

# 3D as a method of geometrical simulation (on the example of combination of a quadric and conic)

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## ABSTRACT

It is shown that on the example of the mentioned problem 3d-computer geometrical models allow in full to solve and investigate applied problems of geometrical simulation.

## Keywords

Geometrical simulation, 3d-computer technologies, AutoCAD.

## 1. INTRODUCTION

A peculiar feature of most applied problems of geometric simulation is that we can't obtain an explicit and theoretically exact projection or analytical solution to these problems. However, if the solution is found, as a rule it is connected with a number of assumptions, it is complicated and cumbersome, has a low clearness and therefore, doesn't allow us to carry out the necessary analysis of the model without further graphic explanation.

The existence of graphical editors, which allow us to create virtual realistic 3D geometric models, highly expands the possibilities of geometric simulation. On this basis a new method of 3D geometric simulation has appeared, combining with computer and software it allows us to study the model and obtain visual target characteristics without analytical and structural (projection) design.

The object of this article is to show the application of 3D method and its possibilities on the example of combination of a quadric and conic. The choice of this problem as an example is due to the availability of its solutions in particular cases [1-3].

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The author has complicated the problem by considering the combination of all types of conic and selectable quadric [4, 5]. A variant of this problem is given in this article: *let us combine the given hyperbola with the given one-sheet elliptic hyperboloid (OH) if the hyperbola must pass through a selectable and given point on the surface of the hyperboloid.*

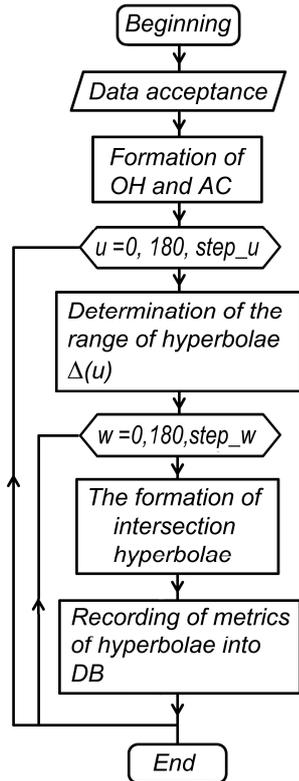
Defining the point as a parameter of the problem hasn't allowed us to obtain accurate geometric or analytical solution. Being a theoretical one, a considered problem demonstrates the possibility of new 3D methods to study and solve similar complicated applied problems.

## 2. METHOD OF STUDY

"Black box" method which is peculiar to experimental research is used to study the objects, internal structure of which is unknown. Algorithm of the solution (fig. 1) is to get a range of hyperbolae, located on the surface of  $OH$  and passing through the given point and to find hyperbola with the required metrics among them.

We shall build the model of  $OH$  (fig. 2) on the framework of two hyperbolae in the planes of symmetry and 10...20 cross-cut ellipsis. We will get hyperbolae to the framework as sections of elliptical cone [4]. We shall set out check point  $B$  on the surface of  $OH$ .

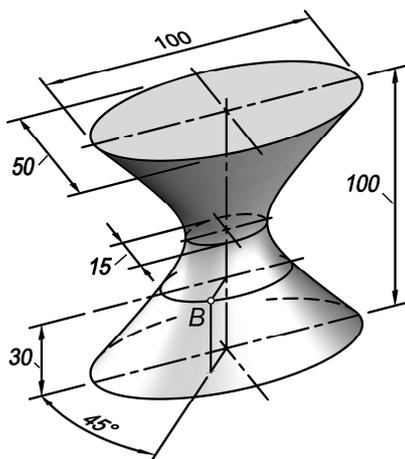
To create a range of hyperbolae we shall introduce a cutting plane  $\psi$  (fig. 3), performing a rotation around two axes passing through the given point  $B$ . The first rotation is around axis  $iI$ , which is parallel to axis  $i$  of



