(0) = z_2(0) = 0$ and (5.12) imply $z_1(t) \le z_2(t)$. Similarly, we prove that $z_2(t) \le z_1(t)$. Therefore, $y_1(t) = y_2(t)$ for $t \in (0, s_*)$. □

Remark 5.12. It follows from Lemma 5.11 that the restriction of the unique positive solution $y_{c_*} = y_{c_*}(t)$, $t \in [0, 1]$, of the b.v.p. (3.1) to the interval $(0, s_*)$ is also the unique solution of the forward i.v.p. (3.2) with $c = c_*$ on $(0, s_*)$.

Thanks to the uniqueness result, we have the following comparison argument.

Lemma 5.13. *Let $f \in L^1(0,1)$ be as in Theorem 3.9, $0 < \theta < s_*$. Assume that the functions $\varphi, \psi \in AC[0,\theta]$ are positive in $(0,\theta)$ and satisfy $\varphi(0) = \psi(0) = 0$,*

$$\varphi'(t) \le h(t, \varphi(t), c_*), \qquad \psi'(t) \ge h(t, \psi(t), c_*) \quad \text{for a.e. } t \in [0,\theta],$$

i.e., $P_{c_*}\varphi \le 0$ and $P_{c_*}\psi \ge 0$ a.e. in $[0,\theta]$. Let $y_{c_*} = y_{c_*}(t)$, $t \in [0,1]$, be the unique solution of the b.v.p. (3.1). Then

$$\varphi(t) \le y_{c_*}(t) \le \psi(t), \quad t \in [0,\theta].$$

Proof. The proof follows from [43, Theorem §10.XXII] combined with Lemma 5.11 and Remark 5.12. □

We can now employ similar reasoning as in Section 5.2.1 to prove the following results.

Theorem 5.14. *Let $\alpha > 0$, $\beta \in \mathbb{R}$ and $g(t) \sim (-t^\alpha)$, $d(t) \sim t^\beta$ as $t \to 0+$. Assume that*

$$-1 < \alpha + \frac{\beta}{p-1} \le \frac{1}{p-1}.$$

If

$$\frac{\alpha - \beta + 1}{p} < 1$$

then $z_1 < +\infty$. If

$$\frac{\alpha - \beta + 1}{p} \ge 1$$

then $z_1 = +\infty$.

Proof. The assumptions on d and g imply that for θ such that $0 < \theta < \min\{s_*, s_1\}$ the function $f(t) = (d(t))^{\frac{1}{p-1}} g(t)$ is continuous in $(0,\theta)$ and $f(t) \sim -t^{\alpha + \frac{\beta}{p-1}}$ is equivalent to

$$f(t) = -\eta(t) t^{\alpha + \frac{\beta}{p-1}}, \quad t \in (0,\theta),$$

where $\eta = \eta(t)$ is a continuous function in $(0,\theta)$, $\lim_{t\to 0+} \eta(t) \in (0,+\infty)$. Since $f \in L^1(0,1)$, the parameters α, β and p must satisfy

$$\alpha + \frac{\beta}{p-1} > -1.$$

Let $-1 < \alpha + \frac{\beta}{p-1} \le \frac{1}{p-1}$. For $\kappa > 0$ we set

$$y_\kappa(t) = \kappa t^{\alpha + \frac{\beta}{p-1} + 1}, \quad t \in [0,\theta].$$

Clearly $\alpha + \frac{\beta}{p-1} + 1 > 0$, $y_\kappa(0) = 0$ and $y_\kappa > 0$ in $(0,\theta]$. Then

$$\begin{aligned} P_{c_*} y_\kappa &= \kappa\left(\alpha + \frac{\beta}{p-1} + 1\right) t^{\alpha + \frac{\beta}{p-1}} - p'\left[c_* \kappa^{\frac{1}{p}} t^{\frac{\alpha + \frac{\beta}{p-1} + 1}{p}} + \eta(t) t^{\alpha + \frac{\beta}{p-1}}\right] \\ &= t^{\alpha + \frac{\beta}{p-1}}\left[\kappa\left(\alpha + \frac{\beta}{p-1} + 1\right) - p'\eta(t)\right] - t^{\frac{\alpha + \frac{\beta}{p-1} + 1}{p}} p' c_* \kappa^{\frac{1}{p}} \end{aligned} \tag{5.13}$$

for a.e. $t \in [0, \theta]$. The assumption $\alpha + \frac{\beta}{p-1} \le \frac{1}{p-1}$ implies

$$\alpha + \frac{\beta}{p-1} \le \frac{\alpha + \frac{\beta}{p-1} + 1}{p}$$

and therefore the power $t^{\alpha+\frac{\beta}{p-1}}$ dominates the power $t^{\frac{\alpha+\frac{\beta}{p-1}+1}{p}}$ near 0. It then follows from (5.13) that we may distinguish between two cases:

(i) There exists $\underline{\kappa} \ll 1$ so small that $P_{c_*} y_{\underline{\kappa}} \le 0$ for a.e. $t \in [0, \theta]$.

(ii) There exists $\overline{\kappa} \gg 1$ so large that $P_{c_*} y_{\overline{\kappa}} \ge 0$ for a.e. $t \in [0, \theta]$.

From Lemma 5.13 we then conclude

$$y_{\underline{\kappa}}(t) \le y_{c_*}(t) \le y_{\overline{\kappa}}(t), \quad t \in [0, \theta],$$

where y_{c_*} is the solution of the b.v.p. (3.1).
Case (i). Let $\frac{\alpha-\beta+1}{p} < 1$. Then there exists $c_1 > 0$ such that

$$\begin{aligned} z_1 &= \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \, \mathrm{d}t \le \int_0^{\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{\underline{\kappa}}(t))^{\frac{1}{p}}} \, \mathrm{d}t + \int_\theta^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \, \mathrm{d}t \\ &\le c_1 \int_0^\theta \frac{t^{\frac{\beta}{p-1}}}{t^{\frac{\alpha+\frac{\beta}{p-1}+1}{p}}} \, \mathrm{d}t + \int_\theta^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \, \mathrm{d}t \\ &= c_1 \int_0^\theta \frac{\mathrm{d}t}{t^{\frac{\alpha-\beta+1}{p}}} + \int_\theta^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \, \mathrm{d}t < +\infty. \end{aligned}$$

Case (ii). Let $\frac{\alpha-\beta+1}{p} \ge 1$. Then there exists $c_2 > 0$ such that

$$\begin{aligned} z_1 &= \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \, \mathrm{d}t \ge \int_0^{\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{\overline{\kappa}}(t))^{\frac{1}{p}}} \, \mathrm{d}t + \int_\theta^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \, \mathrm{d}t \\ &\ge c_2 \int_0^\theta \frac{t^{\frac{\beta}{p-1}}}{t^{\frac{\alpha+\frac{\beta}{p-1}+1}{p}}} \, \mathrm{d}t + \int_\theta^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \, \mathrm{d}t \\ &= c_2 \int_0^\theta \frac{\mathrm{d}t}{t^{\frac{\alpha-\beta+1}{p}}} \int_\theta^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \, \mathrm{d}t = +\infty. \end{aligned}$$

$\square$

Theorem 5.15. *Let $\alpha > 0$, $\beta \in \mathbb{R}$ and $g(t) \sim (-t^\alpha)$, $d(t) \sim t^\beta$ as $t \to 0+$. Assume that*

$$\alpha + \frac{\beta}{p-1} > \frac{1}{p-1}.$$

If $\beta > 2 - p$ then $z_1 < +\infty$. If $\beta \le 2 - p$ then $z_1 = +\infty$.

Proof. As in the proof of Theorem 5.14, we conclude that there exists $0 < \theta < \min\{s_*, s_1\}$ such that

$$f(t) = (d(t))^{\frac{1}{p-1}} g(t) = \eta(t) t^{\alpha + \frac{\beta}{p-1}}, \quad t \in (0, \theta),$$

where $\eta = \eta(t)$ is a continuous function in $(0, \theta)$, $\lim_{t \to 0+} \eta(t) \in (0, +\infty)$.

Let $\alpha + \frac{\beta}{p-1} > \frac{1}{p-1}$. For $\kappa > 0$ we set

$$y_\kappa(t) = \kappa t^{p'}, \quad t \in [0, \theta].$$

Clearly $y_\kappa(0) = 0$ and $y_\kappa(t) > 0$ for $t \in (0, \theta]$. Then

$$\begin{aligned} P_{c_*} y_\kappa &= \kappa p' t^{p'-1} - p' \left[c_* \kappa^{\frac{1}{p}} t^{\frac{p'}{p}} + \eta(t) t^{\alpha + \frac{\beta}{p-1}} \right] \\ &= \left(\kappa p' - p' c_* \kappa^{\frac{1}{p}} \right) t^{\frac{1}{p-1}} - p' \eta(t) t^{\alpha + \frac{\beta}{p-1}}, \end{aligned} \tag{5.14}$$

for a.e. $t \in [0, \theta]$. The assumption $\alpha + \frac{\beta}{p-1} > \frac{1}{p-1}$ implies that the power $t^{\frac{1}{p-1}}$ dominates $t^{\alpha + \frac{\beta}{p-1}}$ near 0. It follows from (5.11) that we may distinguish between two cases:

(i) There exists $\underline{\kappa} \ll 1$ so small that $P_{c_*} y_{\underline{\kappa}} \le 0$ for a.e. $t \in [0, \theta]$.

(ii) There exists $\overline{\kappa} \gg 1$ so large that $P_{c_*} y_{\overline{\kappa}} \ge 0$ for a.e. $t \in [0, \theta]$.

From Lemma 5.13 we then conclude

$$y_{\underline{\kappa}}(t) \le y_{c_*}(t) \le y_{\overline{\kappa}}(t), \quad t \in [0, \theta],$$

where y_{c_*} is the solution of the b.v.p. (3.1).

Case (i). Let $\beta > p - 2$. Then there exists $c_3 > 0$ such that

$$\begin{aligned} z_1 &\le \int_0^\theta \frac{(d(t))^{\frac{1}{p-1}}}{(y_{\underline{\kappa}}(t))^{\frac{1}{p}}} \, \mathrm{d}t + \int_\theta^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \, \mathrm{d}t \le c_3 \int_0^\theta \frac{t^{\frac{\beta}{p-1}}}{t^{\frac{p'}{p}}} \, \mathrm{d}t + \int_\theta^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \, \mathrm{d}t \\ &= c_3 \int_0^{s_*} \frac{\mathrm{d}t}{t^{\frac{1-\beta}{p-1}}} + \int_\theta^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \, \mathrm{d}t < +\infty. \end{aligned}$$

Case (ii). Let $\beta \le p - 2$. Then there exists $c_4 > 0$ such that

$$\begin{aligned} z_1 &\ge \int_0^\theta \frac{(d(t))^{\frac{1}{p-1}}}{(y_{\overline{\kappa}}(t))^{\frac{1}{p}}} \, \mathrm{d}t + \int_\theta^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \, \mathrm{d}t \\ &\ge c_4 \int_0^\theta \frac{\mathrm{d}t}{t^{\frac{1-\beta}{p-1}}} + \int_\theta^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \, \mathrm{d}t = +\infty. \end{aligned}$$

$\square$

Remark 5.16. To visualize conditions from Theorems 5.14 and 5.15, we introduce the sets

$$\mathcal{A}_0^1 := \left\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, -1 < \alpha + \frac{\beta}{p-1} \leq \frac{1}{p-1}, \alpha - \beta + 1 \geq p\right\},$$
$$\mathcal{B}_0^1 := \left\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, -1 < \alpha + \frac{\beta}{p-1} \leq \frac{1}{p-1}, \alpha - \beta + 1 < p\right\},$$
$$\mathcal{A}_0^2 := \left\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, \alpha + \frac{\beta}{p-1} > \frac{1}{p-1}, \beta \leq 2 - p\right\},$$
$$\mathcal{B}_0^2 := \left\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, \alpha + \frac{\beta}{p-1} > \frac{1}{p-1}, \beta > p - 2\right\}.$$

Then $z_1 = +\infty$ if $(\alpha,\beta) \in \mathcal{A}_0^1 \cup \mathcal{A}_0^2$ and $z_1 < +\infty$ if $(\alpha,\beta) \in \mathcal{B}_0^1 \cup \mathcal{B}_0^2$. For different values of p, theses sets are depicted in Figures 5.7 and 5.8.