**Fig. 1. Block-diagram of an algorithm**

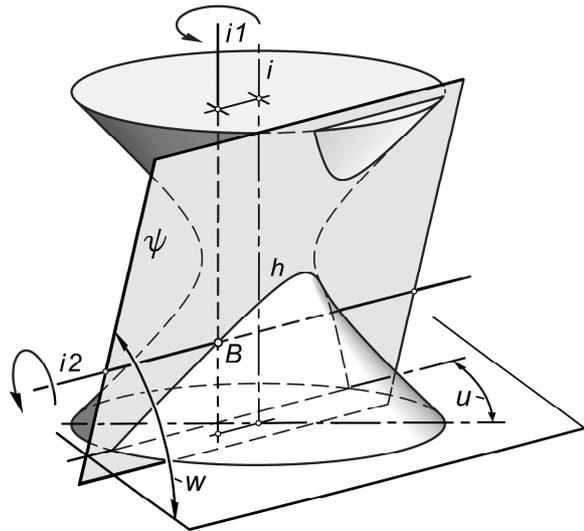
*OH*. The second rotation is around contour curve *i2*, which belongs to the plane  $\psi$  and rotating round *i1* with it. The position of plane  $\psi$  is set with the angle *u* and angle *w*.

Taking into consideration the symmetry of the model, we shall set the interval of each rotation at  $0 \dots 180^\circ$ .



**Fig. 2. Parameters of a model**

The rotary steps which are *step\_u* (at position *u*) and *step\_w* (at position *w*) are taken as  $0.5 \dots 1^\circ$  considering the required accuracy of the solution. In this case the range of conics is  $20 \dots 50$  thousand. This



**Fig. 3. A diagram of formation of a range of hyperbolae**

range is successfully proceeded on a personal computer.

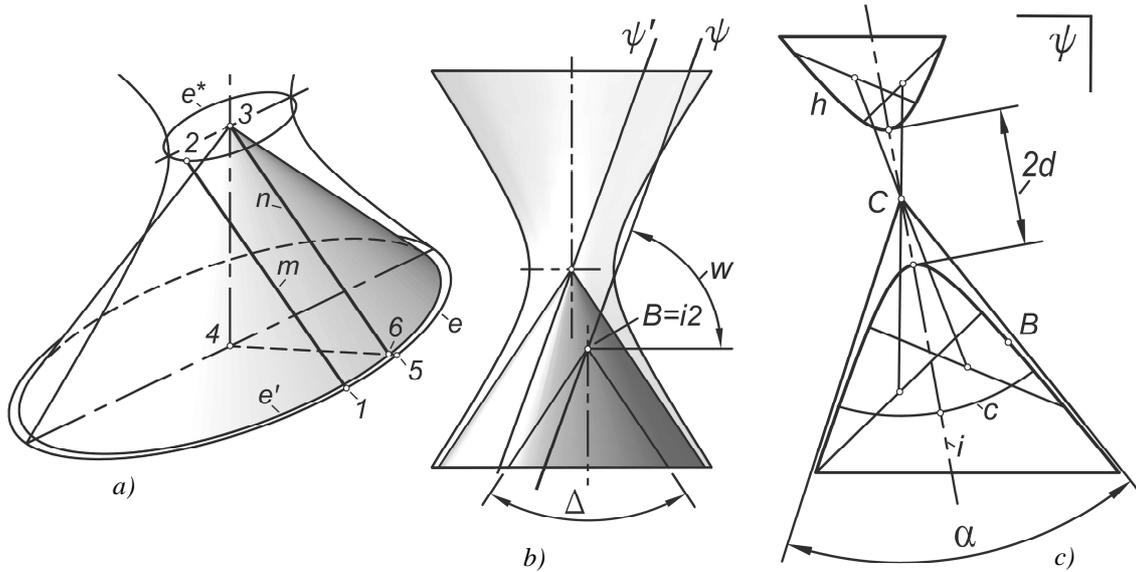
The range of hyperbolae shall be defined at asymptotic cone (*AC*). To build (fig. 4, *a*) it we shall draw tangent *m* to ellipsis *e* of the neck of *OH* from arbitrary point *1* of ellipsis *e* of *OH*. Tangent point 2 is determined by the object snap. Let us place *m* to the center 3 of ellipsis *e*. From the center 4 through point 6 we shall draw the segment till its intersection with ellipsis *e* at point 5. Let us scale ellipsis *e* so that it passes through point 6, thus we obtain ellipsis *e'*. We shall build *AC* at its base *e'* and apex 3.