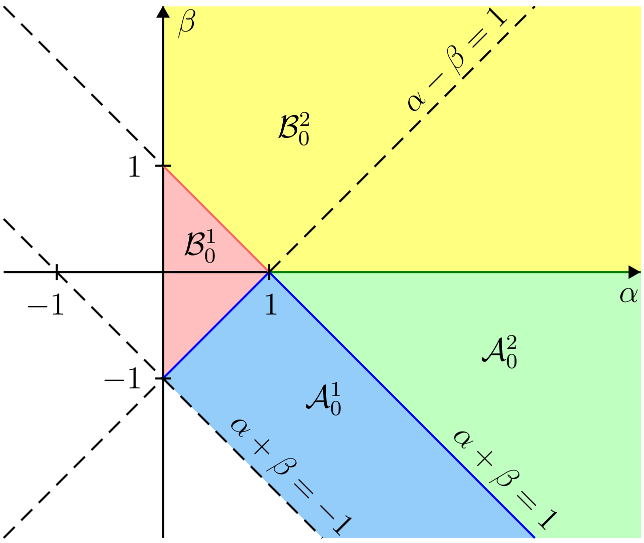

Figure 5.7: Visualization of the sets $\mathcal{A}_0^1$, $\mathcal{A}_0^2$ and $\mathcal{B}_0^1$, $\mathcal{B}_0^2$ for $p = 2$

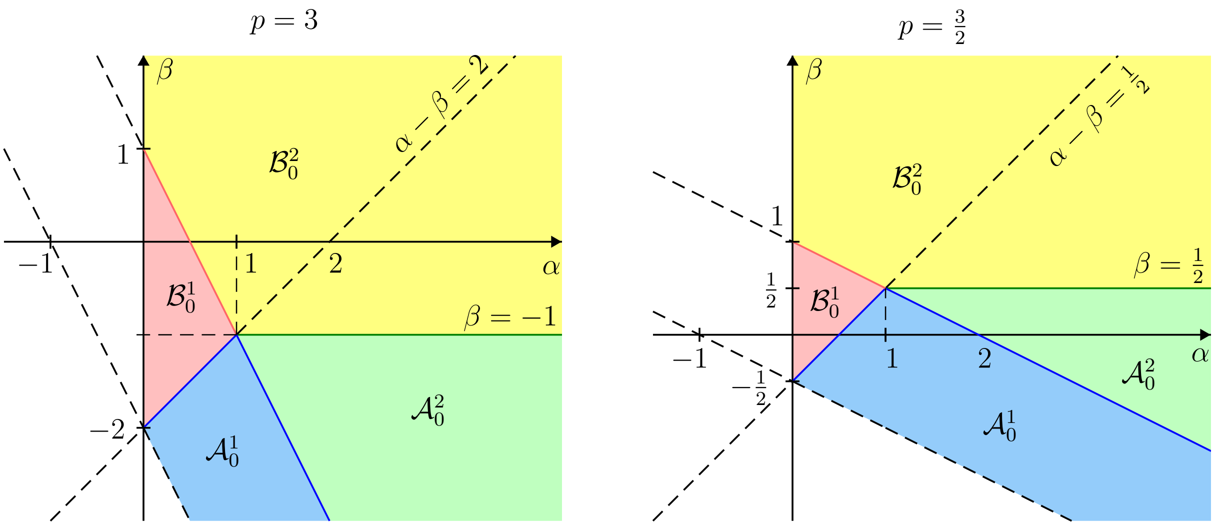

Figure 5.8: Geometric interpretation for values $p \neq 2$

Let us briefly discuss some interesting observations. While the sets $\mathcal{A}_0^1$ and $\mathcal{B}_0^1$ are complete analogues of $\mathcal{A}_1^1$ and $\mathcal{B}_1^1$, the sets $\mathcal{A}_0^2$ and $\mathcal{B}_0^2$ differ significantly, with $\mathcal{B}_0^2$ now being the most prominent, see Figures 5.7 and 5.8 below. In contrast to the asymptotics near 1, sharp-type solutions reaching 0 in a finite z_1 are very common and not restricted to non-Lipschitz reactions. For $p = 2$, degenerate diffusion ($\beta > 0$) always leads to sharp-type profiles. Strictly positive diffusion ($\beta = 0$) yields front-type solutions provided g is Lipschitz.

As p increases, the set $\mathcal{B}_0^2$ expands below the α-axis, meaning that singular diffusion with $0 < \beta < 2 - p$ gives rise to sharp-type solutions independently of α. Conversely, smaller values of p shift the blue and green regions above the α-axis. Consequently, degenerate non-Lipschitz diffusion results more frequently in front-type solutions rather than sharp-type. However, the horizontal line dividing $\mathcal{A}_0^2$ and $\mathcal{B}_0^2$ cannot ascend beyond $\beta = 1$. This implies that degenerate Lipschitz diffusion produces sharp-type profiles independently of both p and α.

Remark 5.17. Let $(\alpha, \beta) \in \mathcal{B}_0^2$. The estimates on z_1 from Theorem 5.15 provide additional information about the smoothness of the profile as it reaches 0. In particular, we have

$$y_{c_*}(t) \geq \underline{\kappa} t^{p'}, \quad t \in [0, \theta]$$

and, therefore, there exist a constant $c_5 > 0$ such that

$$\begin{aligned} 0 \geq z'(0+) &= \lim_{U \to 0+} \frac{\mathrm{d}z}{\mathrm{d}U} = \lim_{U \to 0+} -\frac{(d(U))^{\frac{1}{p-1}}}{(y_{c_*}(U))^{\frac{1}{p}}} \\ &\geq -c_5 \lim_{U \to 0+} \frac{U^{\frac{\beta}{p-1}}}{U^{\frac{p'}{p}}} = -c_5 \lim_{U \to 0+} U^{\frac{\beta-1}{p-1}}. \end{aligned}$$

Hence we are able to distinguish the following cases:

(i) If $\beta > 1$ then $z'(0+) = 0$ and hence $U'(z_1-) = -\infty$.

(ii) If $\beta = 1$ then $0 \geq z'(0+) \geq -c_5$ and hence $0 > U'(z_1-) \geq -\infty$.

In either case, the wave profile U is not smooth at $z_1 \in \mathbb{R}$.

As we can see, the above estimate for $z'(0+)$ cannot provide any information about the existence of profiles with $U'(z_1-) = U'(z_1+) = 0$. In particular, a smooth profile might appear if $(\alpha, \beta) \in \mathcal{B}_0^2$ and $\beta < 1$, or if $(\alpha, \beta) \in \mathcal{B}_0^1$.

For $p = 2$, regions corresponding to the cases (i) and (ii) are highlighted in Figure 5.9. The light yellows part of $\mathcal{B}_0^2$ indicates where no further classification based on the value of $U'(z_1-)$ is available.

Finally, let us recall the discussion from Remark 5.16 to make an interesting observation. The closer p is to 1, the narrower the indeterminate region becomes, as the upper boundary line of $\mathcal{A}_0^2$ approaches the threshold $\beta = 1$. Therefore, the likelihood of obtaining a smooth profile is diminished significantly when considering a Lipschitz reaction ($\alpha \geq 1$). This is another major difference compared to the behaviour near 1. Whenever the profile reaches the equilibrium 1, it does so with a zero derivative.

Remark 5.18 (Classification of profiles)**.** Let us consider the following particular case of the equation (2.1):

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^\beta (1-u)^\delta \left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x}\right) + u^\alpha (1-u)^\gamma (u - s_*), \quad 0 < s_* < 1, \quad p > 1.$$

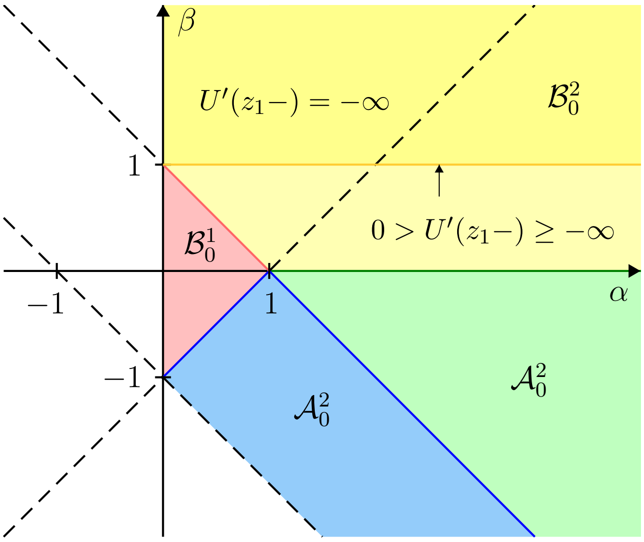

Figure 5.9: Visualization of the sets $\mathcal{A}_0^1$, $\mathcal{A}_0^2$ and $\mathcal{B}_0^1$, $\mathcal{B}_0^2$ for $p = 2$

Here $d(s) = s^\beta(1-s)^\delta$ and $g(s) = s^\alpha(1-s)^\gamma(s-s_*)$, $s \in [0,1]$, are the diffusion and reaction terms, respectively, with $\alpha > 0$, $\gamma > 0$ and $\beta, \delta \in \mathbb{R}$. If

$$\int_0^1 s^\alpha(1-s)^\gamma(s-s_*)\,\mathrm{d}s > 0,$$

then also (4.2) holds true and Theorem 4.1 guarantees the existence of monotone nonincreasing profile with $c_* > 0$.

Combining the results from Theorems 5.7, 5.8 and 5.14, 5.15 together with the notation from Remarks 5.9 and 5.16, we arrive at the following classification of travelling wave profiles $u(x,t) = U(x-ct) = U(z)$. The unique profile $U = U(z)$ is

$$\begin{array}{cccll} \text{front-type} & \text{if} & (\alpha,\beta) \in \mathcal{A}_0^1 \cup \mathcal{A}_0^2 & \text{and} & (\gamma,\delta) \in \mathcal{A}_1^1 \cup \mathcal{A}_1^2; \\ \text{sharp of type I} & \text{if} & (\alpha,\beta) \in \mathcal{B}_0^1 \cup \mathcal{B}_0^2 & \text{and} & (\gamma,\delta) \in \mathcal{A}_1^1 \cup \mathcal{A}_1^2; \\ \text{sharp of type II} & \text{if} & (\alpha,\beta) \in \mathcal{A}_0^1 \cup \mathcal{A}_0^2 & \text{and} & (\gamma,\delta) \in \mathcal{B}_1^1 \cup \mathcal{B}_1^2; \\ \text{sharp of type III} & \text{if} & (\alpha,\beta) \in \mathcal{B}_0^1 \cup \mathcal{B}_0^2 & \text{and} & (\gamma,\delta) \in \mathcal{B}_1^1 \cup \mathcal{B}_1^2. \end{array}$$

In this example, the diffusion coefficient is continuous in $(0,1)$, i.e., $M_U = \emptyset$. It follows from Remarks 5.10 and 5.17 that profiles which are sharp of type II are C^1-functions on $\mathbb{R}$, and sharp of type III profiles are generally of class $C^1(\mathbb{R} \setminus \{z_1\})$.

5.3 Monostable case

Finally, we address the case of monostable reaction term $g \in C[0,1]$, characterized by the property $g(0) = g(1) = 0$, $g(s) > 0$ for $s \in (0,1)$. Let us recall that under the assumptions of Theorem 4.6, there exists a half-line of admissible wave speeds $c \in [c^*, +\infty)$, $c^* > 0$. Unlike in the previous case, we now need to take into account also the non-existence result from Theorem 4.7, which concerns the behaviour of $d(s)$ and $g(s)$ as $s \to 0+$. While the asymptotic analysis near the equilibrium 1 is not affected by this and hence similar to that in the bistable unbalanced case, the analysis near 0 becomes less detailed and an additional assumption on c must be imposed in order to refine some of the estimates.

5.3.1 Asymptotics near 1

Let us start by examining the behaviour of solutions $U = U(z)$ as $z \to -\infty$. The means of the proofs are basically the same as in the bistable case, except for the fact that the assumption (4.5) yields a different necessary condition for the involved parameters.

In what follows, we assume that $c \in [c^*, +\infty)$ is arbitrary but fixed.

Theorem 5.19. *Let $\gamma > 0$, $\delta \in \mathbb{R}$ and $g(t) \sim (1-t)^\gamma$, $d(t) \sim (1-t)^\delta$ as $t \to 1-$. Assume that*

$$0 \le \gamma + \frac{\delta}{p-1} \le \frac{1}{p-1}\,. \tag{5.15}$$

If

$$\frac{\gamma - \delta + 1}{p} < 1$$

then $z_0 > -\infty$. If

$$\frac{\gamma - \delta + 1}{p} \ge 1$$

then $z_0 = -\infty$.

Proof. Set $f(t) = (d(t))^{\frac{1}{p-1}} g(t)$. Then $f(t) \sim (1-t)^{\gamma + \frac{\delta}{p-1}}$ as $t \to 1-$ and we observe that due to the assumption (4.5) from Theorem 4.6, we have the following necessary condition for the parameters γ, δ and p:

$$\gamma + \frac{\delta}{p-1} \ge 0,$$

i.e., the first inequality in (5.15).

Our assumptions on d and g yield the existence of $\theta > 0$ such that $f(t) = (d(t))^{\frac{1}{p-1}}\, g(t)$ is continuous in $(1-\theta, 1)$. Hence $f(t) \sim (1-t)^{\gamma+\frac{\delta}{p-1}}$ is equivalent to

$$f(t) = \eta(t)(1-t)^{\gamma + \frac{\delta}{p-1}}, \quad t \in (1-\theta, 1),$$

where $\eta = \eta(t)$ is a continuous function in $(1-\theta, 1)$ with $\lim_{t\to 1-} \eta(t) \in (0, +\infty)$.

The rest of the proof is then carried out in the same way as in the proof of Theorem 5.7, using upper and lower solutions of the form

$$y_\kappa(t) = \kappa(1-t)^{\gamma + \frac{\delta}{p-1} + 1}, \quad \kappa > 0, \quad t \in [1-\theta, 1].$$

□

Theorem 5.20. *Let $\gamma > 0$, $\delta \in \mathbb{R}$ and $g(t) \sim (1-t)^\gamma$, $d(t) \sim (1-t)^\delta$ as $t \to 1-$. Assume that*

$$\gamma + \frac{\delta}{p-1} > \frac{1}{p-1}\,.$$

If $\gamma < 1$ then $z_0 > -\infty$. If $\gamma \ge 1$ then $z_0 = -\infty$.