For each value of *u*-coordinate we shall project *AC* to the plane (fig. 4, *b*) which is perpendicular to the current position of axis *i2*, and find sector  $\Delta$ . For hyperbolae cutting plane  $\psi$  shall pass within sector  $\Delta$ .

Hyperbola of the section *h* of plane  $\psi$  obtained as spline curve is characterised by metrics *d*,  $\alpha$  (fig. 4, *c*). We shall define center *C* by the method of chords. Let us draw axis *i* through the middle point *c*, then we shall define apex of hyperbola, defining metric *d*. Angle  $\alpha$  between asymptotes of hyperbola is determined by section *AC* with plane  $\Psi' \parallel \Psi$ .

Metrics of every hyperbolae  $\alpha$ , *d*, as well as their angle coordinates *u*, *w*, angles of inclination of the planes of hyperbolae to the planes of symmetry of *OH*, and other parameters which are necessary for the study of the model, are placed into database (*DB*).

Further study of the model is in the extraction of hyperbolae with the given values of metrics  $\alpha$ , *d* from *DB*. We shall introduce the allowance of the search (error)  $del = 0.5 (del1 + del2)$ , where *del1* is the error to metric *d*, *del2* is the error to metric  $\alpha$ . Originally having determined  $del = 5 \dots 10\%$ , we shall extract a



**Fig. 4. Elements of solution algorithm: a – asymptotic cone; b - determination of the range of hyperbolae; c – determination of metrics of hyperbolae**

group of hyperbolae from  $DB$ . The number of groups is the number of solutions. In each group we shall find the hyperbola with a minimum value  $del$ , which is one of the variant of solution.

AutoCAD is used to perform simulation, algorithms and their metrics, analysis of database is performed by means of list-processing of AutoLisp.

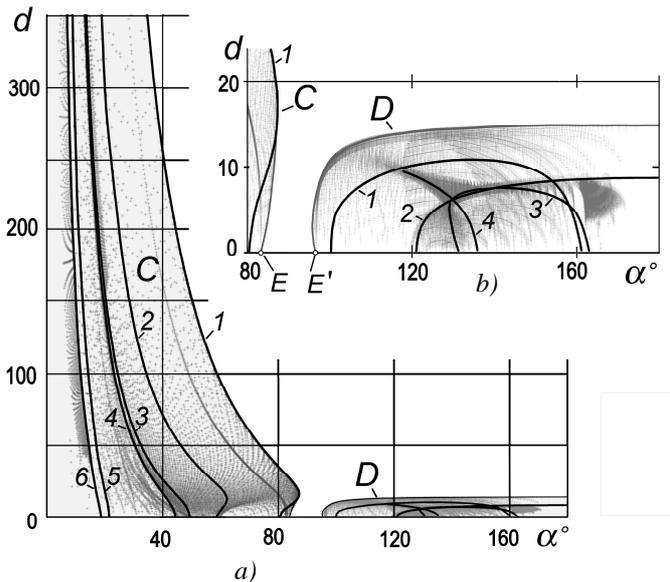
### 3. THE RANGE OF POSSIBLE SOLUTIONS

Let us display a database in the coordinates  $\alpha, d$  (fig. 5, a). Each hyperbola is marked with the point. About  $\approx 36000$  points are displayed all in all. We have two areas  $C$  and  $D$ . Tests show that area  $C$  are the hyperbolae with the exterior center (see fig. 8, b). Area  $D$  (fig. 5, b) is the hyperbolae with the inner center (see fig. 9, b).