Proof. The proof follows the same reasoning as in the proof of Theorem 5.8, employing upper and lower solutions of the form

$$y_\kappa(t) = \kappa(1-t)^{p\left(\gamma + \frac{\delta}{p-1}\right)}, \quad \kappa > 0, \quad t \in [1-\theta, 1].$$

□

Remark 5.21. To visualize conditions from Theorem 5.19 and 5.20, we introduce the sets

$$\mathcal{A}_1^1 := \left\{ (\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, 0 \le \gamma + \frac{\delta}{p-1} \le \frac{1}{p-1}, \gamma - \delta + 1 \ge p \right\},$$
$$\mathcal{B}_1^1 := \left\{ (\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, 0 \le \gamma + \frac{\delta}{p-1} \le \frac{1}{p-1}, \gamma - \delta + 1 < p \right\},$$
$$\mathcal{A}_1^2 := \left\{ (\gamma,\delta) \in \mathbb{R}^2 : \gamma \ge 1, \gamma + \frac{\delta}{p-1} > \frac{1}{p-1} \right\},$$
$$\mathcal{B}_1^2 := \left\{ (\gamma,\delta) \in \mathbb{R}^2 : 0 < \gamma < 1, \gamma + \frac{\delta}{p-1} > \frac{1}{p-1} \right\}.$$

Then $z_0 = -\infty$ if $(\gamma,\delta) \in \mathcal{A}_1^1 \cup \mathcal{A}_1^2$ and $z_0 > -\infty$ if $(\gamma,\delta) \in \mathcal{B}_1^1 \cup \mathcal{B}_1^2$.

Notice that the sets $\mathcal{A}_1^2$ and $\mathcal{B}_1^2$ are the same as in the bistable case, while $\mathcal{A}_1^1$ and $\mathcal{B}_1^1$ differ only in the lower bound for $\gamma + \frac{\delta}{p-1}$. Therefore, in the geometric interpretation (see Figures 5.10 and 5.11), the lower boundary line of the corresponding regions always intersects the origin.

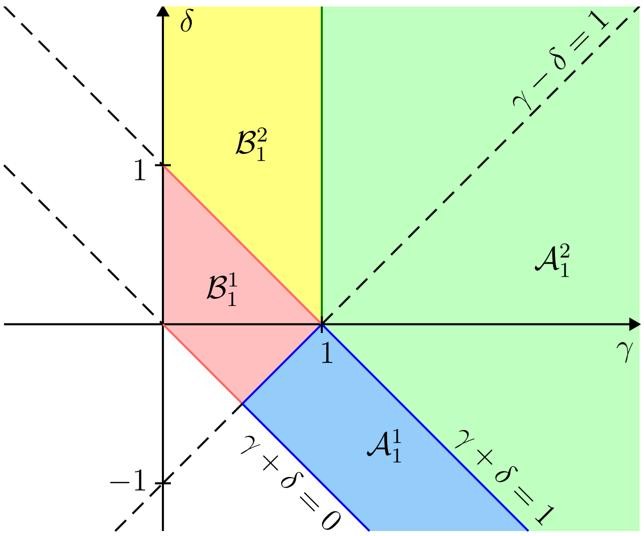

Figure 5.10: Visualization of the sets $\mathcal{A}_1^1$, $\mathcal{A}_1^2$ and $\mathcal{B}_1^1$, $\mathcal{B}_1^2$ for $p = 2$

As for the derivative $U'(z_0+)$, the same reasoning as in Remark 5.10 applies also in the monostable case. In particular, for any $(\gamma,\delta) \in \mathcal{B}_1^1 \cup \mathcal{B}_1^2$, the wave profile $U = U(z)$ is a C^1-function in the neighbourhood of $z_0 \in \mathbb{R}$.

5.3.2 Asymptotics near 0

In this section, we adopt a different approach from the previous cases. We proceed directly with the reasoning and summarize it in the main theorem at the end.

Let us assume that $g(t) \sim t^\alpha$ and $d(t) \sim t^\beta$ as $t \to 0+$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$. It follows from Theorem 4.7 that

$$\alpha + \frac{\beta}{p-1} \ge \frac{1}{p-1} \tag{5.16}$$

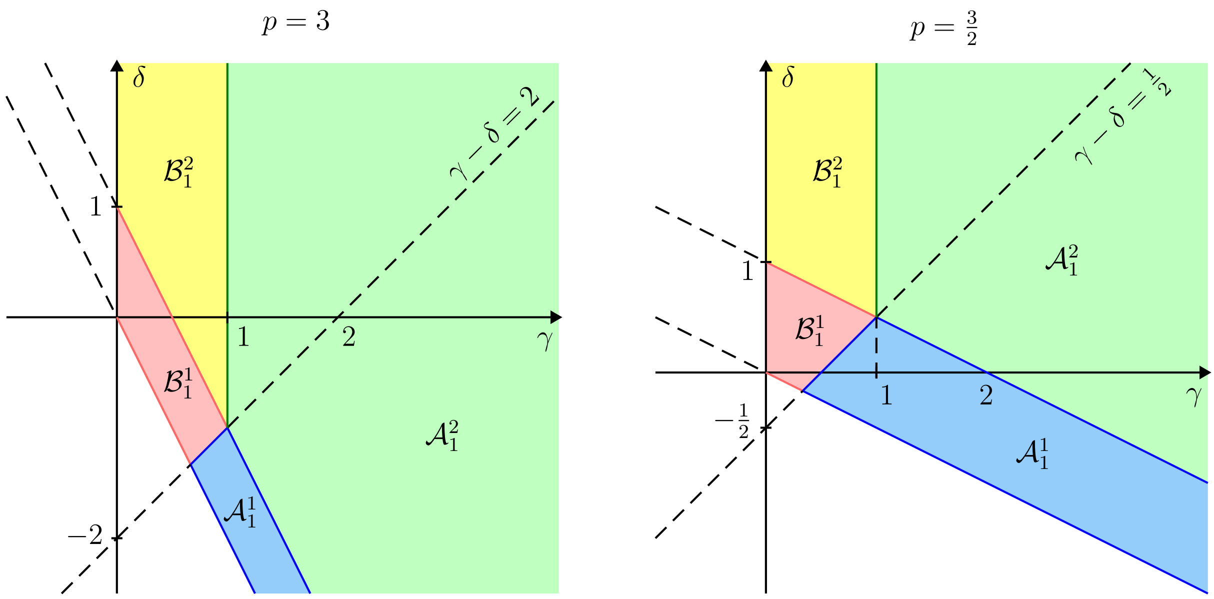

Figure 5.11: Geometric interpretation for values $p \neq 2$

must hold, otherwise there is no solution of (4.1).

The proof of Theorem 3.13 suggests a method to determine when $z_1 = +\infty$. Indeed, there we showed that

$$0 < y_c(t) \le k t^{p'}, \quad t \in (0,1),$$

with $k = (\frac{c}{p})^{p'}$. Together with the expression for z_1 from (5.2), the above inequality yields that there exists a constant $c_1 > 0$ such that

$$z_1 = \int_0^{\frac{1}{2}} \frac{(d(t))^{\frac{1}{p-1}}}{(y_c(t))^{\frac{1}{p}}}\,\mathrm{d}t \ge \int_0^{\frac{1}{2}} \frac{(d(t))^{\frac{1}{p-1}}}{k^{\frac{1}{p}} t^{\frac{p'}{p}}}\,\mathrm{d}t \ge c_1 \int_0^{\frac{1}{2}} \frac{t^{\frac{\beta}{p-1}}}{t^{\frac{1}{p-1}}}\,\mathrm{d}t = c_1 \int_0^{\frac{1}{2}} t^{\frac{\beta-1}{p-1}}\,\mathrm{d}t.$$

From the last integral we conclude that $z_1 = +\infty$ if and only if $\beta \le 2-p$. The values of α and β for which this situation occurs are depicted in Figure 5.12 for $p=2$. The boundary lines of this region are generally given by the expressions $\beta = 1-(p-1)\alpha$ and $\beta = 2-p$, suggesting how the layout changes for different values of p.

However, this estimate is far from being optimal. Indeed, we can refine the asymptotics of y_c near 0 in the case of power-type behaviour of g and d near 0 and prove $z_1 = +\infty$ under more general assumptions on α and β.

Notice that (5.16) is equivalent to $p\alpha + p'\beta \ge p'$ and set $\omega := p\alpha + p'\beta$, $y_\kappa(t) := \kappa t^\omega$, $t \in (0,1)$, with $\kappa > 0$. Let

$$f_1 := \sup_{t\in(0,1)} \frac{(d(t))^{\frac{1}{p-1}} g(t)}{t^{\alpha+\frac{\beta}{p-1}}}. \tag{5.17}$$

It follows from (5.16) and (4.5) that $\mu \le f_1 < +\infty$. In particular, (5.17) yields

$$f(t) \le f_1 t^{\alpha+\frac{\beta}{p-1}} = f_1 t^{\frac{\omega}{p}}, \quad t \in [0,1].$$

Therefore, we have

$$\begin{aligned} P_c y_\kappa &= y_\kappa'(t) - p'\left[c\,(y_\kappa(t))^{\frac{1}{p}} - f(t)\right] \le \omega\kappa t^{\omega-1} - p'c\kappa^{\frac{1}{p}} t^{\frac{\omega}{p}} + p' f_1 t^{\frac{\omega}{p}} \\ &= t^{\frac{\omega}{p}}\left(\omega\kappa t^{\varepsilon} - p'c\kappa^{\frac{1}{p}} + p' f_1\right), \quad t \in [0,1], \end{aligned}$$

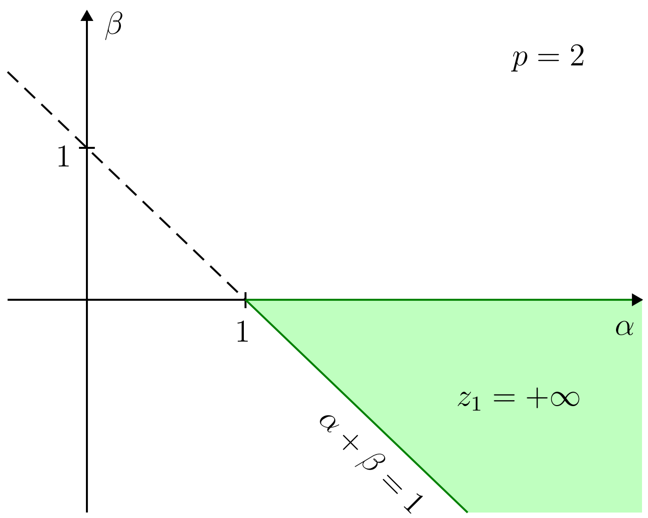

Figure 5.12: Visualization of conditions leading to z_1 infinite

with $\varepsilon = \omega - 1 - \frac{\omega}{p} \geq 0$. Since $t \in [0,1]$, the following inequality

$$\omega\kappa - p' c \kappa^{\frac{1}{p}} + p' f_1 \leq 0 \tag{5.18}$$

would imply that $P_c y_\kappa \leq 0$ a.e. in $[0,1]$. Notice that (5.18) is equivalent to

$$c \geq \frac{\omega\kappa + p' f_1}{p' \kappa^{\frac{1}{p}}} =: H(\kappa), \ \ \kappa > 0. \tag{5.19}$$

Obviously, $H(\kappa) > 0$, $\kappa \in (0,+\infty)$ and $\lim_{\kappa\to 0+} H(\kappa) = \lim_{\kappa\to+\infty} H(\kappa) = +\infty$. The global minimum of H over $(0,+\infty)$ is attained at

$$\kappa_{\min} = \frac{\frac{(p')^2}{p} f_1}{\omega}$$

and, due to $\omega \geq p'$,

$$H(\kappa_{\min}) = (p')^{1-\frac{2}{p}}\, p^{\frac{1}{p}}\, f_1^{\frac{1}{p'}}\, \omega^{\frac{1}{p}} \geq (p')^{\frac{1}{p'}}\, p^{\frac{1}{p}}\, f_1^{\frac{1}{p'}}. \tag{5.20}$$

It follows from (5.18)–(5.20) that for $\overline{\kappa} = \kappa_{\min}$ and all $c \geq (p')^{\frac{1}{p'}}\, p^{\frac{1}{p}}\, f_1^{\frac{1}{p'}}$ we have that $P_c y_{\overline{\kappa}} \leq 0 = P_c y_c$ a.e. in $[0,1]$ and since $y_{\overline{\kappa}}(1) > 0$, by Lemma 3.5, we get $y_{\overline{\kappa}}(t) \geq y_c(t)$ for $t \in [0,1]$. In particular, we deduce

$$z_1 = \int_0^{\frac{1}{2}} \frac{(d(t))^{\frac{1}{p-1}}}{(y_c(t))^{\frac{1}{p}}}\,\mathrm{d}t \geq \int_0^{\frac{1}{2}} \frac{(d(t))^{\frac{1}{p-1}}}{\overline{\kappa}^{\frac{1}{p}} t^{\frac{\omega}{p}}}\,\mathrm{d}t \geq c_2 \int_0^{\frac{1}{2}} \frac{t^{\frac{\beta}{p-1}}}{t^{\frac{\omega}{p}}}\,\mathrm{d}t = +\infty$$

with some $c_2 > 0$ if and only if

$$\frac{\beta}{p-1} - \frac{\omega}{p} \leq -1,$$

which is equivalent to $\alpha \geq 1$.

On the other hand, let $c \in [c^*, +\infty)$ be fixed. Since d and g are strictly positive in $(0,1)$ and $f(t) \sim t^{\alpha + \frac{\beta}{p-1}}$ as $t \to 0+$, there exists $0 < f_2 < +\infty$ such that

$$f(t) \geq f_2 t^{\alpha + \frac{\beta}{p-1}} = f_2 t^{\frac{\omega}{p}}, \quad t \in \left[0, \frac{1}{2}\right].$$

We set $y_{\underline{\kappa}}(t) = \underline{\kappa} t^{\omega}$, $t \in [0, \frac{1}{2}]$, where

$$\underline{\kappa} := \min \left\{ 2^{\omega} y_c \left(\frac{1}{2} \right), \left(\frac{f_2}{c} \right)^p \right\}.$$

Then $y_{\underline{\kappa}}(\frac{1}{2}) \leq y_c(\frac{1}{2})$ and

$$\begin{aligned} P_c y_{\underline{\kappa}} = y'_{\underline{\kappa}}(t) - p' \left[c \left(y_{\underline{\kappa}}(t) \right)^{\frac{1}{p}} - f(t) \right] &\geq \omega \underline{\kappa} t^{\omega - 1} - p' c \underline{\kappa}^{\frac{1}{p}} t^{\frac{\omega}{p}} + p' f_2 t^{\frac{\omega}{p}} \\ &\geq p' t^{\frac{\omega}{p}} \left(f_2 - c \underline{\kappa}^{\frac{1}{p}} \right) \geq 0 = P_c y_c \quad \text{in} \quad \left[0, \frac{1}{2} \right]. \end{aligned}$$

By Lemma 3.5 we conclude $y_{\underline{\kappa}}(t) \leq y_c(t)$, $t \in [0, \frac{1}{2}]$. In particular, we conclude

$$z_1 = \int_0^{\frac{1}{2}} \frac{(d(t))^{\frac{1}{p-1}}}{(y_c(t))^{\frac{1}{p}}} \, \mathrm{d}t \leq \int_0^{\frac{1}{2}} \frac{(d(t))^{\frac{1}{p-1}}}{\underline{\kappa}^{\frac{1}{p}} t^{\frac{\omega}{p}}} \, \mathrm{d}t \leq c_3 \int_0^{\frac{1}{2}} \frac{t^{\frac{\beta}{p-1}}}{t^{\frac{\omega}{p}}} \, \mathrm{d}t < +\infty$$

with some $c_3 > 0$ if and only if

$$\frac{\beta}{p-1} - \frac{\omega}{p} > -1,$$

which is equivalent to $\alpha < 1$.

We can summarize the asymptotics of y_c near 0 as follows.

Theorem 5.22. *Let $\alpha > 0$, $\beta \in \mathbb{R}$ and $g(t) \sim (-t^{\alpha})$, $d(t) \sim t^{\beta}$ as $t \to 0+$. Let f_1 be as in* (5.17) *and assume that*

$$\alpha + \frac{\beta}{\beta} \geq \frac{1}{p-1}.$$

(i) *Let $c \geq c^*$. If $\beta \leq 2 - p$ then $z_1 = +\infty$. If $0 < \alpha < 1$ then $z_1 < +\infty$.*

(ii) *Let $c \geq (p')^{1/p'} p^{1/p} f_1^{1/p'}$. If $\alpha \geq 1$ then $z_1 = +\infty$.*

Remark 5.23. To visualize conditions from Theorem 5.22, we introduce the following sets:

$$\begin{aligned} \mathcal{A}_0 &:= \left\{ (\alpha, \beta) \in \mathbb{R}^2 : \alpha \geq 1, \alpha + \frac{\beta}{p-1} \geq \frac{1}{p-1} \right\}, \\ \mathcal{B}_0 &:= \left\{ (\alpha, \beta) \in \mathbb{R}^2 : 0 < \alpha < 1, \alpha + \frac{\beta}{p-1} \geq \frac{1}{p-1} \right\}, \end{aligned}$$

see Figures 5.13 and 5.14 for geometric interpretation.

If $(\alpha, \beta) \in \mathcal{B}_0$ and $c \geq c^*$, then $z_1 < +\infty$. On the other hand, if $(\alpha, \beta) \in \mathcal{A}_0$ and $c \geq (p')^{1/p'} p^{1/p} f_1^{1/p'}$, then $z_1 = +\infty$. Without the restriction $c \geq (p')^{1/p'} p^{1/p} f_1^{1/p'}$ (notice that $f_1 \geq \mu$), we only know that $z_1 = +\infty$ if $(\alpha, \beta) \in \mathcal{A}_0$ and $\beta \leq 2 - p$. This horizontal line is highlighted in Figure 5.14 and it coincides with the α-axis in Figure 5.13. It is important to note that for $c \in [c^*, (p')^{1/p'} p^{1/p} f_1^{1/p'})$, the type as well as the smoothness (see Remark 5.24 below) of the wave profile U might be very different for $(\alpha, \beta) \in \mathcal{A}_0$ and $\beta > 2 - p$.

For example, in view of the results from [33, Theorem 2] and [22, Theorem 6.3], one should expect that for $1 < p \leq 2$, $\alpha > 0$, $\beta = 1$ and $c = c^*$, the profile U reaches zero in $z_1 < +\infty$ and with $U'(z_1-) < 0$. Since $U'(z_1+) = 0$, U is not a C^1-function in the neighbourhood of $z_1 \in \mathbb{R}$.

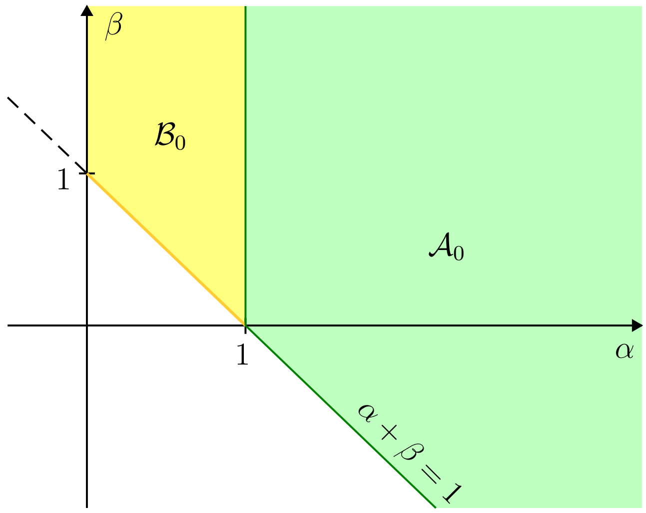

Figure 5.13: Visualization of the sets $\mathcal{A}_0$ and $\mathcal{B}_0$ for $p = 2$

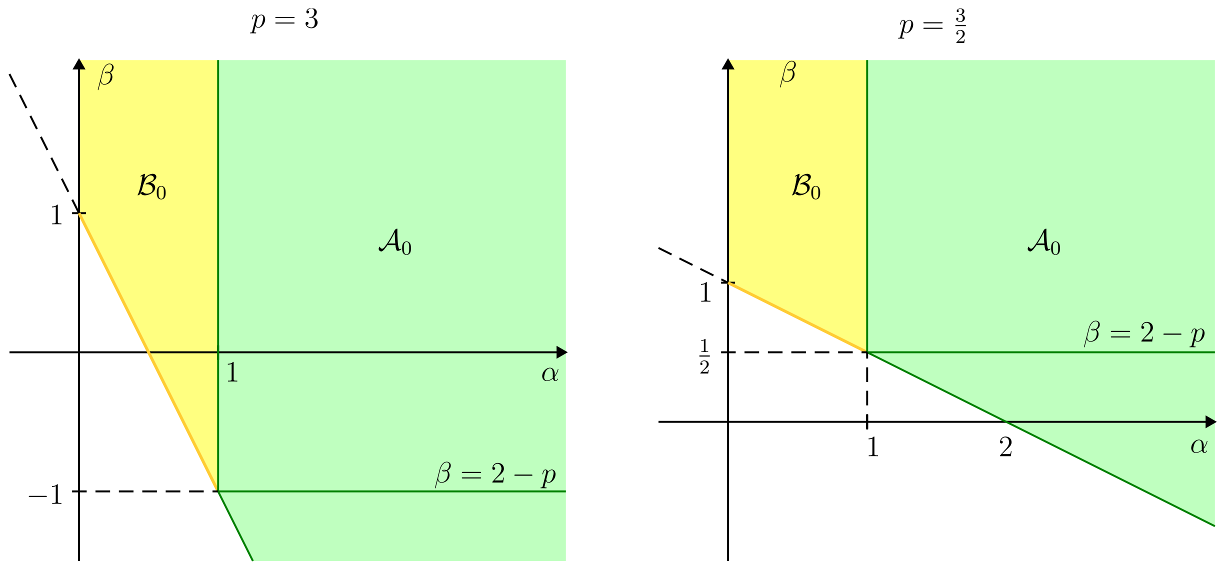

Figure 5.14: Visualization of the sets $\mathcal{A}_0$ and $\mathcal{B}_0$ for $p \neq 2$

Remark 5.24. $(\alpha, \beta) \in \mathcal{B}_0$, i.e., $z_1 < +\infty$. Then it follows from Remark 2.7 that for $\beta \leq 0$ we have $\lim_{z \to z_1-} U'(z) = 0$, i.e., the travelling wave profile U is a C^1-function in a neighbourhood of $z_1 \in \mathbb{R}$. Notice, however, that $(\alpha, \beta) \in \mathcal{B}_0$ with $\beta \leq 0$ occurs only for $1 < p < 2$.

Fortunately, we are able to improve this result provided c is "large enough". Let $c \geq (p')^{1/p'} \, p^{1/p} \, f_1^{1/p'}$ and $y_{\overline{\kappa}}(t) = \overline{\kappa} t^{\omega}$ be as above. Then $y_c(t) \leq \overline{\kappa} t^{\frac{\omega}{p}}$, $t \in [0, 1]$, and there exists c_4 such that

$$\begin{aligned} z'(0+) &= \lim_{U \to 0+} \frac{\mathrm{d}z}{\mathrm{d}U} = \lim_{U \to 0+} - \frac{(d(U))^{\frac{1}{p-1}}}{(y_c(U))^{\frac{1}{p}}} \leq - \lim_{U \to 0+} \frac{(d(U))^{\frac{1}{p-1}}}{\overline{\kappa}^{\frac{1}{p}} U^{\frac{\omega}{p}}} \leq -c_4 \lim_{U \to 0+} \frac{U^{\frac{\beta}{p-1}}}{U^{\frac{\omega}{p}}} \\ &= -c_4 \lim_{U \to 0+} U^{-\alpha} = -\infty, \end{aligned}$$

if $\alpha > 0$. Hence $U'(z_1-) = 0$, i.e., the travelling wave profile U is a C^1-function in a neighbourhood of $z_1 \in \mathbb{R}$ for any $(\alpha, \beta) \in \mathcal{B}_0$ and $c \geq (p')^{1/p'} \, p^{1/p} \, f_1^{1/p'}$.

As we mentioned in the previous remark, the results from [33] and [22] suggest that a different outcome can be expected at least when $1 < p \le 2$, $\beta = 1$ and $c = c^*$.

Remark 5.25 (Classification of profiles)**.** Let us consider the following particular case of the equation (2.1):

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^\beta (1-u)^\delta \left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) + u^\alpha (1-u)^\gamma, \quad p > 1.$$

Here $d(s) = s^\beta(1-s)^\delta$ and $g(s) = s^\alpha(1-s)^\gamma$, $s \in [0,1]$, are the diffusion and reaction terms, respectively, with $\alpha > 0$, $\gamma > 0$ and $\beta, \delta \in \mathbb{R}$.

Combining the results from Theorems 5.19, 5.20 and 5.22 together with the notation from Remarks 5.21 and 5.23, we arrive at the following classification of travelling wave profiles $u(x,t) = U(x-ct) = U(z)$.

Let $c \ge c^*$. The profile $U = U(z)$ is

$$\begin{array}{rlll}
\text{front-type} & \text{if} \;\; (\alpha,\beta) \in \mathcal{A}_0,\ \beta \le 2-p & \text{and} & (\gamma,\delta) \in \mathcal{A}_1^1 \cup \mathcal{A}_1^2; \\
\text{sharp of type I} & \text{if} \;\; (\alpha,\beta) \in \mathcal{B}_0 & \text{and} & (\gamma,\delta) \in \mathcal{A}_1^1 \cup \mathcal{A}_1^2; \\
\text{sharp of type II} & \text{if} \;\; (\alpha,\beta) \in \mathcal{A}_0,\ \beta \le 2-p & \text{and} & (\gamma,\delta) \in \mathcal{B}_1^1 \cup \mathcal{B}_1^2; \\
\text{sharp of type III} & \text{if} \;\; (\alpha,\beta) \in \mathcal{B}_0 & \text{and} & (\gamma,\delta) \in \mathcal{B}_1^1 \cup \mathcal{B}_1^2.
\end{array}$$

If $c \ge (p')^{1/p'}\, p^{1/p}\, f_1^{1/p'}$, we obtain front-type and type II sharp solutions for all $(\alpha,\beta) \in \mathcal{A}_0$, i.e., also for $\beta > 2-p$.

Since $d \in C(0,1)$, we have $U \in C^1(z_0, z_1)$ and it follows from Remark 5.21 that front-type solutions as well as sharp of type II solutions are differentiable for all $z \in \mathbb{R}$. Moreover, if $c \ge (p')^{1/p'}\, p^{1/p}\, f_1^{1/p'}$, we conclude from Remark 5.24 that all types of profiles are C^1-functions on $\mathbb{R}$. Therefore, in this case, Figure 5.4 depicts accurately the derivatives of U at z_0 and z_1.

Chapter 6

The influence of convection in the case of combustion nonlinearity

In this chapter, we present our most recent results concerning the appearance of travelling waves in a scalar reaction-diffusion-convection equation with p-Laplacian type diffusion and combustion-type reaction.

The notation in this section mostly coincides with that used in the previous text except for some minor changes. The diffusion coefficient is denoted D instead of d to emphasize the fact that different assumptions are considered for the problem with convection. Finally, for convenience reasons, the analogue of interval (z_0, z_1) is now denoted as (z_1, z_2).