### 4. HYPERBOLAE OF A PARTICULAR CASE

Defining the angles of inclination of the plane of hyperbola to the planes of symmetry of  $OH$ , we shall get the curves  $1..6$ . The curves  $1, 5$  are front-projecting hyperbolae (their planes are perpendicular to the frontal plane of symmetry of  $OH$  including minor axis of ellipsis); hyperbolae  $1$  and  $5$  differ from each other by the sign of the angle of inclination to the frontal plane of symmetry of  $OH$ . Hyperbola  $2$  and  $4$  are equally inclined to the planes of symmetry of  $OH$  in a number of ways. The curves  $3$  and  $6$  are front-projecting hyperbolae with different angles of inclination to the frontal plane of symmetry.

Provided  $\alpha \rightarrow 0$  asymptotes of hyperbolae are combined, distance  $2d$  between the apexes increases



**Fig. 5. The range of possible solutions**

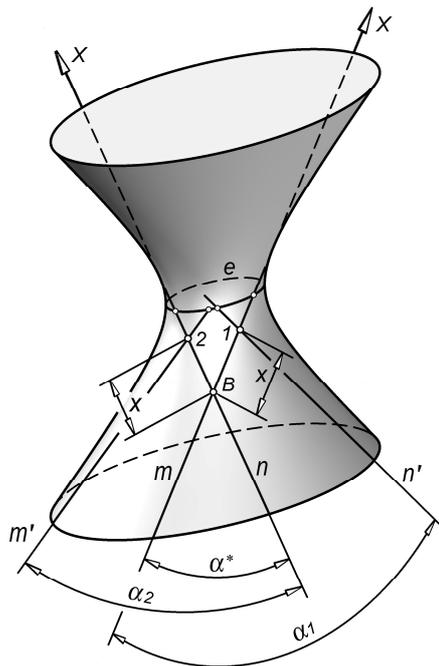
without restrictions, hyperbola degenerates into the straight line. Provided  $\alpha = 180^\circ$ , hyperbola is degenerated into two parallel lines tangent to the neck. One of the lines of each pair passes through point  $B$ . Provided  $d = 0$  (points of axis  $\alpha$ ), the apexes of hyperbolae coincide with the center, hyperbolae degenerate into their asymptotes, that is into two intersection lines.

## 5. THE NUMBER OF POSSIBLE SOLUTIONS

We shall build lines  $m$  and  $n$  passing through point  $B$ , which are tangent to the ellipsis  $e$  of the neck of  $OH$  (fig. 6). We shall take point  $1$  to the tangent  $m$ , and build tangent  $n'$  to the ellipsis  $e$  from this point. We will obtain lines  $m \cap n'$  which can be considered as a degenerated hyperbola with angle  $\alpha_1$  between asymptotes and  $d = 0$ . The move of point  $1$  leads to the change of angle  $\alpha_1$  according to the experimentally defined curve  $1$  (fig. 7). Curve 2 reflects the same dependence while moving point 2 along tangent  $n$ .

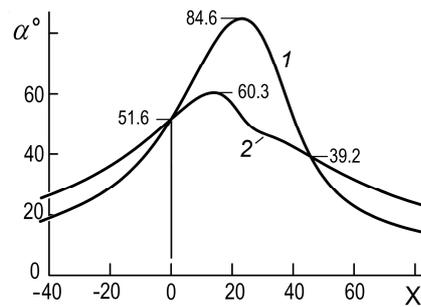
Let us assume that for a group of hyperbolae with the same value of the angle between asymptotes the number of possible solutions is equal to the number of degenerated hyperbolae of this group and doesn't depend on the distance  $d$  between the apex and center. This assumption is proved by the experimental testing (see below).

If we put the horizontal straight line on the level  $\alpha$  (see fig. 7), we shall define the number of solutions of this group with the given value of the angle  $\alpha$ . If  $\alpha > 84.6^\circ$ , there is no solution. Value  $\alpha = 84.6^\circ$  refers to the point  $E$  (see fig. 5,  $b$ ), in which there is the only



**Fig. 6. The tangent planes to the surface of  $OH$  passing through point  $B$**

solution. If  $60.3^\circ < \alpha < 84.6^\circ$  there are two solutions. If  $\alpha < 60.3^\circ$  there are four solutions. And among them there are coinciding solutions. If  $\alpha = 51.6^\circ$  and  $60.3^\circ$  two planes coincide among them and that is why among four solutions we see only three solutions. If  $\alpha$



**Fig. 7. Dependence of the angle between the tangents on the inclination of the point of tangency**

$= 39.2^\circ$  there are four solutions, two of which have equal angles between the asymptotes of hyperbolae (but their planes do not coincide).