6.1 Reaction-diffusion-convection equation

Let us consider the reaction-diffusion-convection equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u)\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right] + \frac{\partial H(u)}{\partial x} + g(u). \tag{6.1}$$

Here $p > 1$, D is a density-dependent diffusion coefficient with $D > 0$ in $(0, 1)$, H represents a nonlinear convective flux function, and g is a combustion-type reaction term, i.e.,

$$g(s) = 0 \text{ in } [0, s_*], \quad g(s) > 0 \text{ in } (s_*, 1), \quad g(1) = 0 \tag{6.2}$$

for some $s_* \in (0, 1)$. Our assumptions on the regularity of these functions will be specified in the next section.

We are concerned with the existence and properties of travelling wave solutions $u(x, t) = U(x - ct)$, $c \in \mathbb{R}$, which connect the stationary states 0 and 1. Clearly, if H is constant, equation (6.1) reduces to the previously discussed reaction-diffusion equation (2.1). Our aim is to investigate how the additional transport term H affects the existence of travelling waves.

Let us start with a brief summary of results in the case $p = 2$, which can be found in [37]. There, the authors assume $H \in C^1[0, 1]$, strictly positive $D \in C^1[0, 1]$ and different types of reaction $g \in C[0, 1]$. For further details and discussions, we refer to the papers [31, 36, 32], on which the survey [37] is based.

For monostable equations, i.e., when g is a type A function, the presence of convective processes does not affect the existence of a continuum of admissible wave speeds. Instead, it simply causes a "shift" of the threshold value c^*, which, consequently, need not be positive. In the bistable case, the situation might change considerably compared to the case without convection. In particular, if g is a type C function, it was shown in [36] that it is possible to have connections from 0 to s_* and from s_* to 1 with the same wave speed, which causes the disappearance of the travelling wave connecting 0 and 1. This phenomenon depends on the behaviour of the product $D(s)g(s)$ at s_*. In particular, it occurs when the derivative of this product vanishes at s_*. Otherwise, there exists a unique wave speed c_* (as in the absence of convection), but this value might be again shifted due to the convective term. Finally, if g is a type B function, i.e., it satisfies (6.2), the travelling wave solution might also disappear, but for a different reason than in the case of type C functions. Now, this phenomenon is linked to the threshold value s_* and the convective term H. In particular, if $H(s_*)$ is "large" with respect to the terms D and g (cf. Section 6.3), no travelling waves exist.

Let us also mention the case of degenerate diffusion $D \in C^1[0,1]$ in monostable equations. In [34], the existence of travelling waves is proved assuming $D(0) = 0$ and $D(1) \geq 0$, instead of $D > 0$ in $[0,1]$. Furthermore, the authors also investigate "sharpness" of the wave profiles (cf. Remark 4.2), providing detailed classification for cases $D'(0) \neq 0$ and $D'(0) = D'(1) = 0$. In the absence of convection, the profiles are either of front-type (when $c > c^*$) or sharp of type I (when $c = c^*$), regardless of whether $D(1) > 0$ or $D(1) = 0$. In other words, the possible degeneracy at 1 does not cause any other sharpness phenomenon. However, the same is not true for reaction-diffusion-convection equations. Now the double degeneracy can determine the appearance of solutions which are sharp of type II or III.

In this chapter, we study how the results for $p = 2$ and combustion nonlinearity (6.2) extend to equations with p-Laplacian type diffusion. Furthermore, we impose weaker assumptions on D and g than those in [32], although, for technical purposes, less general than the assumptions made in Chapter 2, cf. Remark 6.6.

6.2 Preliminaries

We consider the equation (6.1) with $p > 1$ and assume that the functions D, H and g have the following properties: $D \in C^1(0,1)$, $D > 0$ in $(0,1)$, $H \in C^1[0,1]$, $g \in C[0,1]$ is Lipschitz continuous in $[s_*, 1)$ and satisfies (6.2). Notice that the diffusion coefficient D might degenerate or have a singularity at one or both endpoints 0, 1. Without loss of generality, we further assume that $H(0) = 0$ and write

$$H(U) \coloneqq \int_0^U h(s)\,\mathrm{d}s,$$

where $h(U) = \frac{\mathrm{d}}{\mathrm{d}U} H(U)$, $U \in [0,1]$, is the convective velocity.

Formally substituting $u(x,t) = U(z)$ with $z = x - ct$ into (6.1) yields the ordinary differential equation

$$\big(D(U(z))|U'(z)|^{p-2}U'(z)\big)' + (c + h(U(z)))\,U'(z) + g(U(z)) = 0, \quad z \in \mathbb{R},$$

where primes denote differentiation with respect to the wave coordinate z. This equation is autonomous, hence its solutions are invariant under translations and we can normalize them by $U(0) = s_*$. As in the pure reaction-diffusion case, we look for travelling wave profiles U which satisfy boundary conditions $U(-\infty) = 1$, $U(+\infty) = 0$, i.e., we consider the following boundary value problem on the real line

$$\begin{cases} \left(D(U)|U'|^{p-2}U'\right)' + (c+h(U))\,U' + g(U) = 0, \\ \lim\limits_{z\to-\infty} U(z) = 1, \qquad \lim\limits_{z\to+\infty} U(z) = 0 \end{cases} \tag{6.3}$$

Equations involving the p-Laplacian need not have classical solutions, as the second derivative may not exist in general. They are also known for their finite property, meaning that the solution of (6.1) might reach 0 in finite time. Apart from this fact, possible degenerations and singularities of the diffusivity D also require us to adopt a more general concept of solution U to (6.3).

Definition 6.1. Let $I_U = \{z \in \mathbb{R} : 0 < U(z) < 1\}$. A continuous function $U : \mathbb{R} \to [0,1]$ is a solution of (6.3) if

(a) $U \in C^1(I_U)$ and the equation in (6.3) holds at every point of I_U;

(b) the function $z \mapsto D(U(z))|U'(z)|^{p-2}U'(z)$ is continuous on $\mathbb{R}$ and

$$D(U(z))|U'(z)|^{p-2}U'(z) \to 0 \quad \text{as } U(z) \to 0 \ \text{ and } U(z) \to 1;$$

(c) (boundary conditions) $U(z) \to 1$ as $z \to -\infty$ and $U(z) \to 0$ as $z \to +\infty$.

Remark 6.2. It is not difficult to see that the above definition is a simpler form of Definition 2.2 in Chapter 2 with $M_U = \emptyset$. In addition, it explicitly contains the boundary conditions from (6.3). Below we will show that I_U is in fact an open interval and that $U'(z) < 0$ for all $z \in I_U$. If $p = 2$, $D \in C^1[0,1]$ and g is a Lipschitz function in $[0,1]$, then $I_U = \mathbb{R}$ and $U \in C^2(\mathbb{R})$ is a classical solution, cf. [32].

Similarly as in the absence of convection, we would like to establish equivalence of the b.v.p. (6.3) with a first-order one on a bounded interval. To do so, we first derive the following analogue of Proposition 2.10.

Proposition 6.3. *Let U be a solution of* (6.3). *There exist $-\infty \le z_1 < z_2 \le +\infty$ such that $U \equiv 1$ in $(-\infty, z_1]$, $U \equiv 0$ in $[z_2, +\infty)$ and $U'(z) < 0$ for any $z \in (z_1, z_2)$.*

Before proceeding to the proof, we note that it relies on two auxiliary lemmas, which are presented afterward for the sake of clarity in exposition.

Proof. First we show that the derivative of a solution to (6.3) does not vanish in the set I_U. Indeed, let $z_0 \in I_U$ be such that $0 < U(z_0) \le \theta$. If $U'(z_0) = 0$ then it follows from Lemma 6.4 that the boundary conditions in (6.3) are not satisfied, a contradiction. Now consider $z_0 \in I_U$, $s_* < U(z_0) < 1$ with $U'(z_0) = 0$. Then

$$\left(D(U(z)|U'(z)|^{p-2}U'(z))\right)'\Big|_{z=z_0} = -g(U(z_0)) < 0.$$

It follows from Lemma 6.5 that z_0 must be the point of strict local maximum of U and therefore $\lim\limits_{z\to-\infty} U(z) \neq 1$, again a contradiction.

Next we prove that $U'(z) < 0$ for all $z \in I_U$, i.e., the solution cannot "switch" from 0 to 1 and back again finitely many times (while still satisfying the boundary conditions).

To this end, we observe that $c > -H(1)$ is a necessary condition for the existence of solution to (6.3). Indeed, integrating the equation in (6.3) we obtain

$$D(U(z))|U'(z)|^{p-2}U'(z) - D(U(\hat{z}))|U'(\hat{z})|^{p-2}U'(\hat{z}) + c(U(z) - U(\hat{z})) \\ + H(U(z)) - H(U(\hat{z})) + \int_{\hat{z}}^{z} g(U(\xi))\,\mathrm{d}\xi = 0, \quad z, \hat{z} \in \mathbb{R}.$$

Passing to the limits $z \to +\infty$, $\hat{z} \to -\infty$ and taking into account parts (b) and (c) of Definition 6.1 yields

$$c + H(1) - H(0) = \int_{-\infty}^{+\infty} g(U(\xi))\,\mathrm{d}\xi.$$

Since $H(0) = 0$ and the integral on the left-hand side is positive, we conclude that $c > -H(1)$.

Suppose that there exist $\underline{z}, \bar{z} \in \mathbb{R}$ such that $U(\underline{z}) = 0$, $U(\bar{z}) = 1$ and $U'(z) > 0$ for all $z \in (\underline{z}, \bar{z})$. Integrating the equation in (6.3) from $\underline{z}$ to $\bar{z}$ and employing the same arguments as above, we arrive at

$$c + H(1) = -\int_{\underline{z}}^{\bar{z}} g(U(\xi))\,\mathrm{d}\xi < 0,$$

i.e., $c < -H(1)$, a contradiction.

Therefore, there exist $-\infty \leq z_1 < z_2 \leq +\infty$ such that $U \equiv 1$ in $(-\infty, z_1]$, $U \equiv 0$ in $[z_2, +\infty)$ and $U'(z) < 0$ for any $z \in (z_1, z_2)$. This concludes the proof. □

The following lemmas were used in the proof of Proposition 6.3.

Lemma 6.4. *Let $U \in C^1(\mathbb{R})$ be a solution of the initial value problem*

$$\begin{cases} (D(U)|U'|^{p-2}U')' = -\left(c + h(U)\right) U', \\ U(z_0) = U_0 \in (0,1),\ U'(z_0) = 0. \end{cases} \tag{6.4}$$

Then U does not verify part (c) of Definition 6.1.

Proof. Integrating the equation in (6.4) and using the initial conditions yields

$$D(U(z))|U'(z)|^{p-2}U'(z) = c(U_0 - U(z)) + H(U_0) - H(U(z)), \quad z \in \mathbb{R}. \tag{6.5}$$

Put

$$\mathcal{S}_p(\nu) := |\nu|^{p-2}\nu \ \text{ for } \ \nu \neq 0,\ \mathcal{S}_p(0) = 0, \quad p > 1.$$

Since $\mathcal{S}_{p'}$ is the inverse function to $\mathcal{S}_p$, equation (6.5) is for $D(U(z)) \neq 0$ equivalent to

$$U'(z) = \mathcal{S}_{p'}\left(\frac{1}{D(U(z))}\left[c(U_0 - U(z)) + H(U_0) - H(U(z))\right]\right). \tag{6.6}$$

If $1 < p \leq 2$ then, due to $D \in C^1(0,1)$ and $H \in C^1[0,1]$, the right-hand side of (6.6) is Lipschitz continuous in U. Hence $U(z) = U_0$, $z \in \mathbb{R}$, is a unique solution of (6.4) in $\mathbb{R}$, and therefore does not verify part (c) of Definition 6.1.

If $p > 2$, i.e., $1 < p' < 2$, then the right-hand side of (6.6) is not Lipschitz continuous only at one point $U = U_0$, but it is one-sided Lipschitz continuous there. Therefore, either $U(z) = U_0$, $z \in (-\infty, z_0]$ is a unique solution of (6.4) in $(-\infty, z_0]$, or $U(z) = U_0$, $z \in [z_0, +\infty)$, is a unique solution of (6.4) in $[z_0, +\infty)$. In either case, part (c) of Definition 6.1 is not satisfied. □

Lemma 6.5. *Let U be a solution of* (6.3) *and let $z_0 \in \mathbb{R}$ be such that $U(z_0) \in (0,1)$, $U'(z_0) = 0$ and*

$$\big(D(U(z))|U'(z)|^{p-2}U'(z)\big)'\Big|_{z=z_0} < 0.$$

Then U has a strict local maximum at z_0.