For the circular  $OH$  the curves 1 and 2 coincide, that is the reason of existence of maximum two solutions.

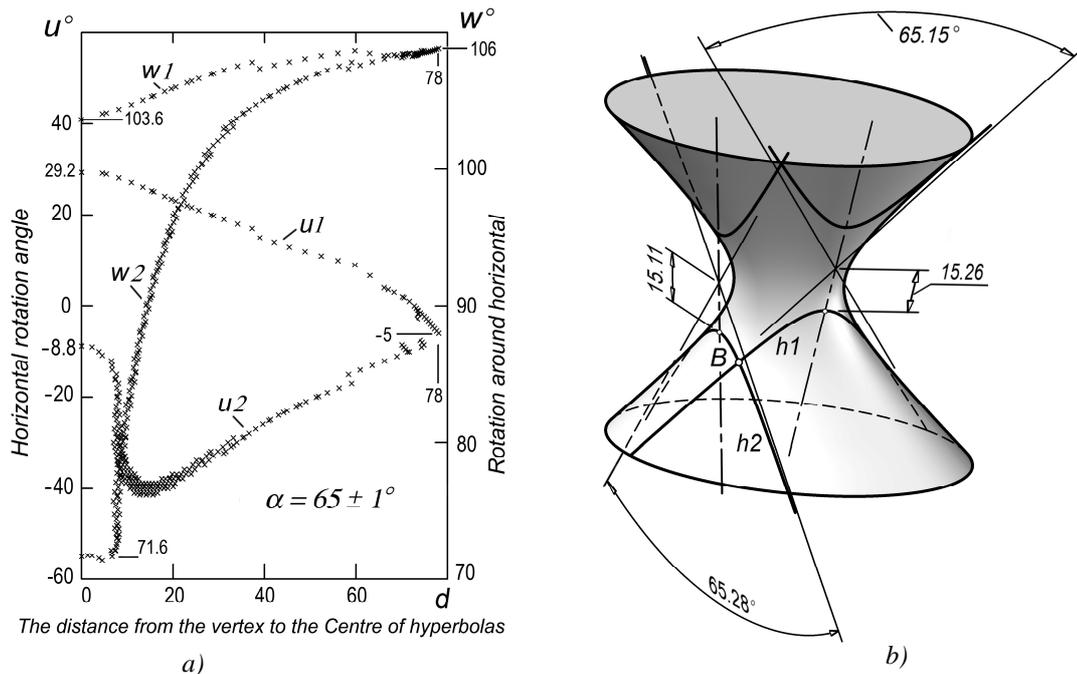
## 6. GROUPS OF HYPERBOLAE

Let us consider the formation of hyperbolae with the given angle  $\alpha$  between the asymptotes. Let  $\alpha = 65^\circ$ . Let us draw the sample of hyperbolae with  $\alpha = 65 \pm 1^\circ$  from database. We shall extract angular coordinates  $u, w$  of the planes of hyperbola and build dependency of their changes on metric  $d$  (fig. 8,  $a$ ).

We see that if  $d = 0$  there are two degenerate hyperbolae. This corresponds to fig. 7, according to which if  $\alpha = 65^\circ$  there are two solutions. To the first degenerate hyperbola there are coordinates  $u_1=29.2^\circ, w_1=103.6^\circ$  (see fig. 8,  $a$ ), to the second hyperbola the coordinates are  $u_2 = -8.8^\circ, w_2 = 71.6^\circ$ . Having planes with the given coordinates  $u, v$ , in sections we will get two pairs of concurrent straight lines, the angle between the straight lines of which is  $65^\circ$ . The increase of metric  $d$  requires the rotation of sectional plane  $\psi$  toward axes  $u$  and  $v$  in accordance with the dependence  $u_1(d), w_1(d)$  to the first hyperbola. To the second hyperbola the dependence is  $u_2(d), w_2(d)$ . In the interval  $0 < d < d_{max} \approx 78$  the original number of hyperbolae is kept. The value of  $d_{max}$  corresponds to the upper value of the area  $C$  (see fig. 5,  $a$ ). At  $d=d_{max}$  hyperbolae coincide and their coordinates take the values  $u \approx -5^\circ, w \approx 106^\circ$ .