Proof. We have

$$0 > \big(D(U(z))|U'(z)|^{p-2}U'(z)\big)'\Big|_{z=z_0} = \frac{\mathrm{d}D}{\mathrm{d}U}\Big|_{U=U(z_0)} \underbrace{|U'(z_0)|^p|}_{=0} + D(U(z_0))\left(|U'(z)|^{p-2}U'(z)\right)'\Big|_{z=z_0}.$$

Since $D(U(z_0)) > 0$, we get $(|U'(z)|^{p-2}U'(z))'\big|_{z=z_0} < 0$, and therefore, $|U'(z)|^{p-2}U'(z)$ is strictly decreasing in z_0 and equal to 0 at $z = z_0$. Since the power $\mathcal{S}_p(\nu) = |\nu|^{p-2}\nu$ is strictly increasing, $U'(z)$ is strictly decreasing at $z = z_0$. Hence z_0 is the point of strict local maximum of U. □

It follows from Proposition 6.3 that $I_U = (z_1, z_2)$, i.e., I_U is an open interval, bounded or unbounded. As in Section 2.3, we now follow the substitutions from [22] and set

$$-w(U) \coloneqq D(U)|U'|^{p-2}U'. \tag{6.7}$$

Since $U'(z) < 0$ for all $z \in (z_1, z_2)$, we have $w = w(U) > 0$ in $(0,1)$ and w satisfies

$$\frac{1}{p'D^{p'-1}(U)}\frac{\mathrm{d}}{\mathrm{d}U}w^{p'}(U) - (c + h(U))\left(\frac{w(U)}{D(U)}\right)^{p'-1} + g(U) = 0, \quad U \in (0,1).$$

where $p' = \frac{p}{p-1}$ is the exponent conjugate. Put

$$y(U) \coloneqq w^{p'}(U) > 0$$

and write t instead of U. Then $y = y(t)$ solves

$$y'(t) = p'\left[(c + h(t))(y^+(t))^{\frac{1}{p}} - f(t)\right], \quad t \in (0,1), \tag{6.8}$$

where $f(t) \coloneqq D^{p'-1}(t)g(t)$. In terms of y, part (b) of Definition 6.1 translates to

$$y(0) = y(1) = 0. \tag{6.9}$$

It follows from (6.7) that

$$\frac{\mathrm{d}z}{\mathrm{d}U} = \left(\frac{D(U)}{w(U)}\right)^{p'-1}$$

and therefore

$$z(U) = -\int_{s_*}^{U}\left(\frac{D(s)}{w(s)}\right)^{p'-1}\mathrm{d}s = -\int_{s_*}^{U}\frac{(D(s))^{p'-1}}{(y(s))^{\frac{1}{p}}}\mathrm{d}s, \quad U \in (0,1). \tag{6.10}$$

Since $z = z(U)$ maps $(0,1)$ onto (z_1, z_2), we have

$$z_1 = -\int_{s_*}^{1}\frac{(D(s))^{\frac{1}{p-1}}}{(y(s))^{\frac{1}{p}}}\mathrm{d}s \quad \text{and} \quad z_2 = \int_{0}^{s_*}\frac{(D(s))^{\frac{1}{p-1}}}{(y(s))^{\frac{1}{p}}}\mathrm{d}s. \tag{6.11}$$

It follows from the above calculations that the existence of a monotone solution to (6.3) implies the existence of a positive solution to (6.8), (6.9) which, in addition, satisfies (6.11). Actually, it is possible to show that these problems are equivalent, cf. Section 2.3. This allows us to derive existence and non-existence results for (6.3) by investigating existence and non-existence of positive solutions to (6.8), (6.9).

Remark 6.6. Let us note that the results presented in the following sections could be derived under weaker assumptions on D and g. In particular, we may assume $D \in C(0,1)$ and $g \in C[0,1]$. By appropriately modifying Definition 6.1, functions D and h with jump discontinuities in $(0,1)$ could also be considered. However, the assertion in Proposition 6.3 would not hold due to the lack of uniqueness of the associated Cauchy problem. Therefore, as in Section 2.3, we would need to assume the monotonicity property of solutions in order to transform the second-order problem (6.3) into a first-order one.

6.3 Non-existence results

In what follows, we denote

$$h_m := \min_{s\in[0,1]} h(s),$$

and assume that the integral

$$\int_0^1 (D(s))^{\frac{1}{p-1}} g(s)\,\mathrm{d}s$$

exists finite.

Theorem 6.7 (Non-existence)**.** *Let*

$$H(s_*) \geq s_* h_m + \left(p' \int_0^1 (D(s))^{\frac{1}{p-1}} g(s)\,\mathrm{d}s\right)^{1/p'}. \tag{6.12}$$

Then the b.v.p. (6.3) *has no solution for any* $c > -h_m$*. If strict inequality holds in* (6.12)*, there is no solution for any* $c \geq -h_m$*.*

Proof. It suffices to show that the first-order b.v.p. (6.8), (6.9) does not admit positive solutions for the given values of c.

Assume by contradiction that $c > -h_m$ and $y_c = y_c(t)$ is a positive solution of (6.8), (6.9). Integrating the equation (6.8) over $(s_*, 1)$ and using (6.9) yields

$$y_c(s_*) = -p' \int_{s_*}^1 (c + h(\tau))\,(y_c(\tau))^{\frac{1}{p}}\,\mathrm{d}\tau + p' \int_{s_*}^1 f(\tau)\,\mathrm{d}\tau < p' \int_0^1 f(\tau)\,\mathrm{d}\tau, \tag{6.13}$$

where $f(t) = D^{\frac{1}{p-1}}(t)g(t)$. On the other hand, since $f \equiv 0$ on $(0, s_*)$ the equation (6.8) is separable on $(0, s_*)$. Using (6.9) we obtain

$$(y_c(s_*))^{\frac{1}{p'}} = cs_* + H(s_*). \tag{6.14}$$

It follows from (6.13), (6.14) and the condition (6.12) that

$$\left(p' \int_0^1 f(\tau)\,\mathrm{d}\tau\right)^{1/p'} > (y_c(s_*))^{\frac{1}{p'}} = cs_* + H(s_*) > -h_m s_* + H(s_*) \geq \left(p' \int_0^1 f(\tau)\,\mathrm{d}\tau\right)^{1/p'},$$

a contradiction.

Assuming strict inequality in (6.12) and $c \geq -h_m$, we would arrive at

$$\left(p' \int_0^1 f(\tau)\,\mathrm{d}\tau\right)^{1/p'} \geq (y_c(s_*))^{\frac{1}{p'}} = cs_* + H(s_*) \geq -h_m s_* + H(s_*) > \left(p' \int_0^1 f(\tau)\,\mathrm{d}\tau\right)^{1/p'},$$

again a contradiction. This concludes the proof. □

We notice that

$$c \geq -h(0) \tag{6.15}$$

is a necessary condition for the existence of a positive solution of (6.8), (6.9). Indeed, if $c < -h(0)$ then, by the continuity of h, there exists $\delta > 0$ such that $c < -h(u)$ for all $U \in [0, \delta]$. Integrating the equation (6.8) over $[0, \delta]$ and using $y_c(0) = 0$ together with $c + h(u) < 0$ in $[0, \delta]$, we arrive at

$$y_c(\delta) = p' \int_0^\delta (c + h(\tau))(y_c^+(\tau))^{\frac{1}{p}}\,\mathrm{d}\tau < 0,$$

a contradiction with the positivity of solution $y_c = y_c(t)$.

Taking into account the necessary condition (6.15) in Theorem 6.7, we obtain the following corollary, which addresses the non-existence of a wave profile for any $c \in \mathbb{R}$.

Corollary 6.8. *If strict inequality holds in* (6.12) *and* $h_m = h(0)$, *then* (6.3) *has no solution for any* $c \in \mathbb{R}$.

6.4 Existence results

Let

$$k = k(p) = \begin{cases} \frac{1}{2^{p'-1}-1} & \text{if } 1 < p < 2, \\ 1 & \text{if } p = 2, \\ \dfrac{p'}{p'-1+\frac{1+p'(p'-1)^{\frac{1}{p'-2}}+(p'-1)^{\frac{p'}{p'-2}}}{\left(1+(p'-1)^{\frac{1}{p'-2}}\right)^{p'}}} & \text{if } p > 2. \end{cases} \tag{6.16}$$

Then $k = k(p)$ is a continuous function in $(1, +\infty)$ and

$$\lim_{p \to 1+} k(p) = 0 \quad \text{and} \quad \lim_{p \to +\infty} k(p) = \frac{1}{2}.$$

Theorem 6.9 (Existence)**.** *Let*

$$H(1) \leq h_m + \left(k(p) \int_0^1 (D(s))^{\frac{1}{p-1}} g(s)\,\mathrm{d}s\right)^{1/p'}. \tag{6.17}$$

Then there exists a unique $c = c_* > -h_m$ *such that the b.v.p.* (6.3) *has a unique (up to translation) solution* $U = U(z)$. *Moreover, the solution* U *is strictly decreasing on* I_U *and* c_* *satisfies*

$$c_* < \frac{1}{s_*}\left[\left(p' \int_0^1 (D(s))^{\frac{1}{p-1}} g(s)\,\mathrm{d}s\right)^{1/p'} - H(s_*)\right] - h_m. \tag{6.18}$$

Proof. We first prove the statement of Theorem 6.9 assuming that $h_m = \min_{s\in[0,1]} h(s) = 0$, i.e., we will show that if

$$H(1) \le \left(k(p) \int_0^1 (D(s))^{\frac{1}{p-1}} g(s)\,\mathrm{d}s\right)^{1/p'},$$

then there exists a unique positive value $c = c^*$ for which (6.3) admits a solution. This result can then be applied to the case of a more general $h \in C[0,1]$ with $h_m \neq 0$ by means of a suitable shift, discussed at the end of the proof.

Thanks to the equivalence established in Section 6.2, we proceed by investigating the initial value problem

$$\begin{cases} y_c'(t) = p'\left[(c+h(t))(y_c^+(t))^{\frac{1}{p}} - f(t)\right], & t \in (0,1), \\ y_c(1) = 0. \end{cases} \tag{6.19}$$

Let $c \ge 0$. Since $c + h(t) \ge 0$ for all $t \in [0,1]$, the function

$$y \mapsto (c+h(t))(y^+)^{\frac{1}{p}}, \quad t \in [0,1],$$

satisfies one-sided Lipschitz condition and it follows from Lemma 3.1, where we replace c by $c+h(t)$, that (6.19) has a unique global solution $y_c = y_c(t)$ defined on $[0,1]$. Our aim is to show that there exists $c > 0$ such that $y_c(t) > 0$ for $t \in (0,1)$ and $y_c(0) = 0$.

First, let us observe that $f(t) > 0$ in $(s_*, 1)$ implies that

$$y_c(t) > 0 \quad \text{for} \quad t \in (s_*, 1), \tag{6.20}$$

and

$$y_c(s_*) = -p' \int_{s_*}^1 (c+h(\tau))(y_c(\tau))^{\frac{1}{p}}\,\mathrm{d}\tau + p' \int_{s_*}^1 f(\tau)\,\mathrm{d}\tau < p' \int_0^1 f(\tau)\,\mathrm{d}\tau. \tag{6.21}$$

According to Lemma 6.12 (see the end of this section), for any $p > 1$ we have

$$(y_0(s_*))^{\frac{1}{p'}} > H(s_*). \tag{6.22}$$

In particular, $y_0(s_*) > 0$ and hence there exists $0 < \delta \le s_*$ such that $y_c(t) > 0$ for $t \in (s_* - \delta, s_*)$. Since $f \equiv 0$ on $(0, s_*)$, $y_0 = y_0(t)$ solves the equation

$$y_0'(t) = p' h(t)(y_0(t))^{\frac{1}{p}}, \quad t \in (s_* - \delta, s_*).$$

Separating variables, we obtain for $t \in (s_* - \delta, s_*)$

$$(y_0(s_*))^{\frac{1}{p'}} - (y_0(t))^{\frac{1}{p'}} = H(s_*) - H(t),$$

i.e.,

$$(y_0(t))^{\frac{1}{p'}} - H(t) = (y_0(s_*))^{\frac{1}{p'}} - H(s_*) > 0$$

by (6.22). It follows that $\delta = s_*$ and

$$(y_0(t))^{\frac{1}{p'}} > 0 \quad \text{for all } t \in [0, s_*].$$

Therefore,

$$y_0(t) > 0 \quad \text{for all} \;\; t \in [0,1). \tag{6.23}$$

Set

$$c_* := \sup\{c > 0 : y_c(t) > 0 \text{ for all } t \in (0,1)\}.$$

It follows from (6.20), (6.23) and the continuous dependence of the solution to (6.19) on the parameter c that the set $\{c > 0 : y_c(t) > 0 \text{ for all } t \in (0,1)\}$ is non-empty and $c_* > 0$. If $c_* = +\infty$ then there exist $c_n \to +\infty$ and corresponding $y_{c_n} = y_{c_n}(t) > 0$, $t \in (0,1)$, which satisfy

$$y'_{c_n}(t) = p'(c_n + h(t))(y_{c_n}(t))^{\frac{1}{p}}, \quad t \in (0, s_*).$$

Separating variables yields

$$(y_{c_n}(t))^{\frac{1}{p'}} = (y_{c_n}(s_*))^{\frac{1}{p'}} + c_n(t - s_*) + H(t) - H(s_*), \quad t \in (0, s_*), \tag{6.24}$$

and from (6.21) we get

$$y_{c_n}(s_*) < p' \int_0^1 f(\tau)\,\mathrm{d}\tau < +\infty.$$

Therefore, the right-hand side in (6.24) tends to $-\infty$, a contradiction. Hence

$$0 < c_* < +\infty.$$

Next we prove that $y_{c_*}(t) > 0$, $t \in (0,1)$, $y_{c_*}(0) = 0$. Indeed, by the continuous dependence of (6.19) on the parameter c and the definition of c_*, the solution $y_{c_*} = y_{c_*}(t)$ must vanish somewhere in the interval $[0, s_*]$. Let $\eta \in [0, s_*]$ be the largest zero of y_{c_*}. It follows from the comparison argument that solutions of (6.19) decrease with c. This can be easily shown as in Lemma 3.5 and Corollary 3.6 by replacing c with $c + h(t)$. If $\eta > 0$ then for $c < c_*$ we have $y_c(t) > 0$ on $(0, \eta)$ and hence from

$$y'_c(t) = p'(c + h(t))(y_c(t))^{\frac{1}{p}}, \quad t \in (0, \eta),$$

we again deduce

$$0 < (y_c(t))^{\frac{1}{p'}} = (y_c(\eta))^{\frac{1}{p'}} + c(t - \eta) + H(t) - H(\eta). \tag{6.25}$$

Since for $c \to c_*$ we have $y_c(\eta) \to y_{c_*}(\eta) = 0$ by continuous dependence on parameter, for any fixed $t \in (0, \eta)$ there exists $c < c_*$, $(c_* - c)$ sufficiently small, such that

$$(y_c(\eta))^{\frac{1}{p'}} + c(t - \eta) + H(t) - H(\eta) < 0$$

($h_m = 0$ implies that H is nondecreasing), which contradicts (6.25). Hence $\eta = 0$.

Finally, we show that positive solutions of (6.19) do not vanish at 0 for values of $c \neq c_*$. Assume by contradiction that there exists $\hat{c} \neq c_*$ such that $y_{\hat{c}} = y_{\hat{c}}(t) > 0$ solves (6.19) in $(0,1)$, $y_{\hat{c}}(0) = 0$. The definition of c_* yields $\hat{c} < c_*$. Separating variables in the equation in (6.19) on $(0, s_*)$, we obtain

$$(y_{\hat{c}}(s_*))^{\frac{1}{p'}} = \hat{c}s_* + H(s_*)$$

and also

$$(y_{c_*}(s_*))^{\frac{1}{p'}} = c_* s_* + H(s_*). \tag{6.26}$$

Hence $y_{\hat{c}}(s_*) < y_{c_*}(s_*)$. On the other hand, the comparison argument applied to (6.19) yields $y_{\hat{c}}(t) \geq y_{c_*}(t)$, $t \in [0,1]$. This follows from Corollary 3.6 with $c_1 = \hat{c} + h(t)$ and $c_2 = c_* + h(t)$. In particular, $y_{\hat{c}}(s_*) \geq y_{c_*}(s_*)$, a contradiction.

It follows from (6.26) together with (6.21) that

$$c_* = \frac{1}{s_*}\left((y_{c_*}(s_*))^{\frac{1}{p'}} - H(s_*)\right) < \frac{1}{s_*}\left[\left(p'\int_0^1 (D(s))^{\frac{1}{p-1}} g(s)\,\mathrm{d}s\right)^{1/p'} - H(s_*)\right],$$

i.e., (6.18) holds. This concludes the proof for $h_m = 0$.

If $h_m \neq 0$, we can consider a new convective velocity $\tilde{h}(s) := h(s) - h_m$, $s \in [0,1]$. Then $\tilde{h}_m := \min_{s\in[0,1]} \tilde{h}(s) = 0$ and $\tilde{H}(U) = \int_0^U \tilde{h}(s)\,\mathrm{d}s = H(U) - h_m U$ is a nondecreasing function. Setting $\tilde{c} := c + h_m$, the equation in (6.3) becomes

$$\left(D(U)|U'|^{p-2}U'\right)' + (\tilde{c} + \tilde{h}(U))U' + g(U) = 0$$

and we can apply the above reasoning to prove the existence of a unique positive value $\tilde{c}_*$ assuming that

$$\tilde{H}(1) \leq \left(k(p)\int_0^1 (D(s))^{\frac{1}{p-1}} g(s)\,\mathrm{d}s\right)^{1/p'}.$$

Since $\tilde{H}(1) = H(1) - h_m$, we immediately see that condition (6.17) yields a unique value $c_* = \tilde{c}_* - h_m > -h_m$, corresponding to the problem with convective velocity h, and the estimate (6.18) holds. □

Notice that if $h_m \leq 0$, from Theorem 6.9 we immediately see that the unique wave speed c^* is positive. The following result addresses the existence of a positive wave speed in the case $h_m > 0$.

Theorem 6.10 (Positive wave speed c). *If $h(s) > 0$ for $s \in [0,1]$ and*

$$H(1) \leq \left(k(p)\int_0^1 (D(s))^{\frac{1}{p-1}} g(s)\,\mathrm{d}s\right)^{1/p'}, \tag{6.27}$$

then $c^* > 0 > -h_m$.

Proof. If $h_m > 0$, the proof can be carried out exactly as in the case $h_m = 0$. In particular, statements concerning the i.v.p. (6.19) remain valid and the positivity of h justifies the use of Lemma 6.12. Therefore, if (6.27) holds we conclude that $c^* > 0 > -h_m$. □

To establish the inequality (6.22) in the proof of Theorem 6.9, we employ the following lemma.

Lemma 6.11 (Technical inequalities). *Let $a > 0$, $b > 0$. Then*

(i) *for $r \geq 2$ we have*

$$a^r + ra^{r-1}b + b^r \leq (a+b)^r;$$

(ii) *for $1 < r < 2$ we have*

$$a^r + ra^{r-1}b + b^r \leq \hat{k}(r)(a+b)^r,$$

where

$$\hat{k}(r) = \frac{1 + r(r-1)^{\frac{1}{r-2}} + (r-1)^{\frac{r}{r-2}}}{\left(1 + (r-1)^{\frac{1}{r-2}}\right)^r}.$$

Proof. We put $t = \frac{b}{a} > 0$ and write the inequality in an equivalent form

$$f(t) := \frac{1 + rt + t^r}{(1+t)^r} \le \hat{k}(r),$$

where we set $\hat{k}(r) = 1$ for $r \ge 2$. Then the optimal choice for $\hat{k}(r)$ would be $\hat{k}(r) = \max_{t \ge 0} f(t)$, if this maximum exists. Indeed, it does. Namely, f is a continuously differentiable function on $[0, +\infty)$ satisfying $f(0) = 1 = \lim_{t \to +\infty} f(t)$. An elementary calculation yields that $t_1 = (r-1)^{\frac{1}{r-2}}$ is the only stationary point of f in $(0, +\infty)$.

Part (i). It is clear that equality holds for $r = 2$. Let $r > 2$. Then $f(1) = \frac{2+r}{2^r} < 1$. Hence $t_1 = (r-1)^{\frac{1}{r-2}}$ is the point of global minimum of f, $0 < f(t_1) \le f(1) < 1$ and therefore $\max_{t \ge 0} f(t) = f(0) = 1$.

Part (ii). Let $1 < r < 2$. Then $f(1) = \frac{2+r}{2^r} > 1$ and hence t_1 is the point of global maximum of f in $[0, +\infty)$ with

$$\hat{k}(r) = f(t_1) = \frac{1 + r(r-1)^{\frac{1}{r-2}} + (r-1)^{\frac{r}{r-2}}}{\left(1 + (r-1)^{\frac{1}{r-2}}\right)^r}.$$

□

Lemma 6.12 (Inequality (6.22))**.** *Assume that $h(t) \ge 0$ in $[0,1]$ and let $y_0 = y_0(t)$ be a solution of the i.v.p.* (6.19) *with $c = 0$. If*

$$H^{p'}(1) \le k(p) \int_0^1 f(t)\,\mathrm{d}t, \tag{6.28}$$

where $k = k(p)$ is given by (6.16)*, then*

$$(y_0(s_*))^{\frac{1}{p'}} > H(s_*).$$

Proof. We proceed by contradiction, that is, we assume that

$$(y_0(s_*))^{\frac{1}{p'}} \le H(s_*).$$

Since $f > 0$ on $(s_*, 1)$, it follows from the equation in (6.19) that $y_0(t) > 0$ for all $t \in (s_*, 1)$. Set $z(t) := (y_0(t))^{\frac{1}{p'}}$. Then $z(t) > 0$ in $(s_*, 1)$, $z(1) = 0$,

$$z(s_*) \le H(s_*) \tag{6.29}$$

and $z = z(t)$ satisfies the equation

$$[z^{p'}(t)]' = p'h(t)z^{p'-1}(t) - p'f(t), \quad t \in (s_*, 1), \tag{6.30}$$

or, equivalently,

$$z'(t) = h(t) - \frac{f(t)}{z^{p'-1}(t)}, \quad t \in (s_*, 1). \tag{6.31}$$

Integrating (6.30) and using the mean value theorem, we obtain

$$\begin{aligned} z^{p'}(s_*) &= z^{p'}(1) - p' \int_{s_*}^1 h(t) z^{p'-1}(t)\,\mathrm{d}t + p' \int_{s_*}^1 f(t)\,\mathrm{d}t \\ &= -p' z^{p'-1}(t_0)(H(1) - H(s_*)) + p' \int_0^1 f(t)\,\mathrm{d}t \end{aligned} \tag{6.32}$$

for some $t_0 \in (s_*, 1)$. From (6.31) we have

$$z(t_0) - z(s_*) = \int_{s_*}^{t_0} h(t)\,\mathrm{d}t - \int_{s_*}^{t_0} \frac{f(t)}{z^{p'-1}(t)}\,\mathrm{d}t < H(t_0) - H(s_*)$$

and hence

$$z(t_0) < z(s_*) + H(1) - H(s_*) \tag{6.33}$$

thanks to the monotonicity of H (in particular, $h \geq 0$ implies that H is nondecreasing). It follows from (6.32), (6.33) together with (6.29)

$$H^{p'}(s_*) > -p'\left(H(s_*) + [H(1) - H(s_*)]\right)^{p'-1}(H(1) - H(s_*)) + p' \int_0^1 f(t)\,\mathrm{d}t. \tag{6.34}$$

Next we proceed separately for $p = 2$, $1 < p < 2$ and $p > 2$.
Case 1: $p = 2$. Since $p' = 2$, (6.34) becomes

$$H^2(s_*) > -2\left(H(s_*) + [H(1) - H(s_*)]\right)(H(1) - H(s_*)) + 2\int_0^1 f(t)\,\mathrm{d}t.$$

Reorganizing the terms in the above inequality and using (6.28), we obtain

$$\begin{aligned}
0 &> -H^2(s_*) - 2H(s_*)(H(1) - H(s_*)) - (H(1) - H(s_*))^2 - (H(1) - H(s_*))^2 \\
&\quad + 2\int_0^1 f(t)\,\mathrm{d}t = -H^2(1) - (H(1) - H(s_*))^2 + 2\int_0^1 f(t)\,\mathrm{d}t \\
&\quad > 2\left(\int_0^1 f(t)\,\mathrm{d}t - H^2(1)\right) \geq 0,
\end{aligned}$$

a contradiction.
Case 2: $1 < p < 2$. Since $p' > 2$, we use the inequality

$$(a+b)^r \leq 2^{r-1}(a^r + b^r), \quad a, b > 0,\ r > 1,$$

with $a = H(s_*)$, $b = H(1) - H(s_*)$, $r = p' - 1$ in (6.34) and obtain

$$\begin{aligned}
H^{p'}(s_*) &> -p'\left(H(s_*) + [H(1) - H(s_*)]\right)^{p'-1}(H(1) - H(s_*)) + p'\int_0^1 f(t)\,\mathrm{d}t \\
&\geq -p'2^{p'-2}\left(H^{p'-1}(s_*) + [H(1) - H(s_*)]^{p'-1}\right)(H(1) - H(s_*)) + p'\int_0^1 f(t)\,\mathrm{d}t.
\end{aligned}$$

Hence

$$\begin{aligned}
0 >& - H^{p'}(s_*) - p'2^{p'-2}H^{p'-1}(s_*)(H(1) - H(s_*)) - p'2^{p'-2}\left(H(1) - H(s_*)\right)^{p'} + p'\int_0^1 f(t)\,\mathrm{d}t \\
&= -H^{p'}(s_*) - p'H^{p'-1}(s_*)(H(1) - H(s_*)) - (H(1) - H(s_*))^{p'} \\
&\quad + (1 - p'2^{p'-2})(H(1) - H(s_*))^{p'} + (p' - p'2^{p'-2})H^{p'-1}(s_*)(H(1) - H(s_*)) \\
&\quad + p'\int_0^1 f(t)\,\mathrm{d}t
\end{aligned}$$

and, using the inequality from Lemma 6.11 (i) with $a = H(s_*)$, $b = H(1) - H(s_*)$ and $r = p'$,

$$0 > -(H(s_*) + (H(1) - H(s_*)))^{p'} + (1 - p'2^{p'-2})(H(1) - H(s_*))^{p'}$$
$$+ (p' - p'2^{p'-2})H^{p'-1}(s_*)(H(1) - H(s_*)) + p' \int_0^1 f(t)\,\mathrm{d}t.$$

Then $0 \leq H(s_*) \leq H(1)$ implies

$$0 > -H^{p'}(1) + (1 - p'2^{p'-2})H^{p'}(1) + (p' - p'2^{p'-2})H^{p'}(1) + p' \int_0^1 f(t)\,\mathrm{d}t$$

and from (6.28) we conclude

$$0 > -p'(2^{p'-1} - 1)H^{p'}(1) + p' \int_0^1 f(t)\,\mathrm{d}t \geq 0,$$

a contradiction.