Metrics  $\alpha = 65^\circ$  and  $d = 15$  are set as an example. Two hyperbolae  $h_1, h_2$  are found in database (fig. 8,  $b$ ). The values of  $u, v$  coordinates of these hyperbolae correspond to fig. 8,  $a$ . For  $h_1$  the error of  $del$  is  $(0.26:15 + 0.15:65) \cdot 50 = 0.98\%$ . For  $h_2$  the value of  $del = 0.58\%$ .

Obtuse angle between the asymptotes leads to the hyperbolae with inner center (see fig. 5, area  $D$ ). As the tangent planes for the angles  $\alpha$  and  $(180 - \alpha)$  are



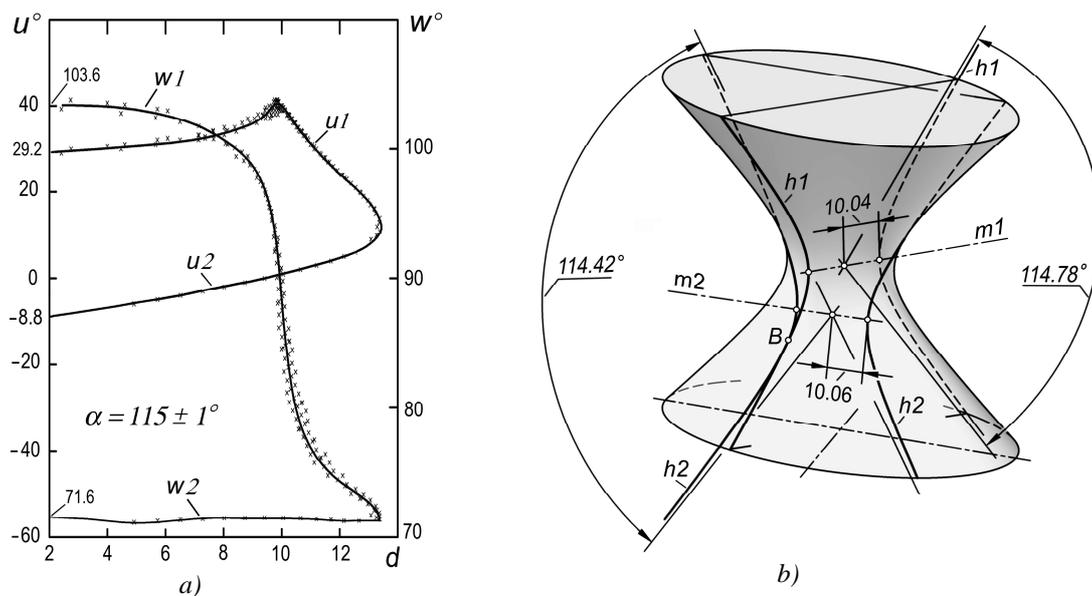
**Fig. 8. Hyperbolae with exterior center for  $\alpha = 65^\circ$ : a – is the change of angular coordinates of a sectional plane; b – hyperbolae for  $d = 15$  (two solutions)**

equal, than the number of solutions for obtuse angle can be determined in accordance with fig. 7 for additional axis. For example, the number of solutions for  $\alpha = 115^\circ$  and  $\alpha = 65^\circ$  is the same and equals two. Limiting point  $E$  has a responsive point  $E'$  (see fig. 5).

Let us draw a sample to  $\alpha = 115 \pm 1^\circ$  (fig. 9). The number and coordinates of degenerated hyperbolae

are the same (see the values of  $u, v$  if  $d = 0$ ) as for the angle  $\alpha = 65^\circ$  (see fig. 8, a above). However the cinematics of movement of a sectional plane is considerably different (fig. 9, a). Throughout the whole interval  $0 \leq d < d_{max} \approx 13.5$  two solutions are kept. At  $d_{max}$  two combined hyperbolae are obtained.

As an example of hyperbola with inner center from database we shall get hyperbolae  $h1, h2$  if  $d = 10$  (fig.

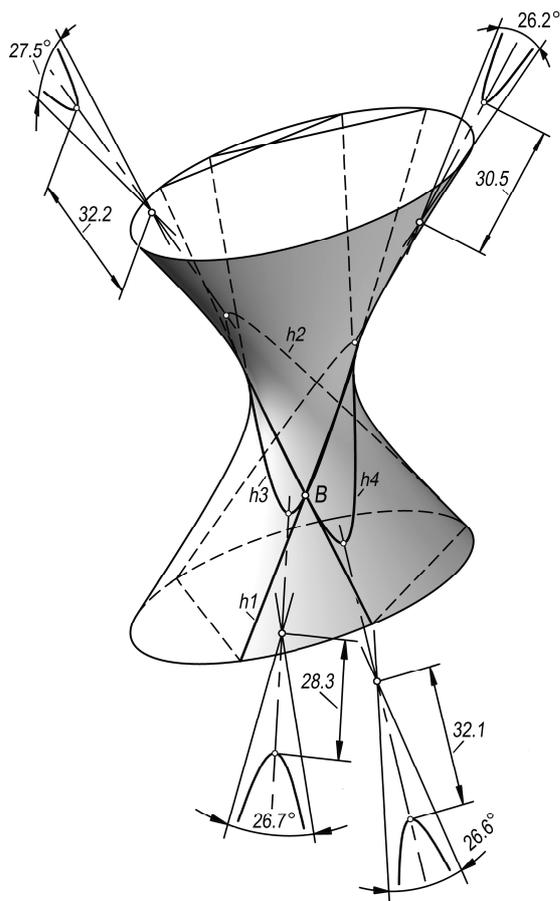


**Fig. 9. Hyperbolae with inner center: a – angular coordinates of a sectional plane; b – hyperbolae  $\alpha = 115^\circ$  and  $d = 10$  (two solutions)**

9, b). We see that real axes of these hyperbolae  $m1$ ,  $m2$  are perpendicular to the axis of  $OH$ . The errors  $del$  of hyperbolae  $h1$ ,  $h2$  are 0.55% and 0.30% correspondingly. Hyperbola  $h2$  is front-projecting (for it  $u2 \approx 0$ , see fig. 9, a).

Having taken metrics  $\alpha = 27^\circ$ ,  $d = 30$  we will get the solution with four hyperbolae with inner center (fig. 10). This corresponds to fig. 7, where the horizontal straight line  $\alpha = 27^\circ$  crosses the curves 1, 2 summarily in 4 points. Only one half-hyperbola is on the bay of  $OH$  of each hyperbolae  $h1 \dots h4$ . The second half-hyperbola is built on the basis of five points of the first half-hyperbola [4]. Because of insufficient density of database in this area the found hyperbolae have serious errors which are having  $del = 2.3, 4.6, 3.4$  and  $4.2\%$  for hyperbolae  $h1 \dots h4$  correspondingly.

Improved accuracy of solution to any desired value is achieved by creating the  $DB$  higher density, the use of interpolation or the transition to straight line skeleton from  $OG$  [6].



**Fig. 10. Four solutions of hyperbola with  $\alpha = 27^\circ$ ,  $d = 30$**

## SUMMARY

1. 3D method of computer geometric simulation allows us to find the solution of the problem under consideration with the required accuracy and studying its mechanism including the sphere and number of solutions.

2. 3D method in a combination with the model of a “black box” and programming can be recommended for practical problems of geometric simulation, in which the construction of analytical and geometric models is complicated or unreasonable due to their difficulty.

3. 3D method doesn't require projection and analytical constructions and can be considered as an independent method of geometric simulation.

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