Case 3: $p > 2$. Since $1 < p' < 2$, we now use the inequality

$$(a + b)^r \leq a^r + b^r, \quad a, b > 0,\ 0 < r < 1,$$

with $a = H(s_*)$, $b = H(1) - H(s_*)$, $r = p' - 1$ in (6.34) and obtain

$$0 > -H^{p'}(s_*) - p' \left(H^{p'-1}(s_*) + [H(1) - H(s_*)]^{p'-1}\right)(H(1) - H(s_*)) + p' \int_0^1 f(t)\,\mathrm{d}t,$$

i.e.,

$$0 > -H^{p'}(s_*) - p'H^{p'-1}(s_*)(H(1) - H(s_*)) - p'\,(H(1) - H(s_*))^{p'} + p' \int_0^1 f(t)\,\mathrm{d}t,$$

or equivalently

$$0 > -H^{p'}(s_*) - p'H^{p'-1}(s_*)(H(1) - H(s_*)) - (H(1) - H(s_*))^{p'}$$
$$-(p' - 1)\,(H(1) - H(s_*))^{p'} + p' \int_0^1 f(t)\,\mathrm{d}t.$$

For $a > 0$, $b > 0$, $r \in (1, 2)$ we have $a^r + ra^{r-1}b + b^r \leq \hat{k}(r)(a + b)^r$ by the technical Lemma 6.11 (ii) below. We apply it with $a = H(s_*)$, $b = H(1) - H(s_*)$, $r = p'$:

$$0 > -\hat{k}(p')\,(H(s_*) + (H(1) - H(s_*)))^{p'} - (p' - 1)\,(H(1) - H(s_*))^{p'} + p' \int_0^1 f(t)\,\mathrm{d}t.$$

But (6.28) yields

$$0 > -(\hat{k}(p') + (p' - 1))H^{p'}(1) + p' \int_0^1 f(t)\,\mathrm{d}t \geq 0,$$

a contradiction. □

6.5 Asymptotic analysis of the wave profile

In this section, we discuss asymptotic behaviour of the solution $U = U(z)$ to (6.3) as $z \to \pm\infty$. Our aim is to determine whether the solution attains 0 and/or 1 (or neither of them). To this end, we study the convergence of the integrals from (6.11), and hence the finiteness of the values z_1, z_2. For technical reasons, we assume power-type behaviour of D and g near equilibria 0 and 1.

In what follows, we consider $H(U) > 0$, $U \in (0, 1]$, and profiles with $c_* > 0$. For the sake of brevity, we will use the following notation, introduced in Chapter 5: for $s_0 \in \mathbb{R}$ we write

$$\phi_1(s) \sim \phi_2(s) \text{ as } s \to s_0 \quad \text{if and only if} \quad \lim_{s\to s_0} \frac{\phi_1(s)}{\phi_2(s)} \in (0, +\infty).$$

Asymptotics near 0

Let us assume that $D(t) \sim t^\alpha$ as $t \to 0+$ for some $\alpha \in \mathbb{R}$. Thanks to $f \equiv 0$ in $[0, s_*]$, we have

$$(y_{c_*}(t))^{\frac{1}{p'}} = c_* t + H(t), \quad t \in (0, \theta),$$

and due to the assumption $H \in C^1[0, 1]$, $H > 0$ together with $c_* > 0$, we have $(y_{c^*}(t))^{\frac{1}{p'}} \sim t$ as $t \to 0+$. Let us recall that

$$z_2 = \int_0^{s_*} \frac{(D(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \,\mathrm{d}t.$$

Since

$$\int_0^U \frac{(D(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \,\mathrm{d}t \sim \int_0^U \frac{t^{\alpha(p'-1)}}{t^{p'-1}} \,\mathrm{d}t = \int_0^U t^{\frac{\alpha-1}{p-1}} \,\mathrm{d}t \quad \text{as } U \to 0+, \tag{6.35}$$

we conclude that the following two cases occur:

(a) $z_2 = +\infty$ if and only if $p + \alpha \le 2$;

(b) $z_2 < +\infty$ if and only if $p + \alpha > 2$,

see Figure 6.1 for geometric interpretation.

Observe that for any $\alpha > 1$, the profile $U = U(z)$ is always sharp of type I (cf. Figure 4.1), i.e., $z_2 \in \mathbb{R}$, $U \equiv 0$ in $[z_2, +\infty)$. If $\alpha = 0$ and $1 < p \le 2$, the profile does not attain 0 for any finite z. This result is consistent with that from [32] for $p = 2$ and $D \in C^1[0, 1]$ strictly positive in $[0, 1]$.

In case (b), we can also study the one-sided derivative $U'(z_2-)$. In particular, differentiating (6.10) yields

$$\frac{\mathrm{d}z}{\mathrm{d}U} = -\frac{\mathrm{d}}{\mathrm{d}U} \int_{s_*}^U \frac{(D(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \,\mathrm{d}t = -\frac{(D(U))^{\frac{1}{p-1}}}{(y_{c_*}(U))^{\frac{1}{p}}}, \quad U \in (0, 1).$$

Since $D(U) \sim U^\alpha$, $y(U) \sim U^{p'}$ as $U \to 0+$, we have

$$\left.\frac{\mathrm{d}z}{\mathrm{d}U}\right|_{U=0+} \sim -U^{\frac{\alpha-1}{p-1}} \to \begin{cases} 0 & \text{if } \alpha > 1 \\ \text{const.} < 0 & \text{if } \alpha = 1 \\ -\infty & \text{if } \alpha < 1 \end{cases} \quad \text{as } U \to 0+\,.$$

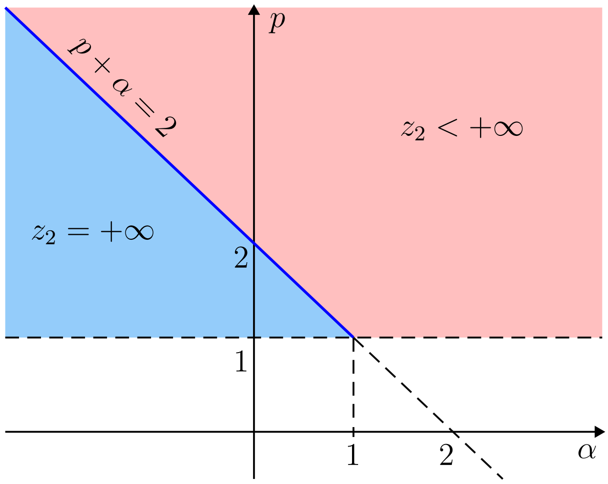

Figure 6.1: Visualization of cases (a) and (b), leading to z_2 finite or infinite

Employing an inverse perspective, we obtain the following classification for the profile $U = U(z)$:

$$U'(z_2-) = \begin{cases} -\infty & \text{if} \quad \alpha > 1, \\ \text{const.} < 0 & \text{if} \quad \alpha = 1, \\ 0 & \text{if} \quad \alpha < 1. \end{cases}$$

Therefore, if $p + \alpha > 2$ and $\alpha < 1$ we have $U'(z_2-) = U'(z_2+) = 0$.

Asymptotics near 1

Let us assume that $D(t) \sim (1-t)^\beta$ and $g(t) \sim (1-t)^\gamma$ as $t \to 1-$ for some $\beta \in \mathbb{R}$, $\gamma > 0$. Since the equation (6.8) is not separable on $(s_*, 1)$, the asymptotic analysis becomes more involved than in the previous case. However, we can apply the same reasoning as in Section 5.2.1, where we investigated asymptotic properties of solutions near 1 in the absence of convection. In fact, this technique yields the same results also when $h(t) \geq 0$ instead of $h \equiv 0$. Replacing c by $c + h(t)$ in the proofs of Theorems 5.7, 5.8, we derive the same conditions leading to $z_1 = -\infty$ and $z_1 > -\infty$. In our current notation, these theorems read as follows.

Theorem 6.13. *Let $\beta \in \mathbb{R}$, $\gamma > 0$ and $D(t) \sim (1-t)^\beta$, $g(t) \sim (1-t)^\gamma$ as $t \to 1-$. Assume that*

$$-1 < \gamma + \frac{\beta}{p-1} \leq \frac{1}{p-1}$$

If

$$\frac{\gamma - \beta + 1}{p} < 1,$$

then $z_1 > -\infty$. If

$$\frac{\gamma - \beta + 1}{p} \geq 1,$$

then $z_1 = -\infty$.

Theorem 6.14. *Let $\beta \in \mathbb{R}$, $\gamma > 0$ and $D(t) \sim (1-t)^\beta$, $g(t) \sim (1-t)^\gamma$ as $t \to 1-$. Assume that*

$$\gamma + \frac{\beta}{p-1} > \frac{1}{p-1}\,.$$

If $\gamma < 1$ then $z_1 > -\infty$. If $\gamma \geq 1$ then $z_1 = -\infty$.

For geometric interpretation, we refer to Figures 5.5 and 5.6 in Section 5.2.1, where we used δ instead of β. As in the case of $h \equiv 0$ discussed therein, the profile $U = U(z)$ is a C^1-function in the neighbourhood of $z_1 \in \mathbb{R}$, cf. Remark 5.10.

Conclusion

In this thesis, we studied travelling waves to a class of reaction-diffusion equations on the real line with a density-dependent diffusion involving the p-Laplacian. Our assumptions on the diffusion and reaction terms were motivated by applications as well as their mathematical generalizations, which were previously considered in literature. An overview of relevant models together with basic existence results was provided in Chapter 1.

The aim of our research was to generalize the well-known theory of admissible wave speeds for monostable and bistable equations by proving the existence of travelling waves under weakest possible assumptions. To this end, we developed a broad theoretical background in Chapter 2. A new definition of a continuous, generally non-smooth, travelling wave profile was introduced in Section 2.1, allowing us to consider a piecewise continuous diffusion coefficient with degenerations or singularities at 0 and 1. In the case of monostable reaction, we have shown in Section 2.2 that the profile is necessarily nonincreasing on $\mathbb{R}$ and strictly decreasing in some maximal interval. Restricting our attention to monotone solutions also in the bistable case, we were then able to follow the substitutions from [22] and reduce the second-order problem to a first-order one (Section 2.3).

Chapter 3 was devoted to the investigation of the equivalent first-order problem in the sense of Carathéodory, which in turn yields the existence of travelling waves to the original problem, presented in Chapter 4. The properties of wave profiles were further explored in Chapter 5, assuming power-type behaviour of the reaction and diffusion terms. Our findings on this topic also constitute one of the main contributions of our work. We derived conditions for the involved parameters which guarantee that the profile reaches (or does not reach) 0 and/or 1. The most detailed classification, which also specified the derivatives at the points of transition from the steady states, was obtained for stationary waves in Section 5.1. Travelling wave profiles were examined in Sections 5.2 and 5.3 for bistable and monostable reactions, respectively. Using suitable upper and lower solutions, we were able to provide a similar classification with less precise information about the smoothness of profiles when reaching 0. Interestingly, if a travelling wave profile reaches 1, it does so with a zero derivative, which is not the case for stationary waves.

In Chapter 6, we explored the effect of convective term on the existence of travelling waves to equations with a special type of nonlinear reaction arising in combustion. Our main results, presented in Theorems 6.7 and 6.9, reveal that if convection dominates over the reaction and diffusion (in the sense of condition (6.12)), the equation ceases to admit travelling wave solutions. Conversely, when the convective term is ''weak'' compared to diffusion and reaction (in the sense of condition (6.17)), a unique wave speed and corresponding profile exist, akin to the pure reaction-diffusion case.

While the general approach employed in this thesis enabled us extend the results obtained in more conventional settings and treat different types of reaction in a largely unified manner, it is, in a sense, too "coarse" to provide more detailed information about solutions. Some of the issues may possibly be addressed via different techniques, yet we expect some inherent complexity to hinder a more comprehensive study.

Despite this, there are many interesting directions for future research. Perhaps the most straightforward task is to study problems with convection and p-Laplacian type diffusion with other than combustion nonlinearities. Our general results derived for the first-order problem also suggest the possibility of considering a discontinuous reaction, which appears in some combustion models [5]. Recently, sign-changing diffusivity has been investigated in the case $p = 2$, see [7, 8], the inspiration for it being drawn from the modelling of collective movements, namely of vehicular flows and crowds dynamics. Another possible extension is to consider equations with diffusion driven by the ϕ-Laplacian operator:

$$u \mapsto \frac{\partial}{\partial x}\left[\phi\left(\frac{\partial u}{\partial x}\right)\right]$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$, see e.g. [12, 10]. Particular examples are for instance the classical p-Laplacian or the relativistic curvature operator

$$\phi(v) = \frac{v}{\sqrt{1 - v^2}}\,.$$

Finally, an interesting and important topic is the study of initial value problems for the equation (1.1). In general settings like ours, it may be too ambitious to prove the convergence of a solution $u(x,t)$ to a travelling wave as $t \to +\infty$. However, even numerical experiments might be of great help in gaining valuable insights.

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Appendix A

List of publications

Publications related to the topic of dissertation

Drábek, P., Zahradníková, M. Bistable equation with discontinuous density dependent diffusion with degenerations and singularities, *Electron. J. Qual. Theory Differ. Equ.* (2021), No. 61, 1–16.

Drábek, P., Zahradníková, M. Traveling waves for unbalanced bistable equations with density dependent diffusion, *Electron. J. Differ. Equ.* (2021), No. 76, 1–21.

Drábek, P., Zahradníková, M. Traveling waves for generalized Fisher–Kolmogorov equation with discontinuous density dependent diffusion, *Math. Meth. Appl. Sci.* 64 (2023), No. 11, 12064–12086.

Drábek, P., Zahradníková, M. Traveling waves in reaction-diffusion-convection equations with combustion nonlinearity, *submitted* (2023).
(Manuscript available at `https://ssrn.com/abstract=4613126`)

Other publications

Eisner, J., Zahradníková, M. Eigenvalues of Boundary Value Problems of Second Order with Jumping Nonlinearities, *2019 9th International Conference On Advanced Computer Information Technologies (ACIT'2019)*, České Budějovice, Czech Republic, 2019, pp. 149–152.

Eisner, J., Zahradníková, M. Unilaterally Supported Beam by an Elastic Obstacle, *2022 12th International Conference on Advanced Computer Information Technologies (ACIT'2022)*, Ružomberok, Slovakia, 2022, pp. 72–75.